Exact methods for string flux vacua

Matthew D. Schwartz*
Department of Physics, Harvard University

Abstract

The string landscape holds infinitely many candidate flux vacua, and while these are usually explored one by one, the deepest questions about the landscape concern all of them at once. This paper puts two tools to work on such questions: the genus theory of quadratic forms from computational number theory, which enumerates flux lattices and proves the list complete, and the interval arithmetic behind computer-assisted proofs, which establishes that critical points exist. On K3×K3 we prove, for every flux whose support has rank at most three, the bound indicated by the searches of Bena, Blåbäck, Graña and Lüst: full stabilization there costs at least 25 of the 24 available flux units. At higher rank we construct a fully stabilizing flux inside the budget. In the symmetric sector of a Hulek–Verrill fourfold, whose lattice admits unit charge, we prove that over a stated range of fluxes and moduli the first vacuum appears at charge 4; vacua of lower charge lie just outside that range. For the benchmark KKLT vacuum we show that the critical point of the superpotential exists and compute its value with all instanton degrees included and a proven error bound.

Contents

*Work done with Anthropic, but results and views are not endorsed by Anthropic.

1 Introduction

A flux vacuum of string theory is a discrete object: integer flux quanta on the cycles of the compact dimensions, together with the shape of those dimensions that the flux holds fixed. Infinitely many candidate vacua exist, and their statistics have been studied extensively [1, 2, 3, 4], but the sharpest questions about the landscape concern all fluxes at once. No finite list of examples can decide whether any flux within a geometry's tadpole bound can stabilize all its moduli, or how small the charge of a vacuum in a given flux sector can be. Other questions concern one reported vacuum: whether it exists at the stated point in moduli space, and with what superpotential.

Such questions reduce to definite computations: locating critical points of the Gukov–Vafa–Witten superpotential [5], enumerating flux vectors in a lattice and evaluating periods. For M-theory on a Calabi–Yau fourfold X with flux G the equations are

$$W(z) \;=\; \int_X G \wedge \Omega(z)\,, \qquad D_I W \;=\; 0\,, \qquad N_{\text{flux}} \;=\; \tfrac{1}{2}\int_X G \wedge G \;\leq\; \frac{\chi(X)}{24}\,, \tag{1}$$

where $\Omega(z)$ is the holomorphic four-form, z the complex-structure moduli, D_I the Kähler-covariant derivative and $\chi(X)$ the Euler number [6, 5]. The last condition is the tadpole bound on the flux charge N_{flux}. We call $\chi(X)/24$ the budget; a flux with $N_{\text{flux}} = B$ is at budget B.

The Tadpole Conjecture of Bena, Blåbäck, Graña and Lüst [7, 8] states that the flux charge needed to stabilize many complex-structure moduli at a generic point grows linearly with their number, fast enough to exceed the tadpole bound. Their primary example is M-theory on K3×K3, where the budget is 24 and the smallest charge found in their numerical search for fully stabilizing fluxes was 25. On that geometry the purely transcendental fluxes on attractive K3 pairs within the bound have been enumerated [9, 10, 11], each leaving a non-abelian gauge group unbroken [11]. The conjecture has since been argued, with strong evidence, in the strict asymptotic regime of moduli space [12]. On the fixed locus of a discrete symmetry, invariant fluxes with nonvanishing on-shell superpotential stabilize at least sixty percent of the moduli [13]. For further tests see [14, 15]. In the 2^6 Landau–Ginzburg (Gepner) model, solutions within the bound with all complex-structure moduli massive were reported in [16]; a recomputation of the orientifold charge halves that bound and places those solutions outside it [17]. Examples inconsistent with the refined form (7) of the conjecture appear in [18].

Among individual vacua, the Kachru–Kallosh–Linde–Trivedi (KKLT) [19] benchmark is the flux vacuum of Demirtas, Kim, McAllister and Moritz (DKMM) [20]. Two later computations [21, 22] found its superpotential $|W_0|$ half a percent above the original value, and to our knowledge the three values had not been reconciled in print.

In this paper we apply two exact tools to such questions: the genus theory of quadratic forms and interval arithmetic. Some flux computations are already exact (closed-form counts of vacua [23], reduction theory [9], vacua at algebraic points [24] and lattice enumeration of flux vectors [25, 26, 27, 28]). Most searches and period evaluations use floating point [29, 30] and cannot prove that the smallest charge found is minimal or, without error bounds, decide between two reported values.

Even lattices L of given signature and discriminant form make up a genus, a finite set of isometry classes. Lattice genera, Nikulin embeddings and Niemeier-lattice complements have been used on the K3×K3 flux problem [31, 10, 32, 11], and the Smith–Minkowski–Siegel mass formula appears elsewhere in string theory [33, 34, 35], for instance in counting flux vacua [36, 37]. The formula gives the mass of a genus, the sum of $1/|\mathrm{Aut}\, L|$ over its classes, without enumeration [38]. Thus an enumeration, like ours on K3×K3, is proven complete once its classes exhaust the mass.

Interval arithmetic carries a proven error bound through every truncation and rounding. With it the Krawczyk operator [39] gives a sufficient condition for a box to contain exactly one zero of a map; such root isolation, by α-theory, was proposed for polynomial string-vacuum models in [40, 41]. Rigorous period computation of this kind has been carried out for K3 surfaces, Feynman motives and one-parameter Calabi–Yau operators [42, 43, 44, 45], resting on the truncation bounds of [46], but we find no earlier use of it, joined to exact lattice arithmetic, on flux vacua.

For an enumerative question a structural theorem reduces the infinitely many fluxes to finitely many cases, which we decide by exact arithmetic. In our results, Table 7 and Table S2 of the Supplementary Material excepted, every real number is carried as an interval proven to contain it, every integer is exact, and a discrete statement that is not proven is left open. Throughout, a vacuum means a solution of the classical flux equations, Eq. (1): an integer flux within the budget together with an F-flat point of its superpotential in the complex-structure moduli. Whether such a solution is realized in the full string theory, with the Kähler moduli stabilized and with or without a positive cosmological constant, is a separate and debated question [47, 48]; this paper does not address it, and no result below depends on it.

First, on K3×K3 we prove, under hypotheses (H1)–(H4) of Section 3.1, that every fully stabilizing flux whose support (the smallest primitive sublattice of $H^2(\mathrm{K3}, \mathbb{Z})$ containing the image of the flux matrix) has rank at most three carries $N_{\mathrm{flux}} \geq 25$. We sort fluxes into strata by the signature of their support; on most strata the signature alone excludes full stabilization. On the stratum of rank at most three, 412,531 candidates lie within the budget, each excluded because on one K3 factor the orthogonal complement of its support contains a root (a norm -2 vector, placing that surface at an orbifold point).

Off the stratum of rank at most three the bound as stated for smooth K3×K3 in [7, 8] does not hold. We construct a fully stabilizing flux of charge 22 and rank 22 (Section 3.4), so the bound is a statement about that stratum alone. Its vacuum lies on the fixed locus of an order-66 lattice symmetry, the kind of special locus anticipated in [13]. On the definite stratum of rank 19 we exhibit three root-free lattices and prove that a fully stabilizing flux supported on a pair of them has charge at least 24 or 25, according to the pair. Attaining the lower of the two bounds amounts to a single Diophantine equation that we state and do not solve.

Second, in the S_6-invariant flux sector of a Hulek–Verrill fourfold [49, 24] with $\chi/24 = 30$, whose flux lattice admits unit charge, we prove within a stated flux cut and modulus box that no vacuum has charge below 4 and exhibit one of charge 4. Both statements assume, as in [49], that the order-5 Picard–Fuchs operator of the S_6-symmetric slice annihilates the periods (hypothesis (O)). Vacua of lower charge occur just outside the box. At a conifold point of the slice the unit-charge flux is the vanishing class, giving a solution with $W = 0$ [50, 51]; whether a unit-charge vacuum exists at a smooth point of the slice remains open.

Third, a Krawczyk argument proves that the critical point of the DKMM superpotential exists and is unique within a small box. We obtain $|W_0| = 2.0371 \times 10^{-8}$ to about 140 digits with all instanton degrees included (in fixed conventions, with the flux quantization of [20] as given). We show that the half-percent spread of the published values comes from one approximation (dropping the $\zeta(3)\chi$ term of the prepotential), and that all three computations [20, 21, 22] are internally consistent. Moreover, for all thirty de Sitter candidates of [52] we find that a rational invariant of the flux data [53, 28, 29] is nonzero, consistent with their construction as conifold vacua [54]. Table 1 summarizes the results.

Section 2 reviews the vacuum equations and the two tools, and Sections 3–5 present the three results. The Supplementary Material contains the proofs of the existence and isolation criteria of Section 3.1, enumerations in ten flux sectors with an attractor-point no-go theorem, and a related conjecture (Sections S1–S2), notes on the KKLT data and the implementation (Sections S3–S4), and the lattice, charge-22 and Hulek–Verrill data (Sections S5–S7).

2 Flux vacua and two exact tools

We use the vacuum equations of flux compactifications and two bodies of standard mathematics, the genus theory of integral quadratic forms (already applied to K3 lattices in [33, 31, 11, 55]) and interval arithmetic with rigorous error bounds. Specific to this paper are only the remarks on our implementation and the form of the tail bound (15), a majorant of the kind due to van der Hoeven [56, 57] and Mezzarobba [58, 46].

2.1 The vacuum equations and the tadpole bound

Consider M-theory on a Calabi–Yau fourfold X with four-form flux G_4, quantized as in [59]. The flux enters the M2-brane tadpole condition

$$n_{M2} + N_{\text{flux}} \;=\; \frac{\chi(X)}{24}\,, \qquad N_{\text{flux}} \;=\; \tfrac{1}{2}\int_X G_4 \wedge G_4\,, \tag{2}$$

which caps the flux charge once anti-M2 branes are excluded [6, 60]: with $n_{M2} \geq 0$, $N_{\text{flux}} \leq \chi(X)/24$.

Supersymmetric Minkowski vacua require G_4 to be primitive and of Hodge type $(2,2)$ [60]. In terms of the Gukov–Vafa–Witten superpotential [5] the conditions on the complex structure read

$$W(z) \;=\; \int_X G_4 \wedge \Omega(z)\,, \qquad D_I W \;=\; 0 \text{ for all } I \iff G_4^{(1,3)} \;=\; 0\,, \qquad W \;=\; 0 \iff G_4^{(0,4)} \;=\; 0\,, \tag{3}$$

where z are complex-structure coordinates, $D_I = \partial_I + (\partial_I K)$ with K the Kähler potential, and the Hodge components of G_4 are taken at z (the conjugate components vanish too, G_4 being real). A primitive flux obeying $D_I W = 0$ has positive charge:

$$\begin{aligned} &G_4 \text{ primitive}, \quad G_4^{(1,3)} = G_4^{(3,1)} = 0 \implies *G_4 \;=\; G_4 \\ \implies &\int_X G_4 \wedge G_4 \;=\; \int_X G_4 \wedge *G_4 \;=\; \|G_4\|^2 \;>\; 0 \quad \text{unless } G_4 = 0\,. \end{aligned} \tag{4}$$

The first implication is the Hodge–Riemann sign rule for primitive four-forms on a fourfold. A supersymmetric vacuum has in addition $W = 0$ [60], and the flux is then of pure type $(2,2)$. Below we impose or exclude $W = 0$ only where we say so.

Integral quadratic forms enter through the intersection pairing $(x,y) = \int_X x \wedge y$ on $H^4(X,\mathbb{Z})$, a symmetric bilinear form that is unimodular by Poincaré duality. The charge of an integral flux is half its norm,

$$N_{\text{flux}} \;=\; \tfrac{1}{2}\,(G_4, G_4)\,, \qquad G_4 \in H^4(X,\mathbb{Z})\,, \tag{5}$$

result	domain and hypotheses	statement	open
K3×K3, support rank ≤ 3	(H1)–(H4) and (H-RANK); budget $\chi/24 = 24$	every fully stabilizing flux has $N_{\text{flux}} \geq 25$ (Theorem 3.4)	none
K3×K3, definite stratum $(0, 19)$	(H1)–(H4); fluxes supported on a pair of the lattices W_{512}, W'_{512}, W_{1024}	$N_{\text{flux}} \geq 24$ on a determinant-512 pair, $N_{\text{flux}} \geq 25$ on any pair involving W_{1024} (Theorem 3.5; lattice data in Section S5 of the Supplementary Material)	$N_{\text{flux}} = 24$ on the determinant-512 pairs, one Diophantine equation (Eq. (39)); further root-free admissible lattices exist at determinants 160 through 224, so the stratum is open
K3×K3, support ranks $4 \leq r \leq 22$	(H1)–(H4)	we construct a fully stabilizing flux of charge 22 and rank 22 (§3.4; arithmetic data in Section S6 of the Supplementary Material), whose vacuum lies on the fixed locus of an order-66 lattice symmetry, so the bound as stated in [7, 8] fails off the rank-≤ 3 stratum	the enumeration of these strata is proven finite and has not been carried out
Hulek–Verrill fourfold [49, 24], S_6-invariant sector	flux cut $n_2 \leq 751$, $N_{\text{flux}} \leq 30$; modulus box R: $\operatorname{Re}\varphi \in [13/1024, 19/1024]$, $\lvert\operatorname{Im}\varphi\rvert \leq 1/256$; hypothesis (O) of Proposition 4.1 (all-orders annihilation of the periods by the order-5 operator of [49])	no vacuum of charge ≤ 3 in R; one of charge 4, $g = (-1, 0, 0, 0, -3)$ at $\varphi = 0.0172802\ldots$ (Proposition 4.1; Section S7 of the Supplementary Material); the lattice alone allows charge 1	vacua of charge 3 and 2 just outside R; charge 1 at a smooth point of the slice (at the conifold point $\varphi = 1/36$ it is realized by the vanishing class, with $W = 0$ [50, 51]); the full six-parameter flux family
DKMM benchmark [20]	conventions of Section 5.1; flux quantization of the source taken as given	the F-term critical point exists and is unique within a small box; $\lvert W_0\rvert = 2.0371 \times 10^{-8}$ and τ to about 140 digits (Eqs. (66), (67)); the published half-percent spread is the $\zeta(3)\chi$ term	none
thirty de Sitter candidates [52]	repository flux data [53]; for one candidate the integral basis distributed with [28, 29]	$K^T N^{-1} K \neq 0$ exactly for all 30 (Eq. (70)): none is perturbatively flat in any basis, consistent with their construction as conifold vacua	the status of any candidate as a vacuum is not addressed

Table 1: The results, the hypotheses under which each holds, and what stays open. Every statement concerns solutions of the classical flux equations, Eq. (1) or its type IIB counterpart Eq. (57). (H1)–(H4) are the hypotheses of Section 3.1: the supergravity dictionary for flux vacua on K3×K3, integral flux quantization, no anti-M2-branes, and full stabilization in the sense of the predicate (FS) defined there; (H-RANK) restricts the flux support to rank at most three. The support of a flux on either K3 factor is the smallest primitive sublattice of $H^2(\mathrm{K3}, \mathbb{Z})$ containing the image of the flux matrix (Section 3.1), and a stratum is the set of fluxes whose support has a given signature (p, q). W_{512}, W'_{512} and W_{1024} are the three root-free rank-19 lattices of Section 3.3, labeled by their determinants; admissible lattices are defined in Section 3.2. In the Hulek–Verrill row, g is the integer flux vector of the rank-5 invariant sector, n_2 the positive form of Eq. (47), φ the modulus of the S_6-symmetric slice, and charge means N_{flux}. In the last row, K and N are the flux vector and flux matrix of Eq. (70).

so the fluxes satisfying (3) within the tadpole bound are short vectors in an integral lattice of indefinite signature, of positive norm by (4) and of norm at most twice that bound.

In the type IIB orientifold limit, with three-form fluxes F_3, H_3 on a Calabi–Yau threefold X_3, axio-dilaton τ and complex-structure moduli U^a, the corresponding equations are [5]

$$\begin{gathered} W \;=\; \sqrt{2/\pi}\int_{X_3} G_3\wedge\Omega\,, \qquad G_3 \;=\; F_3-\tau H_3\,, \qquad \tau \;=\; C_0+i/g_s\,, \\ D_{U^a}W \;=\; D_\tau W \;=\; 0\,, \qquad D \;=\; \partial+(\partial K)\,, \qquad K \;=\; K_{\mathrm{cs}}+K_\tau\,, \\ n_{D3}+Q_{\mathrm{flux}} \;=\; \frac{\chi(X_4)}{24}\,, \end{gathered} \tag{6}$$

with K_{cs}, K_τ the Kähler potentials of the two sectors, n_{D3} the number of D3-branes and X_4 the F-theory fourfold [7]. The flux charge Q_{flux} is positive once the vacuum conditions make G_3 imaginary self-dual, as in (4), and Eq. (57) fixes the units. On K3×K3 the vacuum conditions become the pair of lattice conditions of Eq. (21), on the symmetric slice of the Hulek–Verrill fourfold the single equation (50), and for the KKLT benchmark a system in six real unknowns, which we solve in Section 5.2.

The Tadpole Conjecture of Bena, Blåbäck, Graña and Lüst [7], in its refined form, states that for large $h^{3,1}$ a flux stabilizing all complex-structure moduli of a fourfold at a generic point has charge

$$N_{\mathrm{flux}} \;>\; \alpha\, h^{3,1}\,, \qquad \alpha > 1/3\,, \tag{7}$$

whereas the bound $\chi/24$ grows more slowly with $h^{3,1}$. If (7) holds, fluxes within the bound stabilize all moduli only at special points of enhanced symmetry or degenerate geometry.

2.2 Exact lattice arithmetic

A lattice is a free abelian group $L \cong \mathbb{Z}^c$ with a nondegenerate integral symmetric bilinear form; it is even if every norm (x,x) is even, and $\det L$ denotes its Gram determinant. An even lattice has a signature (s_+, s_-) over $\mathbb{R}$ and a discriminant form, the finite quadratic form

$$D(L) \;=\; L^*/L\,, \qquad |D(L)| \;=\; |\det L|\,, \qquad q_L : D(L)\to\mathbb{Q}/2\mathbb{Z}\,, \quad q_L(x+L) \;=\; (x,x) \bmod 2\mathbb{Z}\,, \tag{8}$$

where $L^* = \{x \in L\otimes\mathbb{Q} : (x,L)\subseteq\mathbb{Z}\}$ is the dual lattice. Two even lattices with the same signature and isomorphic discriminant forms belong to the same genus (isometric over $\mathbb{R}$ and over every $\mathbb{Z}_p$) [61, 38].

The mass of a genus $\mathcal{G}$,

$$\mathrm{mass}(\mathcal{G}) \;=\; \sum_{[L]\in\mathcal{G}} \frac{1}{|\mathrm{Aut}(L)|}\,, \tag{9}$$

is a rational number that the Smith–Minkowski–Siegel formula gives in closed form, as a product of local densities depending only on the signature and q_L [38]. Kneser's neighbor construction produces from one definite lattice others in its genus, each meeting it in a sublattice of prime index [38]. A list of pairwise non-isometric lattices $L_1,\ldots,L_k \in \mathcal{G}$ is complete exactly when [38, 62]

$$\sum_{i=1}^{k} \frac{1}{|\mathrm{Aut}(L_i)|} \;=\; \mathrm{mass}(\mathcal{G})\,, \tag{10}$$

since a missing class would leave a positive deficit on the left.

If A is a primitive sublattice of signature (a_+, a_-) in an even unimodular lattice Γ of signature (s_+, s_-),[1] its orthogonal complement satisfies

$$K \;=\; A^\perp \subset \Gamma\,, \qquad \operatorname{sign} K \;=\; (s_+ - a_+,\; s_- - a_-)\,, \qquad q_K \;\cong\; -q_A\,, \qquad |\det K| \;=\; |\det A|\,. \tag{11}$$

Conversely, A embeds primitively in Γ if and only if an even lattice K with this signature and discriminant form exists [61]. Given A and K, one recovers Γ by *gluing*, as the overlattice of $A \oplus K$ generated by the pairs in $A^* \oplus K^*$ whose classes match under an isomorphism $D(A) \to D(K)$ taking q_A to $-q_K$. The embeddings with complement in a given class $[K]$ are classified by this gluing up to the automorphisms of A and K [61]. Any statement below that an embedding is unique has this scope: it covers uniqueness of the gluing, and multiplicity of the complement's class is counted separately, by enumerating the genus of K.

A root of an even lattice is a vector δ with $\delta^2 = -2$ in the sign convention of K3×K3 below, where the definite lattices are negative-definite. We state lattice facts for the positive-definite versions, in which a root has norm 2 and a lattice is root-free exactly when $\min(L) \geq 4$. For a positive-definite lattice of rank c, Hermite's constant γ_c bounds the minimum, and any sphere-packing bound $\bar{\delta}_c$ bounds the center density $\delta(L)$,

$$\min(L) \;\leq\; \gamma_c\,(\det L)^{1/c}\,, \qquad \delta(L) \;=\; \Big(\frac{\min(L)}{4}\Big)^{c/2}(\det L)^{-1/2} \;\leq\; \bar{\delta}_c\,. \tag{12}$$

Thus a root-free even lattice of rank c has $\det L \geq (4/\gamma_c)^c$ and $\det L \geq \bar{\delta}_c^{-2}$, and a small determinant forces a root, as in [32] at rank 18 (cf. [11, Sec. 4] and [63]). The constant γ_c is known exactly through rank 8 (Gauss, Korkine–Zolotareff, Blichfeldt [64, 38]) and in ranks 9 and 24 [65, 66]; above rank 8 we take $\bar{\delta}_c$ from the linear-programming bounds of Cohn and Elkies [67].

When (12) does not force a root, we prove that every class of a genus has one by a linear system built from the classification of even unimodular lattices of rank 24 [68, 69]. Following the Kneser–Nishiyama method [70, 31, 11, 62], let $\mathcal{G}'$ be a genus of positive-definite even lattices of rank 19 whose discriminant form is q_T for a positive-definite even ternary lattice T (as in Lemma 3.3). The positive-definite even lattices K of rank 5 with $q_K \cong -q_T$ form the *complement genus* C of $\mathcal{G}'$. Gluing T to any $K \in C$ gives an even unimodular lattice of rank 8, hence E_8, so by (11) the classes of C are the complements of the primitive embeddings of T into E_8, which we enumerate. Gluing $L' \in \mathcal{G}'$ to $K \in C$ likewise gives a positive-definite even unimodular lattice M of rank 24, a Niemeier lattice, containing L' and K as mutually orthogonal primitive sublattices.

If K has a root, M is not the Leech lattice, and by Venkov's theorem its root system $R(M)$ has a single Coxeter number h from a finite list, with

$$\sum_{r \in R(M)} r\, r^{\mathsf{T}} \;=\; 2h\, I_{24}\,, \qquad |R(M)| \;=\; 24h\,;$$

$$\mathrm{LP}(K,h): \quad \sum_v m_v\, x_v x_v^{\mathsf{T}} \;=\; 2h\, G_K - Q_K\,, \qquad \sum_v m_v \;=\; 24h - n_K\,, \qquad 0 \;\leq\; m_v \;\leq\; 2\lfloor 2/v^2 \rfloor\,. \tag{13}$$

The second line is the projection of the first onto K when L' is root-free. Here G_K is the Gram matrix of a basis of K, x_y the vector of inner products of y with that basis, and $Q_K = \sum_{r \in R(K)} x_r x_r^{\mathsf{T}}$ the contribution of the n_K roots of K itself. The integer m_v is the number of roots of M projecting to $\pm v$, for the finitely many $v \in K^* \setminus K$ with $0 < v^2 < 2$.

Hence a root-free $L' \in \mathcal{G}'$ yields a solution of $\mathrm{LP}(K,h)$ for every $K \in C$ with a root and some h on the list. Therefore every class of $\mathcal{G}'$ contains a root if $\mathrm{LP}(K,h)$ is infeasible for one such K and

[1] Primitive means here that Γ/A is torsion-free.

every h, which we prove by Farkas certificates, rational multipliers combining the constraints into an inequality that cannot hold. In Section S5.2 of the Supplementary Material we derive this system, write it out for one genus and give a second proof through theta series and linear programming where that route applies.

2.3 Rigorous numerics for periods and critical points

We carry every period, and every number built from periods, as a ball (a midpoint and a radius that bounds its error) in the arithmetic of the Arb library [71, 72]. Each operation rounds the output radius outward, so the true value remains inside the ball through any finite computation. A ball that excludes zero proves the enclosed number nonzero; this is how we establish $W \neq 0$ below.

We compute the periods of each family from its Picard–Fuchs operator,[2] a Fuchsian operator L of order r in θ-form, $\theta = z\,\mathrm{d}/\mathrm{d}z$. We expand its solutions about a regular singular point, in practice a point of maximal unipotent monodromy (MUM) or a conifold point:

$$z^{-s_{\min}} L = \sum_{s=0}^{S} z^s R_s(\theta)\,, \qquad \varpi(z) \;=\; z^{\rho} \sum_{m\geq 0} \sum_{j=0}^{r-1} (b_m)_j\, z^m\, \frac{\log^j z}{j!}\,,$$
$$\Phi(m,t) \;=\; \frac{G(m)}{|c_0|\,(m-\beta)^r} \sum_{s=1}^{S} t^{-s} \sum_{k=0}^{r} |R_s|_k\,(m+c_s)^k\,, \qquad G(m) \;=\; \prod_i \sum_{j=0}^{r-1} \binom{\mu_i + j - 1}{j} (m-\beta)^{-j}\,. \tag{14}$$

Here $s_{\min}$ is the lowest power of z in L, S the number of further powers, ρ a local exponent, $R_0(\theta) = c_0 \prod_i (\theta - \lambda_i)^{\mu_i}$ the indicial polynomial, $|R_s|_k$ the absolute value of the coefficient of θ^k in R_s, $\beta = \max(0, \max_i \operatorname{Re}\lambda_i - \rho)$ and $c_s = \max(0, \rho - s + 1)$. The coefficient jets $b_m = ((b_m)_0, \ldots, (b_m)_{r-1})$ obey a linear recursion of depth S in m, read off from the R_s, which generates them exactly to any order.

We bound the tail of the series $\varpi(z)$ of (14) truncated at order N by a majorant, following [56, 57, 58, 46]. For any truncation order N past a finite threshold set by the local exponents and S, and any $t > 0$ with $\Phi(N+1,t) \leq 1$, the jets and their tail obey

$$\|b_m\| \;\leq\; K\, t^m \quad (m > N), \qquad \Big|\sum_{m>N} (b_m)_j\, x^m\Big| \;\leq\; \frac{K\,(tx)^{N+1}}{1 - tx} \quad \Big(0 < x < \tfrac{1}{t}\Big), \tag{15}$$

with $K = \max_{N-S<m'\leq N} \|b_{m'}\|\, t^{-m'}$ computed from coefficients in hand and $\|b_m\| = \sum_j |(b_m)_j|$. The hypothesis $\Phi(N+1,t) \leq 1$ of this majorant is a single finite inequality in rational arithmetic, and the conclusion covers all jets b_m with $m > N$, logarithmic components included. The proof is an induction on m. By the recursion, $\|b_m\|\, t^{-m}$ is at most $\Phi(m,t)$ times the largest of the S preceding values $\|b_{m-s}\|\, t^{-(m-s)}$, and $\Phi(\cdot,t)$ is non-increasing past the threshold. An empirical rule of bounding the tail by four times the last computed window, used in an earlier version of our own code, has two exact counterexamples (Section S4.1 of the Supplementary Material). Fixed-degree truncation with no tail estimate, as in [73, 74, 29], gives no error bound.

We transport the period vector by re-expansion (analytic continuation): the truncated series and its tail bound, evaluated at a point inside the disk of convergence, enclose the initial data of the expansion about that point. Finitely many such steps give the Frobenius basis at the point of interest as a vector of balls.

We prove that each critical point of W below exists and is locally unique. Write the equations as $F(x) = 0$ for a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ (the real and imaginary parts of the $D_I W$ in real

[2] Here the letters L, K, A, X and r denote new objects.

coordinates). Let $\tilde{x}$ be a numerical candidate zero, X a box (a product of intervals) centered at $\tilde{x}$, Y an approximate inverse of the Jacobian $F'(\tilde{x})$, and $F'(X)$ a ball enclosure of the Jacobian over X. The Krawczyk operator [39] (for D-finite systems see [75]) and the interval Newton test built on it are

$$\begin{aligned} \mathcal{K}(\tilde{x}, X) \;&=\; \tilde{x} - Y F(\tilde{x}) + \big(I - Y F'(X)\big)(X - \tilde{x})\,, \\ \mathcal{K}(\tilde{x}, X) \subset \operatorname{int} X \;&\Longrightarrow\; F \text{ has exactly one zero in } X\,. \end{aligned} \tag{16}$$

Strict containment proves every matrix in $F'(X)$ nonsingular, and iterating $X \mapsto \mathcal{K}(\tilde{x}, X) \cap X$ shrinks the box.

For a single non-holomorphic equation $\Psi(\varphi) = 0$ in one complex modulus, such as Eq. (50), we take $n = 2$, $x = (\operatorname{Re}\varphi, \operatorname{Im}\varphi)$ and $F = (\operatorname{Re}\Psi, \operatorname{Im}\Psi)$ (written out in Section S7.2 of the Supplementary Material). With the test (16) we enclose the vacuum of Proposition 4.1 in a box of radius below 10^{-22}, and every vacuum of the enumeration of Section 4.3 in a ball of radius at most 2.2×10^{-22}. The critical point of Section 5.2 lies in a box with all six coordinate radii below 5×10^{-149}.

Along a curve in a multi-parameter moduli space the periods can also be transported without a scalar operator, by the Gauss–Manin system restricted to the curve,

$$\frac{\mathrm{d}v}{\mathrm{d}s} \;=\; A(s)\, v\,, \qquad A(s) \in \mathbb{Q}(s)^{r\times r}\,, \tag{17}$$

for the vector v of periods and their s-derivatives (with r here the rank of the system). We reconstruct A from evaluations modulo many primes, as in [76, Sec. 7.2]. However, the finite checks it passes (agreement modulo a fresh prime, exact annihilation of a separately computed period series) do not prove that A is the restricted Gauss–Manin connection to all orders, which would require exact elimination from the Picard–Fuchs ideal over $\mathbb{Q}(s)$. No value quoted in this paper is obtained by transporting a reconstructed system. We use such a system only to check period series summed from closed-form coefficients (Section 5.4); every scalar operator we transport is derived exactly, as for the curve of Eq. (63), or taken in closed form from the literature, as in Eq. (51).

2.4 Implementation and checks

Terrier is an instrument for the arithmetic of flux vacua: our implementation of the methods above, used throughout this paper.[3] Our computations reproduce earlier results where they overlap: the flux tables of [11] on attractive K3 pairs, the test fluxes and the charge-25 flux of [7, 8] (Table 2), the exact Hulek–Verrill vacua of Grimm and van de Heisteeg [24] (Section S7.2 of the Supplementary Material), and the three values of $|W_0|$ published for the KKLT benchmark [20, 21, 22] (Table 7). The only differences are the misprints and final-digit residues collected in Section S3.2 of the Supplementary Material.

3 The tadpole bound on K3×K3

M-theory on the product of two K3 surfaces is the founding example of the Tadpole Conjecture (7), with $\chi/24 = 24$, and there the vacuum conditions reduce to integral lattice theory. Searching by differential evolution, Bena, Blåbäck, Graña and Lüst found no fully stabilizing flux with $N_{\text{flux}} \leq 24$ and reached 25 [7, 8], a value established by exploration rather than proof. We prove, under the hypotheses (H1)–(H4) and (H-RANK) stated below, the bound $N_{\text{flux}} \geq 25$ at support rank at most three (Theorem 3.4). We also bound the charge of fluxes supported on three root-free lattices of the definite rank-19 stratum (Theorem 3.5) and construct a fully stabilizing flux of charge 22 and rank 22 (Section 3.4).

[3] Section S4 of the Supplementary Material describes its tests and Section 6 its distribution.

3.1 Fluxes, lattices and the stabilization condition

We write $X = Y \times \tilde{Y}$ with both factors K3 and $\Gamma = H^2(Y,\mathbb{Z}) \cong 3U \oplus 2E_8(-1)$ for the even unimodular lattice of signature $(3,19)$ [77], with Gram matrix d, and tilde-decorated objects belong to $\tilde{Y}$. As K3 has no odd cohomology,

$$H^4(X,\mathbb{Z}) \;=\; U_0 \,\oplus\, \big(\Gamma \otimes \tilde{\Gamma}\big)\,, \tag{18}$$

with U_0 a hyperbolic plane spanned by the two point classes $x \otimes 1$ and $1 \otimes \tilde{x}$. The general flux is $G = N + m\,(x \otimes 1) + \tilde{m}\,(1 \otimes \tilde{x})$ with $N \in \Gamma \otimes \tilde{\Gamma} \cong \mathbb{Z}^{22\times 22}$ an integer matrix and $m, \tilde{m} \in \mathbb{Z}$. Flux quantization has no half-integral shift on this geometry [59]. From N we form mutually adjoint maps and self-adjoint operators, in the convention of [7, 8],

$$\begin{gathered} g = N\tilde{d}:\ \tilde{\Gamma} \to \Gamma\,, \qquad \tilde{g} = N^{\mathsf{T}} d:\ \Gamma \to \tilde{\Gamma}\,, \qquad S = g\tilde{g}\,, \qquad \tilde{S} = \tilde{g}g\,, \\ N_{\text{flux}} \;=\; \tfrac{1}{2}\operatorname{tr}\!\big(N\tilde{d}N^{\mathsf{T}}d\big) + m\tilde{m} \;=\; \tfrac{1}{2}\operatorname{tr} S + m\tilde{m}\,, \end{gathered} \tag{19}$$

with $g(\tilde{y}) \cdot y = \tilde{y} \cdot \tilde{g}(y)$. The tadpole condition (2) is

$$n_{M2} + N_{\text{flux}} \;=\; \frac{\chi(X)}{24} \;=\; 24\,. \tag{20}$$

A three-dimensional Minkowski vacuum with $n_{M2} \geq 0$ requires G self-dual [60, 7]. On $\Gamma \otimes \tilde{\Gamma}$ self-duality is a condition on the positive 3-planes $\Sigma \subset \Gamma \otimes \mathbb{R}$ and $\tilde{\Sigma} \subset \tilde{\Gamma} \otimes \mathbb{R}$ spanned by $\operatorname{Re}\Omega$, $\operatorname{Im}\Omega$ and the Kähler form, the period data of the respective factor:

$$g\big(\tilde{\Sigma}\big) \subseteq \Sigma\,, \qquad g\big(\tilde{\Sigma}^{\perp}\big) \subseteq \Sigma^{\perp}\,, \tag{21}$$

independently of both volumes. These are Eqs. (4.14) and (4.15) of [7], and we write V_N for their solution set, the vacuum locus. Flux in U_0 stabilizes only the volume ratio, at cost $m\tilde{m} \geq 1$ when nonzero, and we set $m = \tilde{m} = 0$, the optimal choice for the bound.

The flux-stabilizable moduli are the $57 + 57$ real shape moduli $(\Sigma, \tilde{\Sigma})$, each plane ranging over the Grassmannian of positive 3-planes in $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^{3,19}$, of dimension 3×19, and the overall volume is never flux-stabilized. We say that the flux N *fully stabilizes*, written (FS), if V_N has a point that is isolated in V_N and smooth, where smooth means

$$\delta \in \Gamma\,,\quad \delta^2 = -2 \quad\Longrightarrow\quad \delta \not\perp \Sigma\,, \qquad \text{and likewise for } \tilde{\Gamma},\ \tilde{\Sigma}\,. \tag{22}$$

The smoothness clause is the conjecture's own genericity condition: a root δ orthogonal to the period plane Σ means an orbifold singularity of the K3 metric [77, 7] and hence enhanced non-abelian gauge symmetry. Fluxes stabilizing only at such orbifold points therefore do not satisfy (FS). Smooth loci fixed by discrete symmetries, on which the invariant fluxes of [13] stabilize moduli, are not excluded by (FS); the charge-22 vacuum of Section 3.4 lies on such a locus. A zero of the potential with positive-definite Hessian is isolated, so any standard reading of “stabilizes all moduli” implies (FS) and the bounds below hold a fortiori.

We have $V_N \neq \emptyset$ if and only if

$$S \text{ and } \tilde{S} \text{ are diagonalizable over } \mathbb{R} \text{ with eigenvalues} \geq 0\,, \qquad \ker\tilde{g} = \ker S\,, \qquad \ker g = \ker\tilde{S}\,, \tag{23}$$

and the vacua of such a flux are isolated precisely when every eigenspace of S and of $\tilde{S}$, kernels included, is definite; Σ is then the sum of the positive eigenspaces of S.

The first clause of (23) is the criterion of Section 2.1 of [8], following [78], and on its own does not imply $V_N \neq \emptyset$. The exception at the zero eigenspace is identified in [7] (footnote 14), imposed in Section 2.2 of [8] as a definite-signature condition on the eigenspace of any degenerate eigenvalue, and stated in words, in both directions, in Appendix B of [7] (footnote 28; cf. footnote 12 of [78]). What is new here is the closed form of that clause, the kernel condition in (23); the conditions stated in [8] imply it, so their searches are unaffected. In Section S1.1 of the Supplementary Material we prove the isolation criterion and the equivalence of (23) with $V_N \neq \emptyset$.

Our theorems assume four hypotheses.

- **(H1)** *Dictionary*: vacua of M-theory on K3×K3 with flux correspond to self-dual G as above [9, 7, 60]. This is the standard supergravity correspondence; we do not re-derive it.
- **(H2)** *Quantization*: $G \in H^4(X, \mathbb{Z})$. Proven for this geometry [59]; listed for completeness.
- **(H3)** *Tadpole*: $n_{M2} \in \mathbb{Z}_{\geq 0}$, no anti-M2 branes, as in [7, 8].
- **(H4)** *Full stabilization* means (FS) above, over the 114 real shape moduli, with $m = \tilde{m} = 0$, as for the charge-22 flux below.

The reduction to a finite lattice problem takes six steps.

Step 1 (supports and strata). The *reduced support*, or simply the support, of N on Y is the smallest primitive sublattice $T \subset \Gamma$ containing $g(\tilde{\Gamma})$, and $\tilde{T} \subset \tilde{\Gamma}$ is defined likewise from $\tilde{g}$. For an FS flux the kernel clause of (23) gives $T \otimes \mathbb{R} = (\ker \tilde{g})^{\perp} = (\ker S)^{\perp}$ with $\ker S$ a definite eigenspace, so that T is nondegenerate. Since $\tilde{S}\tilde{g} = \tilde{g}S$ and $\tilde{g}(v) \cdot \tilde{g}(w) = v \cdot Sw$, the map $\tilde{g}$ carries each eigenspace $E_\lambda(S)$, $\lambda > 0$, isomorphically onto $E_\lambda(\tilde{S})$, scaling the form by λ, and T and $\tilde{T}$ therefore have the same rank $r = \operatorname{rank} N \leq 22$ and the same signature (p, q), with $p \leq 3$, $q \leq 19$. We call the set of fluxes with a given (p, q) a *stratum*.

Step 2 (the coherent tuple). We choose bases $f_1, \ldots, f_r$ of T and $\tilde{f}_1, \ldots, \tilde{f}_r$ of $\tilde{T}$, with Gram matrices $A, \tilde{A}$ (even, nondegenerate), $d_T = |\det A|$, $d_{\tilde{T}} = |\det \tilde{A}|$. Since g vanishes on $\tilde{T}^{\perp}$ and takes values in T, it is determined by the *reduced flux* $B \in \mathbb{Z}^{r\times r}$, the matrix of $g|_{\tilde{T}} : \tilde{T} \to T$ in these bases, with $\det B \neq 0$. By adjointness $\tilde{g}|_T$ and $S|_T$ have matrices $\tilde{B}$ and S_0 with

$$\tilde{B} \;=\; \tilde{A}^{-1} B^{\mathsf{T}} A\,, \qquad S_0 \;=\; B\tilde{B}\,, \qquad \operatorname{tr} S_0 \;=\; \operatorname{tr} S \;=\; 2N_{\text{flux}}\,. \tag{24}$$

The last equality holds because S vanishes on $\ker S = (T \otimes \mathbb{R})^{\perp}$ and preserves $T \otimes \mathbb{R}$. Every FS flux with $m = \tilde{m} = 0$ and $n_{M2} \geq 0$ thus gives a *coherent tuple*, of trace bounded by (20):

$$(A, \tilde{A}, B)\,, \qquad r \leq 22\,, \qquad \operatorname{tr} S_0 \;\leq\; 2\,(24 - m\tilde{m}) \;=\; 48\,. \tag{25}$$

Step 3 (dual integrality). Because $\tilde{\Gamma}$ is unimodular and $\tilde{T}$ is primitive, orthogonal projection maps $\tilde{\Gamma}$ onto the dual lattice $\tilde{T}^* \subset \tilde{T} \otimes \mathbb{Q}$, and g factors through this projection. Hence $g(\tilde{\Gamma}) = g(\tilde{T}^*) \subseteq T$, which in the dual basis says that

$$C \;:=\; B\tilde{A}^{-1} \in \mathbb{Z}^{r\times r}\,, \qquad \tilde{C} \;:=\; \tilde{B}A^{-1} \in \mathbb{Z}^{r\times r}\,, \tag{26}$$

the second being the analogous condition for $\tilde{g}$ on T^*. Equivalently, $N = \sum_{i,j} M_{ij}\, f_i \otimes \tilde{f}_j$ with $M \in \mathbb{Z}^{r\times r}$, since $T \otimes \tilde{T}$ is primitive and contains N, and by (19) $C = M$, $\tilde{C} = M^{\mathsf{T}}$, $B = M\tilde{A}$, $\tilde{B} = M^{\mathsf{T}}A$: a flux has integer coefficients on its pair of supports, with no fractional glue vectors.

Step 4 (glue determinant and determinant chain). We call the positive integer $c := [T : g(\tilde{T}^*)] = |\det C| = |\det B|/d_{\tilde{T}}$ the *glue determinant*. Substituting $|\det B| = c\, d_{\tilde{T}}$ into (24) gives the *determinant chain*

$$\det S_0 \;=\; \det B\, \det \tilde{B} \;=\; \frac{(\det B)^2\, d_T}{d_{\tilde{T}}} \;=\; c^2\, d_T\, d_{\tilde{T}}\,. \tag{27}$$

Step 5 (spectral conditions). By (23) and the isolation clause, S_0 is diagonalizable over $\mathbb{R}$ with every eigenspace A-definite and with $\tilde{A}$-definite image under $\tilde{g}$. Zero is not an eigenvalue, because $\ker S = (T \otimes \mathbb{R})^{\perp}$ meets $T \otimes \mathbb{R}$ only in 0, and the eigenvalues of S_0 therefore lie in $(0, 48]$. We call a tuple passing the tests of Steps 2–5 *FS-capable.* The smoothness clause (22) is a further condition on the embeddings.

Step 6 (equivalence and entries). Changes of the two bases act by

$$(A, \tilde{A}, B) \longmapsto \left(P^{\mathsf{T}} A P,\ \tilde{P}^{\mathsf{T}} \tilde{A} \tilde{P},\ P^{-1} B \tilde{P}\right), \qquad P, \tilde{P} \in GL_r(\mathbb{Z}), \tag{28}$$

and equivalent tuples describe the same flux on the same ordered pair of supports. For Theorem 3.4 we list FS-capable tuples of rank $r \leq 3$ within the trace budget such that every such tuple is equivalent under (28) to a listed one, and call a listed tuple an *entry.* Theorem 3.4 requires one further hypothesis.

(H-RANK) The FS vacuum datum reduces to a coherent tuple of rank $r \leq 3$.

For each rank $4 \leq r \leq 22$ the enumeration of FS-capable tuples is proven to terminate, with computable limits on their matrix entries (Section S1.1 of the Supplementary Material), and that computation has not been run. A direct search is impractical, since already at $r = 4$ the region within those limits holds 98 orders of magnitude more points than a search could examine. Hence closing the indefinite strata requires a fundamental-domain engine for the infinite unit groups rather than a larger run. On the rank $r \geq 4$ strata the bound itself is known to fail (Section 3.4).

Earlier enumerations of fluxes on K3×K3, by Aspinwall and Kallosh [9] (the 13 saturating solutions), by Braun, Kimura and Watari [10] and, reaching $N_{\text{flux}} \leq 30$, by Braun, Fraiman, Graña, Lüst and Parra de Freitas [11], concern the purely transcendental fluxes on attractive K3 pairs, and all in-budget solutions of that class leave a non-abelian gauge group [11]. Their gauge-rank criterion is inequivalent to the stabilization predicate used here, and in those works no theorem is claimed on the full shape-moduli predicate (FS) of (H4), which a purely transcendental flux never satisfies; restricting to their class and criterion, we reproduce their tables (Section S4.1 of the Supplementary Material).

3.2 The signature argument and the rank-≤ 3 theorem

Most strata are excluded without enumeration, because the kernel of S is indefinite there.

Theorem 3.1 (signature exclusion)**.** *Assume (H1)–(H4). No FS vacuum has reduced flux support of signature (p, q) with $p \leq 2$ and $q \leq 18$. In particular, for $4 \leq q \leq 18$ there is no FS vacuum with reduced support of signature $(0, q)$, at any value of N_{flux}.*

Indeed, the kernel of S contains the orthogonal complement of the reduced support, of signature $(3 - p,\ 19 - q)$, which is indefinite for $p \leq 2$ and $q \leq 18$; an indefinite kernel violates the eigenspace-definiteness clause of the isolation criterion stated after (23), so no vacuum of such a flux is isolated. In particular no FS flux has support of rank $r \leq 2$ $(p, q \leq 2)$. The strata not excluded form the $p = 3$ row and the $q = 19$ column of Figure 1: the finite stratum $(3, 0)$, the definite stratum $(0, 19)$ and the indefinite strata of ranks $4 \leq r \leq 22$.

We enumerate the complete list of 412,531 in-budget rank-≤ 3 candidates, the entries of Step 6, together with one rank-6 flux of signature $(3, 3)$ added as a check, 412,532 entries in all. Since (FS) imposes the smoothness clause (22) on both K3 factors, an obstruction on either factor excludes the entry, and we let A denote the Gram matrix of the support of smaller determinant, $\det A \leq \det \tilde{A}$. We exclude an entry only if smoothness fails on *every* primitive embedding of that support into Γ, since an embedding on which it held could carry a fully stabilizing flux within the budget.

excluded structurally (gray, 57 strata): kernel indefinite, no isolated vacuum (Theorem 3.1)
enumerated, every entry excluded (black, $(3,0)$): all 412,531 entries (Theorem 3.4, Table 2)
enumeration proven finite, not run (outlined: $p=3$, $q \geq 1$; and $q=19$, $p=1,2$)
open (bold, $(0,19)$): the definite rank-19 stratum of Theorem 3.5

Figure 1: Under (H1)–(H4), full stabilization within the budget is impossible off the $p=3$ row and the $q=19$ column (Theorem 3.1), and the one finite FS-capable stratum, $(3,0)$, is excluded entry by entry within $N_{\text{flux}} \leq 24$. Shown is the status of every stratum of reduced flux support on K3×K3 by signature (p,q), with p the number of positive and q the number of negative directions of the support (Section 3.1, Step 1). In the 4×20 grid of signatures the 57 gray cells with $p \leq 2$, $q \leq 18$ (the empty support $(0,0)$ among them) are excluded by Theorem 3.1; the black cell holds the 412,531 entries of $(3,0)$, all excluded (Table 2); outlined cells are indefinite strata whose enumeration is proven finite and has not been run; the bold cell is the definite stratum $(0,19)$, which holds the three lattices $W_{512}, W'_{512}, W_{1024}$ of Theorem 3.5, with proven charge bounds at or above the budget, and which stays open.

On the stratum $(3,0)$ the support is positive definite and $\Sigma = T \otimes \mathbb{R}$ is forced. By Nikulin's theorem [61] in the form (11), on any primitive embedding of the support into Γ the orthogonal complement K has invariants fixed by those of A,

$$K \;=\; A^{\perp} \subset \Gamma\,, \qquad \operatorname{sign} K \;=\; (0,19)\,, \qquad q_K \;=\; -q_A\,, \qquad \det K \;=\; \det A\,, \tag{29}$$

and $K \perp \Sigma$. Hence a root in K violates the smoothness clause (22) on that embedding.

The three eigenvalues of S_0 are positive with sum at most 48 (Step 5), so the arithmetic–geometric mean inequality and the determinant chain (27) give

$$c^2\, d_T\, d_{\tilde{T}} \;=\; \det S_0 \;\leq\; \left(\tfrac{48}{3}\right)^3 \;=\; 4096\,, \qquad \text{hence} \qquad \min(d_T, d_{\tilde{T}}) \;\leq\; 64 \tag{30}$$

as $c \geq 1$, and the minimum is $\det A = \det K$ by the choice of A. Only the smaller of the two determinants is bounded in this way. Thus for each entry we must decide whether every class in the genus of K contains a root.

Lemma 3.2 (root-forcing by packing density)**.** *There is no root-free even positive-definite lattice of rank 19 with determinant ≤ 22. Consequently every complement class of every candidate with* $\det A \leq 22$ *contains a root, and all such candidates are rejected.*

For the proof, we follow the rank-18 argument of [32] (see also [63, 11]): a root-free even lattice L of rank 19 has minimum ≥ 4, so by (12) its center density satisfies $\delta(L) \geq (\det L)^{-1/2}$, and $\det L \leq 22$ would give

$$\delta(L) \;\geq\; (\det L)^{-1/2} \;\geq\; 1/\sqrt{22} \;=\; 0.21320\ldots \;>\; 0.21202\,, \tag{31}$$

where 0.21202 is the rank-19 center-density bound of Cohn and Elkies [67], a contradiction.

In the window $\det K \in \{24, 26, \dots, 64\}$ beyond the packing bound we proceed genus by genus. We call a genus $\mathcal{G}$ of rank-19 lattices *admissible* if it can occur as the genus of a complement K in (29), that is, if

$$\begin{aligned}&\mathcal{G} \text{ is even of signature } (0,19) \text{ with discriminant group on } l \le 3 \text{ generators, and}\\ &\text{some rank-3 even positive-definite form } T \text{ has } q_T = -q_{\mathcal{G}},\\ &\text{so that } K \cong T^{\perp} \subset \Gamma \text{ for } K \in \mathcal{G} \text{ by (11);}\end{aligned} \tag{32}$$

for the complement of an entry, T is the support itself. A rank-19 lattice is admissible when its genus is. We read off the admissible genera from the reduced even positive-definite ternary forms T, 90 classes in 67 genera at determinant ≤ 64, of which 13 genera lie at determinant ≤ 22 and 54 in the window:

det	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64
genera	4	1	2	2	3	1	4	1	4	2	2	1	5	1	2	3	4	1	4	1	6

(33)

The complements of the candidates fall into 48 of these 54 genera.

Lemma 3.3 (rooting)**.** *Every class of each of the* 54 *admissible complement genera with determinant in* $[24, 64]$ *contains a root.*

For the proof we apply the Niemeier-complement argument of Section 2.2, a form of the Kneser–Nishiyama method [70, 31, 11]. For each admissible genus $\mathcal{G}$ with ternary partner T as in (32) we enumerate the complement genus C of T in E_8 by Kneser neighbors; the 54 lists, 98 classes in all, are complete by the mass identity (10), and every class has at least 4 roots. A root-free lattice of $\mathcal{G}(-1)$ glued to a class of C would be a Niemeier lattice with roots, with Coxeter number h among the 18 values $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 22, 25, 30, 46\}$, and for each of the 98 classes and each h a rational Farkas vector proves the linear system (13) infeasible, $1{,}764 = 18 \times 98$ vectors in all. This covers all 54 admissible genera, including the six that no candidate populates (two at determinant 60, four at 64); for 32 of them a theta-series system, Eq. (S5.13) of the Supplementary Material, constrained at both cusps gives a second proof, and for the other 22 the Niemeier argument is the only proof. Section S5.2 of the Supplementary Material gives the argument in full, with the 54 genera and the 98 complement classes (the Farkas vectors are not printed), and shows that genus averaging alone does not force a root.

Theorem 3.4 (rank-stratified tadpole bound)**.** *Assume (H1)–(H4) and (H-RANK). Then every FS flux on K3×K3 has* $N_{\mathrm{flux}} \ge 25$.

For the proof, suppose an FS flux N satisfying (H-RANK) has $N_{\mathrm{flux}} \le 24$. By Step 1 and Theorem 3.1 its supports are positive definite of rank 3 with $\operatorname{tr} S_0 \le 48$, so (30) bounds the smaller support determinant $\det A$ by 64. By (29) the complement K of that support, on whichever factor it lies, has $\det K \le 64$ and admissible genus. Lemma 3.2 or Lemma 3.3 gives a root in K, orthogonal to the period plane of that factor, contrary to the smoothness clause of (FS), which is imposed on both factors. The full enumeration is a consistency check, from which we read off the 48 populated genera.

Thus the conjecture's prediction for K3×K3 holds at support rank at most three, and the value 25 that the searches of [7, 8] reached is not undercut on this stratum. The charge-25 flux of [7], one unit above the budget, fully stabilizes at support rank 17 (Table 2). It lies outside the rank-≤ 3 stratum on which the bound is proven. The charge-22 flux of Section 3.4 is inside the budget, and

outcome	entries	mechanism
rejected: root on every embedding class	368,068	Lemma 3.2 ($\det A \le 22$)
rejected: root on every embedding class	44,463	Lemma 3.3 (48 populated genera, $\det A \in [24, 64]$)
counterexample candidates	0	
total, rank-≤ 3 stratum	412,531	
rank-6 check flux, rejected: explicit roots	1	$2E_8$ complement, 480 roots; decided alongside, 412,532 in all
charge-25 flux of [7]: passes, above the budget	1	support signature $(3, 14)$, rank 17, trace 50; not in the total

Table 2: All 412,531 rank-≤ 3 FS-capable candidates with $N_{\text{flux}} \le 24$ are rejected, 89.2% by the packing lemma and the rest by the rooting lemma on their genera, and so is the one rank-6 flux added as a check. An entry is an FS-capable tuple $(A, \tilde{A}, B)$ of Section 3.1 with the two supports ordered, and in the mechanism column $\det A$ is the smaller of an entry's two support determinants, bounded by 64 through Eq. (30). The rank-6 check flux, of signature $(3, 3)$, has complement $2E_8$ with 480 explicit roots and shows that passing the tests of Steps 2–5 does not imply (FS). We recover the test fluxes in the first four rows of the differential-evolution table of [7, 8], whose searches attained charges 5, 6, 6 and 7; their published charge-25 flux (last row), of support signature $(3, 14)$ and rank 17, passes the stabilization test at trace 50, above the budget.

under (H1)–(H4) the founding prediction, that no flux within the budget fully stabilizes, fails off that stratum. On the rank-≤ 3 stratum every in-budget candidate is excluded by a root orthogonal to the period plane, so its vacua are points of enhanced gauge symmetry. In Section S1.3 of the Supplementary Material we count, for ten other flux sectors, how often the vacua lie on such loci.

3.3 The definite rank-19 stratum

On the stratum $(0, 19)$ the support is a primitive negative-definite sublattice $T \subset \Gamma$ of rank 19, the positive 3-plane Σ is forced to be $T^{\perp} \otimes \mathbb{R}$, and the smoothness clause of (FS) says precisely that T is root-free. In this subsection $W = -T$ denotes a lattice, not the superpotential, and we call W admissible when T is. A fully stabilizing in-budget flux on this stratum is then a pair of even positive-definite root-free admissible rank-19 lattices $W, \tilde{W}$, one on each factor, with an integer flux matrix of charge $N_{\text{flux}} \le 24$.

Every admissible lattice of determinant at most 126 contains a root (Section S1.2 of the Supplementary Material), and a root-free admissible lattice exists at determinant 160, the coinvariant lattice $\Lambda_{M_{20}}$ of [79, 80], found as a Euclidean lattice in [81], so that the smallest root-free admissible determinant lies in $[128, 160]$. The rank-3 stratum could be excluded by roots because there $\det K \le 64$, below that range.

We found three root-free admissible lattices of rank 19 that support fluxes satisfying all stabilization conditions, with certified charge floors that equal or exceed the budget:

$$\begin{aligned} W_{512},\, W'_{512}\ :\ \ & \text{even, positive definite, rank 19},\ \det = 512,\ \min = 4,\ |\mathrm{Aut}| = 2,\\ & \text{kissing numbers 5200 and 5204 (hence not isometric)},\\ & \text{discriminant group } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{128}\,;\\ W_{1024}\ :\ \ & \text{the same at } \det = 1024,\ \text{kissing number 3664},\ |\mathrm{Aut}| = 2,\\ & \text{discriminant group } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{256}\,. \end{aligned} \tag{34}$$

Each lattice is admissible, with rank-3 partner $\mathrm{diag}(2, 2, c')$ in the sense $q_W \cong q_{\mathrm{diag}(2,2,c')}$, $c' = 128, 128, 256$, and $W(-1)$ embeds primitively in Γ with orthogonal complement exactly $\mathrm{diag}(2, 2, c')$;

Section S5.1 of the Supplementary Material gives the three Gram matrices, obtained by Kneser neighbor descent from $E_8 \oplus E_8 \oplus \mathrm{diag}(2,2,c')$, with the gluing data.

By Step 3, a flux supported on a pair (W_A, W_B) of these lattices, with $A = -W_A$ on Y and $\tilde{A} = -W_B$ on $\tilde{Y}$, is an integer matrix $M = C \in M_{19}(\mathbb{Z})$ with $\det M \neq 0$ and no fractional glue vectors, whose $\mathrm{tr}\, S_0$ is $q(M)$ in (35) below and whose glue determinant is $c = |\det M|$. The three lattices give nine ordered pairs (W_A, W_B), six up to exchange of the factors, and on each of the six some flux supported on the embedded pair satisfies the stabilization criterion (23).

A flux M on supports W_A, W_B has charge $N_{\text{flux}} = \frac{1}{2}\, q(M)$ with

$$q(M) \;=\; \mathrm{tr}\Big(M\, W_B\, M^{\mathsf{T}} W_A\Big)\,, \qquad M \in M_{19}(\mathbb{Z})\,,\ \det M \neq 0\,, \tag{35}$$

and q is even because both Gram matrices are, whence the budget (20) becomes $q \leq 48$. The arithmetic–geometric mean inequality for the positive-definite matrix $W_A^{1/2} M W_B M^{\mathsf{T}} W_A^{1/2}$ and the parity of q give

$$q(M) \;\geq\; 19\left(\det W_A\, \det W_B\, (\det M)^2\right)^{1/19} \;\geq\; 19\left(512^2\right)^{1/19} \;=\; 36.6386\ldots \quad\Longrightarrow\quad q(M) \geq 38 \tag{36}$$

at determinant 512^2, ten units below the budget.

We sharpen (36) with two constraints linear in the Gram matrix $G = M^{\mathsf{T}} W_A M$, for which $q(M) = \mathrm{tr}(W_B\, G)$: root-freeness of W_A gives $y^{\mathsf{T}} G y \geq 4$ for every nonzero integer vector y, and Hermite's constant $\gamma_2^2 = 4/3$ [38] gives $\det(Y^{\mathsf{T}} G Y) \geq 16/(4/3) = 12$ for every 19×2 integer matrix Y of rank 2. Choosing such y_α and Y_p with rational weights $w_\alpha \geq 0$ and rational 2×2 matrices $\Lambda_p \succeq 0$ such that $\mathcal{R} = W_B - \sum_\alpha w_\alpha\, y_\alpha y_\alpha^{\mathsf{T}} - \sum_p Y_p \Lambda_p Y_p^{\mathsf{T}} \succ 0$ and applying (36) to $\mathrm{tr}(\mathcal{R}\, G)$ gives

$$q(M) \;=\; \mathrm{tr}(W_B\, G) \;\geq\; 19\left(\det \mathcal{R}\, \det W_A\right)^{1/19} + \; 4\sum_\alpha w_\alpha \;+\; \sum_p 2\sqrt{12\, \det \Lambda_p} \;=:\; V \tag{37}$$

for every integer M with $\det M \neq 0$ (derivation and data in Section S5.3 of the Supplementary Material).

In all five bounds of Table 3 we use $\det W_A \geq 512$ and replace the two radicals by rational lower bounds, which makes V an exact rational number. We found the weights by numerical optimization, but each inequality (37) has exact rational data, so nothing numerical enters the five bounds. We call column j of M *minimal* when it is a minimal vector of W_A ($G_{jj} = 4$), and rows likewise with respect to W_B.

Theorem 3.5 (charge bounds on the three lattices)**.** *Assume (H1)–(H4). On every K3×K3 gluing of the three lattices: $N_{\text{flux}} \geq 25$ for every flux supported on any pair involving W_{1024}; $N_{\text{flux}} \geq 24$ for every flux supported on a determinant-512 pair; and on the determinant-512 pairs, $N_{\text{flux}} = 24$ requires at least two non-minimal columns and at least two non-minimal rows of M, with $|\det M| \leq 13$.*

We read off the first two clauses of the theorem from the first three rows of Table 3, which together cover the nine ordered pairs, since exchanging W_A and W_B in (35) replaces M by M^{T}. We obtain the two-column, two-row clause from 38 further bounds, one for either determinant-512 lattice as W_B and each way of fixing 18 of the 19 diagonal entries of G at the minimum. Every one exceeds 48, the smallest being $48.3747\ldots$ (for W'_{512}) and $48.9376\ldots$ (for W_{512}). The determinant clause follows from (36): $q \leq 48$ forces

$$(\det M)^2 \;\leq\; \frac{(48/19)^{19}}{512^2} \;=\; 169.3265\ldots\,, \qquad \text{hence} \qquad |\det M| \leq 13\,. \tag{38}$$

scope	bound V	consequence
every M, any pair involving W_{1024}	$48.18595\ldots$	$q \geq 50$: $N_{\text{flux}} \geq 25$
every M, $W_B = W_{512}$	$46.61682\ldots$	$q \geq 48$: $N_{\text{flux}} \geq 24$
every M, $W_B = W'_{512}$	$46.60719\ldots$	$q \geq 48$: $N_{\text{flux}} \geq 24$
all columns of M minimal, $W_B = W_{512}$	$51.06729\ldots$	$q \geq 50$ for such M
all columns of M minimal, $W_B = W'_{512}$	$49.18410\ldots$	$q \geq 50$ for such M

Table 3: Every gluing involving W_{1024} is excluded below charge 25, since q is even and $V > 48$ forces $q \geq 50$ and $N_{\text{flux}} \geq 25$; on the determinant-512 pairs $V > 46$ forces $q \geq 48$ and $N_{\text{flux}} \geq 24$, exactly the budget. Listed are the five exact-rational lower bounds $V \leq q(M)$ of the form (37), each valid for every integer M with $\det M \neq 0$ in the scope named in its row, while the consequences for N_{flux} hold under (H1)–(H4); the lattice named in the scope column is W_B in the notation of Eq. (35), the one from which the residual $\mathcal{R}$ is formed, and a column of M is minimal when it is a minimal vector of W_A. V is the exact rational value of the right side of Eq. (37) with the radicals replaced by rational lower bounds, shown as a truncated prefix, so that the number shown does not exceed V; Section S5.3 of the Supplementary Material gives the fractions, the rational lower bounds used for the two radicals, and the numbers of terms. The last two rows restrict to fluxes whose columns are all minimal; applied to M and M^{T} they show that charge 24 is possible only for fluxes with non-minimal columns and non-minimal rows.

The one in-budget possibility left on the three lattices is exact saturation. Does there exist $M \in M_{19}(\mathbb{Z})$, $\det M \neq 0$, with

$$\operatorname{tr}\Big(M\, W_B\, M^{\mathsf{T}} W_A\Big) \;=\; 48\,, \qquad W_A, W_B \in \{W_{512}, W'_{512}\}\,? \tag{39}$$

A solution would be a tadpole-saturating flux, $N_{\text{flux}} = 24$ with $n_{M2} = 0$, satisfying all stabilization conditions, hence a second in-budget counterexample, on the definite stratum, to the K3×K3 bound as stated for smooth compactifications in [7, 8]. If none exists, all three lattices are excluded below charge 25, the value reached by search [7, 8].

Theorem 3.5 constrains a solution of (39) but does not exclude one, and neither can congruence arguments at $p \in \{2, 3\}$, since there is no local obstruction there (Section S5.3 of the Supplementary Material proves local solvability at $p \in \{2, 3\}$ in all 156 cases). The question is finite and mentions no physics. Besides W_{512}, W'_{512} and W_{1024}, further root-free admissible lattices exist, among them seven at determinants 160, 180, 196, 208, 216, 220 and 224, three of them coinvariant lattices of symplectic automorphism groups of K3 surfaces [82, 83, 79, 84, 80] and a fourth listed in [81]. These examples show that bounding the determinant cannot by itself rule out the stratum $(0, 19)$, and excluding it remains an open problem (Section S1.2 of the Supplementary Material identifies the seven lattices and describes what an exclusion would require).

3.4 A fully stabilizing flux of charge 22

In this subsection we construct a fully stabilizing flux of charge 22, inside the budget and off the rank-≤ 3 stratum, and show that it satisfies every condition of (FS) (Table 4). The cyclotomic field $\mathbb{Q}(\zeta_{66})$ has degree 20 and ring of integers $\mathbb{Z}[\zeta_{66}]$; we let $\sigma_k : \zeta_{66} \mapsto e^{2\pi i k/66}$, $k \in \mathcal{P} = \{1, 5, 7, 13, 17, 19, 23, 25, 29, 31\}$, represent its ten pairs of complex embeddings and write Tr for the trace to $\mathbb{Q}$. The element

$$\delta = (\zeta_3 - \zeta_3^{-1})(\zeta_{11} - \zeta_{11}^{-1})^9 \,, \qquad \zeta_3 = \zeta_{66}^{22} \,, \quad \zeta_{11} = \zeta_{66}^{6} \,, \tag{40}$$

is real and generates the different of $\mathbb{Z}[\zeta_{66}]$. Any real unit ε of $\mathbb{Z}[\zeta_{66} + \zeta_{66}^{-1}]$ with $\sigma_k(\varepsilon\delta^{-1}) > 0$ for exactly one $k \in \mathcal{P}$ serves for the construction, and we take $\varepsilon_S = -u_7 u_{17} u_{19} u_{23} u_{29}$ with $u_a = (\zeta_{66}^a - \zeta_{66}^{-a})/(\zeta_{66} - \zeta_{66}^{-1})$, for which $\sigma_k(\varepsilon_S \delta^{-1}) > 0$ exactly at $k = 1$.

The trace form

$$L = \big(\mathbb{Z}[\zeta_{66}],\, b\big) \,, \qquad b(x,y) = \mathrm{Tr}\big(\varepsilon_S\, \delta^{-1} x \bar{y}\big) \,, \tag{41}$$

is integral because δ generates the different, even because every diagonal entry of its Gram matrix in the power basis equals $\mathrm{Tr}(\varepsilon_S \delta^{-1}) = 12726$, and unimodular of signature $(2, 18)$, the two positive directions coming from the one place $k = 1$ at which $\sigma_k(\varepsilon_S \delta^{-1}) > 0$. Hence $L \cong 2U \oplus 2E_8(-1)$ [38], and $\Lambda = U \oplus L$, even unimodular of signature $(3, 19)$, is isometric to $\Gamma \cong H^2(\mathrm{K3}, \mathbb{Z})$ on each factor [77]. The flux acts blockwise,

$$\begin{gathered} \tilde{g} = \tilde{g}_U \oplus \big(\text{multiplication by } 1 + \zeta_{66}\big) \,, \qquad g = g_U \oplus \big(\text{multiplication by } 1 + \zeta_{66}^{-1}\big) \,, \\ \tilde{g}_U = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} , \qquad g_U = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} , \qquad S|_U = g_U \tilde{g}_U = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} , \end{gathered} \tag{42}$$

with g and $\tilde{g}$ integral and mutually adjoint for $d = d_U \oplus b$, $d_U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so that $N = g\tilde{d}^{-1}$ in the convention of (19) is an integer 22×22 matrix with $\frac{1}{2}\,\mathrm{tr}(N\tilde{d}N^{\mathsf{T}} d) = 22$ (Section S6 of the Supplementary Material gives the matrices).

For this flux we find $\det S = 1$ and $\mathrm{tr}\, S = 6 + 38 = 44$, where $38 = \mathrm{Tr}\big((1+\zeta_{66})(1+\zeta_{66}^{-1})\big) = \mathrm{Tr}(2 + \zeta_{66} + \zeta_{66}^{-1})$, and the kernel clause of (23) holds trivially. The characteristic polynomial is

$$\begin{aligned} \det(x - S) \;&=\; (x^2 - 6x + 1)\, m(x)^2 \,, \\ m(x) \;&=\; x^{10} - 19x^9 + 152x^8 - 666x^7 + 1742x^6 - 2782x^5 \\ &\qquad + 2665x^4 - 1443x^3 + 390x^2 - 40x + 1 \,, \end{aligned} \tag{43}$$

with m the minimal polynomial of $2+2\cos(2\pi/66)$, irreducible of degree ten. The minimal polynomial $(x^2 - 6x + 1)\, m(x)$ of S is squarefree, since $\gcd(x^2 - 6x + 1, m) = 1$, so that S is diagonalizable over $\mathbb{R}$ with twelve distinct eigenvalues, all real and positive.

The spectrum is $3 \pm 2\sqrt{2}$ on $U \otimes \mathbb{R}$ and $\lambda_k = 2 + 2\cos(2\pi k/66)$, $k \in \mathcal{P}$, doubly on $L \otimes \mathbb{R}$, on planes P_k on which b is definite, positive-definite only for $k = 1$. Every eigenspace is therefore definite, exactly three directions are positive (the criterion of Section 2.2 and Appendix A of [8]), and the vacuum is the isolated point $\Sigma = \ell_{3+2\sqrt{2}} \oplus P_1$, with $\ell_{3+2\sqrt{2}}$ the positive eigenline of $S|_U$, and likewise for $\tilde{\Sigma}$. No nonzero vector of Λ is orthogonal to Σ on either factor (Section S6.2 of the Supplementary Material), so that the smoothness clause (22) holds vacuously and the generic complex structure in the twistor family of the vacuum metric has Picard number zero. All conditions of (FS) therefore hold at $N_{\text{flux}} = \frac{1}{2}\,\mathrm{tr}\, S = 22 \leq 24$, with two M2-branes accounting for the remaining units.

Multiplication by ζ_{66} is an order-66 isometry of L commuting with S and $\tilde{S}$, so $\mathrm{id}_U \oplus \zeta_{66}$ preserves Σ and $\tilde{\Sigma}$, and the vacuum lies on the fixed locus of an order-66 lattice symmetry. The pair (L, ζ_{66}) is an ideal lattice in the sense of Gross and McMullen [85, 86]. Moreover $(\Lambda, \mathrm{id}_U \oplus \zeta_{66})$ is isometric to entry 0.66.0.1 of the database of [87, 88], the lattice $(H^2(X, \mathbb{Z}), \theta^*)$ of Kondō's K3 surface X with a purely non-symplectic automorphism θ of order 66 [89, 90, 91] (see also [92, Sec. 3.2]); X is the algebraic member, with Picard lattice U, of the twistor family of the vacuum metric (Section S6.3 of the Supplementary Material gives the change of basis).

object	value
field	$\mathbb{Q}(\zeta_{66})$, degree 20, discriminant of absolute value $3^{10}\,11^{18}$, ring of integers $\mathbb{Z}[\zeta_{66}]$
different	generated by $\delta = (\zeta_3 - \zeta_3^{-1})(\zeta_{11} - \zeta_{11}^{-1})^9$, real
unit	any real $\varepsilon \in \mathbb{Z}[\zeta_{66} + \zeta_{66}^{-1}]^\times$ with $\sigma_k(\varepsilon\delta^{-1}) > 0$ for exactly one $k \in \mathcal{P}$;
	here $\varepsilon_S = -u_7 u_{17} u_{19} u_{23} u_{29}$, positive place $k = 1$
lattice L	$(\mathbb{Z}[\zeta_{66}], \mathrm{Tr}(\varepsilon_S \delta^{-1} x\bar{y}))$: even, $\det = 1$, signature $(2, 18)$, diagonal entries 12726; $L \cong 2U \oplus 2E_8(-1)$
lattice Λ	$U \oplus L$: even unimodular, signature $(3, 19)$, $\cong H^2(\mathrm{K3}, \mathbb{Z})$ on each factor
flux on U	$\tilde{g}_U = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, $g_U = \left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, $S\vert_U = \left(\begin{smallmatrix}3&4\\2&3\end{smallmatrix}\right)$
flux on L	$\tilde{g}\vert_L =$ multiplication by $1 + \zeta_{66}$, $g\vert_L =$ multiplication by $1 + \zeta_{66}^{-1}$
trace, determinant	$\mathrm{tr}\, S = 6 + 38 = 44$, $\det S = 1$
characteristic polynomial	$(x^2 - 6x + 1)\, m(x)^2$ with m as in Eq. (43); minimal polynomial $(x^2 - 6x + 1)\, m(x)$, squarefree
eigenvalues	twelve, distinct, real and positive: $3 \pm 2\sqrt{2}$ on U; $\lambda_k = 2 + 2\cos(2\pi k/66)$, $k \in \mathcal{P}$, doubly on L, with $b\vert_{P_k} > 0$ only for $k = 1$
positive directions	3: $\Sigma = \ell_{3+2\sqrt{2}} \oplus P_1$, likewise $\tilde{\Sigma}$; every eigenspace definite, vacuum isolated
smoothness	$\Lambda \cap \Sigma^\perp = \{0\}$ on both factors
support	rank 22, signature $(3, 19)$
charge	$N_{\text{flux}} = 22$, $m = \tilde{m} = 0$, $n_{M2} = 2$
Section S6 of the Supplementary Material	δ and ε_S in the power basis, the Gram matrix of L, g, $\tilde{g}$, the flux matrix N, the spectrum
whether $W = 0$	not decided here
flux matrix in the basis $3U \oplus 2E_8(-1)$ of [7]	not given

Table 4: The flux of charge 22, inside the budget of 24, satisfies every condition of (FS). Listed are the arithmetic data of this fully stabilizing flux. Notation as in Section 3.4: $\zeta_3 = \zeta_{66}^{22}$, $\zeta_{11} = \zeta_{66}^6$, $u_a = (\zeta_{66}^a - \zeta_{66}^{-a})/(\zeta_{66} - \zeta_{66}^{-1})$, $\mathcal{P} = \{1, 5, 7, 13, 17, 19, 23, 25, 29, 31\}$ indexes the embeddings $\sigma_k : \zeta_{66} \mapsto e^{2\pi i k/66}$, U is the hyperbolic plane, $S = g\tilde{g}$ as in Eq. (19), ℓ_λ the eigenline of $S|_U$ with eigenvalue λ and P_k the eigenplane of $S|_L$ with eigenvalue λ_k. The conditions verified are those of (FS) under (H1)–(H4); W is the flux superpotential.

Since the support of this flux is all of Λ, of rank 22 and signature $(3, 19)$, the flux does not contradict Theorem 3.4, which applies only at support rank ≤ 3. It also lies outside the scope of Theorem 3.5 on the definite stratum $(0, 19)$, so whether any bin ≤ 24 holds a fully stabilizing flux there stays open. Hence, under (H1)–(H4), the conjecture's remaining content on this geometry is the stratum statement of Theorem 3.4 together with the open definite stratum.

Every statement of this section assumes (H1)–(H4), and a vacuum here is a solution of the classical vacuum equations (21); whether such solutions realize all the physics expected of them is a separate question the lattice arithmetic does not touch. Three items remain open: the saturation equation (39) on the determinant-512 pairs, the existence of any other flux on the stratum $(0, 19)$ that fully stabilizes within the budget, and the enumeration of the indefinite strata of ranks $4 \leq r \leq 22$. Nothing in this section upgrades any of the three.

4 The Hulek–Verrill fourfold

Our second example is the six-parameter Hulek–Verrill family of Calabi–Yau fourfolds [49, 24, 93], whose budget is $\chi/24 = 30$, with fluxes in the S_6-invariant sector of its flux lattice and moduli on the S_6-symmetric slice. This sector admits fluxes with $N_{\text{flux}} = 1$. A period-level computation proves that no vacuum exists below budget 4 inside a stated cut on the flux and a stated box in the slice modulus φ, and budget 4 is attained. Beyond the box, at smooth points of the slice, the first vacuum has charge at most 2, and a unit-charge vacuum is neither found nor excluded. At the conifold point $\varphi = 1/36$ a unit-charge flux has $W = \partial_\varphi W = 0$ [50, 51].

4.1 Flux lattice, period equation and search box

The fourfolds of the family are the small resolutions of the hypersurfaces

$$\big(X_1 + X_2 + \cdots + X_6\big)\Big(\frac{\varphi^1}{X_1} + \frac{\varphi^2}{X_2} + \cdots + \frac{\varphi^6}{X_6}\Big) \;=\; 1 \tag{44}$$

in the torus T^5 (the complement of $X_1X_2\cdots X_6 = 0$ in $\mathbb{P}^5$), with complex-structure moduli $\varphi^1,\ldots,\varphi^6$, Hodge numbers $h^{3,1}=6$, $h^{2,1}=0$, $h^{1,1}=106$, $h^{2,2}=492$ and $\chi=720$ [24]. The group S_6 permutes the X_i and the φ^i; the symmetric slice is the diagonal $\varphi^1=\cdots=\varphi^6=\varphi$. In the integral frame of [24, 49], whose integrality we take from those works (Section S7.1 of the Supplementary Material lists the points we leave open), a horizontal flux is a vector $G\in\mathbb{Z}^{29}$ whose coordinates S_6 permutes in orbits of sizes $1+6+15+6+1$. The intersection form, denoted Σ in this section, is even, of determinant $2^{16}3^6$ and signature $(17,12)$, and $N_{\text{flux}}=\frac{1}{2}\,G^{\mathsf T}\Sigma G\in\mathbb{Z}$.

The S_6-invariant fluxes form the primitive rank-5 sublattice of vectors $G=Bg$, $g=(g_1,\ldots,g_5)\in\mathbb{Z}^5$, with B the 29×5 matrix of orbit indicator vectors, and

$$B^{\mathsf T}B=\operatorname{diag}(1,6,15,6,1),\quad \mathrm{Gram}_5=B^{\mathsf T}\Sigma B=\begin{pmatrix}2&0&30&0&1\\0&-60&0&6&0\\30&0&180&0&0\\0&6&0&0&0\\1&0&0&0&0\end{pmatrix},\quad \det\mathrm{Gram}_5=6480, \tag{45}$$

of signature $(3,2)$. The charge of an invariant flux is

$$N_{\text{flux}} \;=\; Q(g)/2 \;=\; \tfrac{1}{2}\,g^{\mathsf T}\mathrm{Gram}_5\,g \;=\; g_1^2-30g_2^2+90g_3^2+30g_1g_3+g_1g_5+6g_2g_4, \tag{46}$$

and the Euclidean norm of G in the frame is $n_2(g)=G^{\mathsf T}G=g_1^2+6g_2^2+15g_3^2+6g_4^2+g_5^2$.

We impose a cut on the fluxes and restrict the modulus to a box R, the *census box*, centered on the position $\varphi=1/64$ of an exact vacuum of [24],

$$\begin{gathered}n_2(g)=g_1^2+6g_2^2+15g_3^2+6g_4^2+g_5^2\le 751,\qquad 1\le N_{\text{flux}}\le 30,\qquad |g|\le(27,11,7,11,27),\\ R:\quad \operatorname{Re}\varphi\in[13/1024,\,19/1024],\qquad |\operatorname{Im}\varphi|\le 1/256.\end{gathered} \tag{47}$$

The lower bound $N_{\text{flux}}\ge 1$ holds at every F-flat point, because Q is even and, by self-duality, positive there. Self-duality also gives the cut on n_2. At an F-flat point $Q(g)=\|G\|^2_{\text{Hodge}}\ge\mu(\varphi)\,n_2(g)$, where $\mu(\varphi)$ is the smallest eigenvalue of the Hodge-norm Gram matrix on the invariant sector relative to $B^{\mathsf T}B$, and a lower bound $\mu_{\text{lo}}\le\mu$ valid on R gives

$$\begin{gathered}n_2(g)\;\le\;\frac{Q(g)}{\mu_{\text{lo}}}\;\le\;\frac{2\chi/24}{\mu_{\text{lo}}}=\frac{60}{\mu_{\text{lo}}}\,,\\ \mu_{\text{lo}}\in[0.0798867543765,\,0.0798867623663]\,,\qquad \frac{60}{\mu_{\text{lo}}}\in(751.0631076,\,751.0631828)\,,\end{gathered} \tag{48}$$

with integer part 751 at both ends. Because the bound $\mu\ge\mu_{\text{lo}}$ is proven only on a grid of 21 points spanning R (Section S7.1), we carry the cut as hypothesis (i) of Proposition 4.1.

We count fluxes up to the monodromy T of the slice about $\varphi=0$, which on the invariant sublattice is unipotent, with $(T-1)^5=0$, and of infinite order; S_6 acts trivially on the sector, and g and $-g$ are distinct classes. We call the smallest positive charge the lattice admits the *arithmetic floor*. The form (46) represents the value 1 within the caps of Eq. (47), in 626 classes with $N_{\text{flux}}=1$, and the arithmetic floor of the sector is exactly 1.

On the slice the period vector of Ω in the integral frame is S_6-invariant, hence of the form $B\,\Pi(\varphi)$ with $\Pi(\varphi)\in\mathbb{C}^5$ the *period vector of the sector*. With the pairing $\langle x,y\rangle=x^{\mathsf T}\mathrm{Gram}_5\,y$, the superpotential (3) of g on the slice is $W=\langle g,\Pi\rangle=u\cdot\Pi$, where $u=\mathrm{Gram}_5\,g$ is the flux in the period frame, and the Kähler potential K satisfies $e^{-K}=\langle\Pi,\bar\Pi\rangle>0$. For an invariant flux the six covariant derivatives of W coincide on the diagonal by symmetry, and F-flatness there is the one condition $D_\varphi W=0$. Multiplying by e^{-K} we define

$$G_g(\varphi)\;=\;e^{-K}D_\varphi W\;=\;\langle\Pi,\bar\Pi\rangle\,(u\cdot\Pi')-\langle\Pi',\bar\Pi\rangle\,(u\cdot\Pi)\,,\qquad A_a(\varphi)\;=\;\langle\Pi,\bar\Pi\rangle\,\Pi'_a-\langle\Pi',\bar\Pi\rangle\,\Pi_a\,, \tag{49}$$

(here $\Pi' = d\Pi/d\varphi$), and F-flatness on the sector becomes

$$G_g(\varphi) \;=\; \sum_{a=1}^{5} u_a\, A_a(\varphi) \;=\; 0,$$
$$u = \mathrm{Gram}_5\, g = \big(2g_1 + 30g_3 + g_5,\ 6g_4 - 60g_2,\ 30g_1 + 180g_3,\ 6g_2,\ g_1\big), \tag{50}$$

linear in the integer vector u; the modulus enters only through the five *period fields* $A_a(\varphi)$, and proportional fluxes, in particular g and $-g$, give the same equation.

The Picard–Fuchs operator of the slice, which annihilates Π, reads, with $\theta = \varphi\, d/d\varphi$ kept to the right of the powers of φ,

$$\begin{aligned} L_5 \;=\; &\theta^5 - 2\varphi\,(2\theta+1)\big(14\theta(\theta+1)(\theta^2+\theta+1)+3\big) + 4\varphi^2(\theta+1)^3\big(196\theta(\theta+2)+255\big) \\ &- 1152\varphi^3(\theta+1)^2(\theta+2)^2(2\theta+3)\,. \end{aligned} \tag{51}$$

The coefficient of θ^5 is $(1-4\varphi)(1-16\varphi)(1-36\varphi)$, so the singular points are 0, of maximal unipotent monodromy, the conifold points 1/36, 1/16, 1/4, and ∞ (local exponents and series in Section S7.1). The operator coincides with the one given for this slice in [49, 50, 51].

4.2 Exclusion below $N_{\text{flux}} = 4$

We evaluate the fields A_a on R in ball arithmetic by transport from $\varphi = 0$ with the tail bound (15). We tile R by 3,072 sub-boxes (192 base cells, each subdivided 4×4) and tabulate on each sub-box r outward-rounded interval hulls $[A_a](r)$ of the fields. A zero of G_g in a sub-box r satisfies the two interval memberships

$$0 \in \sum_a u_a\,[\mathrm{Re}\,A_a](r) \quad \text{and} \quad 0 \in \sum_a u_a\,[\mathrm{Im}\,A_a](r). \tag{52}$$

The cut contains 55,516 signed flux vectors with $(g_2, g_3) \neq (0,0)$, of which 3,872 have $N_{\text{flux}} \leq 3$ (forming 3,818 classes under T). For each of the 55,516 vectors and each sub-box both memberships fail, by a distance of roughly half the size of the fields there or more (Section S7.1 of the Supplementary Material). Hence every flux in the cut with an F-flat point in R lies on the cone

$$g_2 \;=\; g_3 \;=\; 0, \tag{53}$$

a codimension-2 sublattice of the sector. We call this constraint *cone necessity.*

The cut cannot be dropped. For example, the family

$$g = (N, 1, 0, 0, 3N), \qquad N_{\text{flux}} = 4N^2 - 30, \qquad u = N\,(5, 0, 30, 0, 1) + (0, -60, 0, 6, 0) \tag{54}$$

has $g_2 = 1$, and u/N tends to $(5, 0, 30, 0, 1)$, the period-frame ray of the charge-4 flux of Proposition 4.1, whose zero in R is transverse. At large N these off-cone fluxes therefore have an F-flat point in R while $N_{\text{flux}} = 4N^2 - 30$ exceeds the cut, and without the cut cone necessity is false.

On the cone the form (46) reduces to

$$N_{\text{flux}} \;=\; g_1\,(g_1 + g_5) \;=\; g_1^2\,(1 + r), \qquad r = g_5/g_1, \tag{55}$$

so every on-cone flux of charge at most 3 has $r \leq 2$. The on-cone fluxes in the cut with $1 \leq N_{\text{flux}} \leq 3$ are

$$\begin{gathered} (\pm1, 0, 0, g_4, g_5) \text{ with } g_5 \in \{0, \pm1, \pm2\},\ g_1 g_5 \geq 0; \qquad (\pm2, 0, 0, g_4, \mp1); \qquad (\pm3, 0, 0, g_4, \mp2); \\ \text{all with } |g_4| \leq 11, \end{gathered} \tag{56}$$

230 in number, each a class by itself under T. We exclude 228 of them by the interval test (52). The remaining two, $\pm(1, 0, 0, 0, 2)$ with $r = 2$, lie on the ray $u = \pm(4, 0, 30, 0, 1)$, which we exclude by a first-order bound on 48 blocks tiling R, with a margin of at least 675.9 on every block (Section S7.2 of the Supplementary Material). Thus no flux in the cut with $N_{\text{flux}} \leq 3$ has a vacuum in R, and $N_{\text{flux}} = 4$ is attained (Table 5, first column).

Proposition 4.1 (No vacuum below $N_{\text{flux}} = 4$ in R)**.** *Assume that* (i) *the flux g lies in the S_6-invariant sector and satisfies the cut of Eq.* (47)*,* (ii) *φ lies in the box R of Eq.* (47)*, and* (O) *the order-5 operator of [49] annihilates the periods of the sector on the diagonal slice. Then no flux with $N_{\text{flux}} \leq 3$ has an F-flat point in R, and $g = (-1, 0, 0, 0, -3)$, with $N_{\text{flux}} = 4$, has one at $\varphi = 0.0172802\ldots$, inside a Krawczyk ball of radius below 10^{-22} that contains exactly one solution, with $W \neq 0$.*

Hypothesis (O) enters every period evaluation in this section. For the fundamental period the annihilation by (51) is a theorem [94, Thm. 1] (see also [95, 96, 97]). Our evaluation of the recurrence on the series confirms it through order 400 and does not prove it; for the other four periods of the sector we take the annihilation from [49].

Along the on-cone fluxes $(g_1, 0, 0, 0, g_5)$ the real zero of G_g increases as r decreases, and the $r = 5/2$ member $g = (2, 0, 0, 0, 5)$, of charge 14, has its zero about 2.8×10^{-4} beyond the upper edge $\operatorname{Re}\varphi = 19/1024$ of R. The fluxes $g = (1, 0, 0, 0, 2)$ and $g = (1, 0, 0, 0, 1)$, of charges 3 and 2, have vacua at distances 0.0020 and 0.0061, respectively, beyond the upper edge of R, each proven to exist inside a Krawczyk ball of radius below 10^{-22}, with $W \neq 0$ (Table 5). For the budget-1 flux $g = (1, 0, 0, 0, 0)$ no zero is located at smooth points of the reachable slice (not a certified absence), and there budget 1 remains undecided: inside R the flux is among the 228 excluded above, and beyond R we have only a floating-point scan.

4.3 The row-by-row enumeration and the special points

Independently of Proposition 4.1 we treat each of the 58,388 orbit classes across budgets 1–30 under T, either excluding a vacuum in R in ball arithmetic or proving one in a Krawczyk ball [39]. We find 28 vacua in R, all with $W \neq 0$ and on the cone $g_2 = g_3 = 0$, as cone necessity requires, at twelve distinct positions (proportional fluxes share a vacuum): two each at charges 4 and 5 and six each at 16, 18, 20 and 22 (Table S15 of the Supplementary Material). No vacuum occurs among the 4,048 off-cone and on-cone classes of charges 1–3.

We compare the twelve positions with a catalog of eleven special points of algebraic φ, the three conifold points and the positions of the ten S_6-breaking vacua of Grimm and van de Heisteeg [24] (two at conifold points), and with the point $\varphi = 1$ of [49] (Section S7.2 of the Supplementary Material). Only the special point $\varphi = 1/64$, at the center of R, lies in R, and no vacuum in R coincides with it: the nearest, those of $\pm(2, 0, 0, 0, 7)$, have $|\varphi - 1/64| \geq 2.8 \times 10^{-4}$. The statement concerns position only, and it does not decide whether the positions are transcendental. At the three conifold points, outside R, Eq. (50) as used here does not apply, and the vanishing classes of charges 1, 6 and 15 give solutions with $W = 0$ [50, 51] (Section S7.3 of the Supplementary Material).

The *surplus* is the charge of the first vacuum in R minus the arithmetic floor. Against the arithmetic floor of 1 the measured surplus on the box is 3, and on the box it is dynamical, in that the F-flatness equation (50) has no solution for any candidate of charge at most 3 inside R. The slice, however, contains the charge-3 and charge-2 vacua just beyond the upper edge of R, so the slice-wide surplus at smooth points is at most 1, with budget 1 undecided there and realized at the conifold point 1/36 [50, 51]. Because we have not set up rigorous continuation for the full six-parameter family, what the full box does to the cone, the floor and the first vacuum is measured

	$g = \pm(1,0,0,0,3)$	$g = \pm(1,0,0,0,2)$	$g = \pm(1,0,0,0,1)$
N_{flux}	4	3	2
$u = \mathrm{Gram}_5\, g$	$\pm(5,0,30,0,1)$	$\pm(4,0,30,0,1)$	$\pm(3,0,30,0,1)$
n_2	10	5	2
$r = g_5/g_1$	3	2	1
$\mathrm{Re}\,\varphi^*$	$0.0172802004\ldots$	$0.0205828761\ldots$	$0.0246365920\ldots$
radius in $\mathrm{Re}\,\varphi$	6.706×10^{-23}	5.634×10^{-23}	5.857×10^{-23}
radius in $\mathrm{Im}\,\varphi$	7.853×10^{-23}	6.273×10^{-23}	6.138×10^{-23}
radius, enumeration ball (Table S15)	8.143×10^{-23}	(outside R)	(outside R)
$\mathrm{Re}\,\varphi^* - 19/1024$	$-0.0012744870\ldots$	$+0.0020281886\ldots$	$+0.0060819045\ldots$
position relative to R	inside	above	above
$\lvert W\rvert \geq$	$4.071993\ldots$	$2.656690\ldots$	$1.296953\ldots$
e^{-K}	$95.5736\ldots$	$81.4473\ldots$	$68.6140\ldots$

Table 5: The first vacuum inside R has charge 4; vacua of charge 3 and 2 exist on the same slice just above the upper edge $\mathrm{Re}\,\varphi = 19/1024$ of R. Each column is an on-cone flux $g = (g_1, 0, 0, 0, g_5)$ of the invariant sector with its charge N_{flux} of Eq. (46), its period-frame vector $u = \mathrm{Gram}_5\, g$, its norm n_2 of Eq. (47), the ratio r of Eq. (55), and a zero φ^* of G_g of Eq. (50) proven by the Krawczyk test (16) to exist and to be the only zero in the box about this midpoint with the radii listed for $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$. The fluxes g and $-g$ have the same zeros, and the first column is the vacuum of Proposition 4.1. In every column the enclosure of $\mathrm{Im}\,\varphi^*$ contains 0 (as it must for a vacuum on the real slice), $|W| = |u \cdot \Pi|$ is bounded below by the positive value listed, and $e^{-K} = \langle \Pi, \bar{\Pi} \rangle$ is positive (the point lies in the physical domain); all three zeros lie below the conifold point $1/36$. All columns assume the cut of Eq. (47) and hypothesis (O) of Proposition 4.1. The second and third vacua lie outside the box R, where no exclusion is claimed; whether a vacuum of charge 1 exists at smooth points of the slice outside R is open, and at the conifold point $1/36$ the unit-charge flux is the vanishing class, with $W = \partial_\varphi W = 0$ (see the text). The radii below 10^{-22} given in the text are those of these three boxes; the row labeled "radius, enumeration ball" is the radius of the independent enclosure of the first vacuum by the row-by-row enumeration (Table S15 of the Supplementary Material), which encloses zeros inside R only. The bound 2.2×10^{-22} of Section 2.3 refers to the largest radius in Table S15 of the Supplementary Material, 2.149×10^{-22}. Digit strings ending in dots are truncated; Section S7.2 of the Supplementary Material gives the values in full.

nowhere. In Sections S1.4 and S2 of the Supplementary Material we compare this surplus with a lattice bound across ten enumerated flux sectors and state a conjecture; neither is used in this section.

Proposition 4.1 and the enumeration hold under the following conditions. The result is census-universe-conditional: the cut $n_2 \leq 751$, $N_{\text{flux}} \leq 30$ is a hypothesis, and the counterexample family shows the theorem cannot drop it. It is domain-conditional: absence is certified on the census box only, and a vacuum of budget at most 3 outside the box is not excluded. It is restricted to the S_6-invariant sector and its diagonal slice: the full six-parameter flux box is open. It stands at one geometry: nothing in this section claims the pattern persists elsewhere. Every period evaluation rests on hypothesis (O). Two more conditions concern only the comparison with the special points: the catalog is fixed at eleven points, and we have not extended it to further complex-multiplication points; neither catalog condition is discharged here.

5 The KKLT benchmark and the de Sitter candidates

Demirtas, Kim, McAllister and Moritz (DKMM) [20] constructed a type IIB flux vacuum on an orientifold of the degree-18 hypersurface in $\mathbb{CP}^4_{[1,1,1,6,9]}$ with $|W_0| = 2.037 \times 10^{-8}$. In this perturbatively flat flux vacuum (PFFV) the integer fluxes make the perturbative superpotential vanish along one direction, on which a two-term instanton racetrack leaves an exponentially small $|W_0|$. Later constructions with the same mechanism, computed like the original in floating point with truncated instanton sums, gave AdS_4 vacua with $|W_0|$ down to 1.1×10^{-95} [98, 99] and thirty candidate de Sitter vacua [52].

We prove that the DKMM superpotential has exactly one F-term critical point in an explicit small box (Eq. (65)) and enclose $|W_0|$ there (Eq. (66)). We then trace the spread of the three published values of $|W_0|$ to one omitted term of the prepotential, evaluate at fixed inputs the superpotential of a later AdS_4 vacuum, and classify the thirty de Sitter candidates by one exact invariant of their published fluxes. All thirty have a nonzero invariant and are therefore not perturbatively flat in any basis; the statement concerns the recorded flux data and says nothing about the candidates as vacua.

5.1 Conventions and the DKMM flux data

The papers on this benchmark and on its successors [98, 99] normalize W_0 differently, by up to four orders of magnitude (the size of the factor $e^{(K_{\rm cs}+K_\tau)/2}$ defined below at the vacuum of [99], where $\operatorname{Im}\tau \sim 90$). We fix each paper's normalization by reproducing one of its numbers. The flux superpotential is

$$W \;=\; \sqrt{2/\pi}\int G_3\wedge\Omega\,, \qquad G_3 \;=\; F_3 - \tau H_3\,, \qquad \tau \;=\; C_0 + i/g_s\,, \tag{57}$$

with period vector $\Pi = (F_\Lambda; X^\Lambda)$ in the gauge $X^0 = 1$, symplectic pairing $\Sigma = \left(\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}\right)$, and units $\ell_s^2 = (2\pi)^2\alpha' = 1$ with reduced $M_{\rm pl} = 1$ (hence W is in $M_{\rm pl}^3$). Writing F and H for the integer six-vectors of flux quanta, with the components of F, H and Π ordered as $(F_0, F_1, F_2, X^0, X^1, X^2)$ and numbered 0 to 5, we have

$$W \;=\; \sqrt{2/\pi}\,(F-\tau H)\cdot\Sigma\cdot\Pi\,, \qquad U^1 \;=\; \Pi_4/\Pi_3\,, \qquad U^2 \;=\; \Pi_5/\Pi_3\,, \tag{58}$$

where $U^a = X^a/X^0$ ($a = 1, 2$) are the complex-structure moduli in this integral symplectic frame. The periods are series in algebraic coordinates z_a about the large-complex-structure point, related to U^a by the mirror map.

Following Eq. (1) of [20], the benchmark value $|W_0|$ denotes the Kähler-invariant combination

$$|W_0| \;\equiv\; e^{(K_{\rm cs}+K_\tau)/2}\,|W|\,, \qquad e^{-K_{\rm cs}} \;=\; -i\,\Pi^\dagger\cdot\Sigma\cdot\Pi\,, \qquad e^{-K_\tau} \;=\; -i(\tau-\bar\tau)\,, \tag{59}$$

evaluated at the F-term critical point. Here $e^{-K_{\rm cs}}$ is built from the exact periods and thus contains the constant ξ of Eq. (62) below and all instanton corrections, and there is no Kähler-moduli factor. We call $|W_0|$ the *dressed* value and $|W|$ the undressed one. Unlike $|W|$, $|W_0|$ does not depend on the normalization of Π. The F-term critical point is the common zero of the three covariant derivatives

$$D_aW \;\equiv\; \partial_{U^a}W + \big(\partial_{U^a}K_{\rm cs}\big)\,W \;=\; 0 \quad (a=1,2)\,, \qquad D_\tau W \;\equiv\; \partial_\tau W + \big(\partial_\tau K_\tau\big)\,W \;=\; 0\,, \tag{60}$$

three complex equations in (U^1, U^2, τ), or six real ones.

We take the DKMM flux data

$$M \;=\; (-16, 50)\,, \qquad K \;=\; (3, -4)\,, \qquad p \;=\; (2/5,\, 3/10)\,, \tag{61}$$

where M and K enter the flux vectors $F = (7, 3, -24, 0, -16, 50)$ and $H = (0, 3, -4, 0, 0, 0)$ of [20], while p is their perturbatively flat direction. With $N_{ab} = \kappa_{abc} M^c$ (κ_{abc} the classical triple intersection numbers), one has $p = N^{-1} K$ and $K \cdot p = 0$, and the perturbative superpotential vanishes identically along $U = \tau p$. The $\zeta(3)$ term of the prepotential enters $|W_0|$ through K_{cs}. At the perturbative level,

$$\xi \;=\; \frac{\zeta(3)\,\chi}{2\,(2\pi i)^3}\,, \qquad e^{-K_{\mathrm{cs}}} \;=\; \tfrac{4}{3}\,\kappa_{abc} x^a x^b x^c \;-\; 4\,\mathrm{Im}\,\xi\,, \tag{62}$$

with χ (the χ_A of Table 7) the Euler number of the A-side threefold, whose Gopakumar–Vafa invariants enter the prepotential, and $x^a = \mathrm{Im}\,U^a$; for the DKMM geometry $\mathrm{Im}\,\xi = -540\zeta(3)/(16\pi^3) = -1.30842$. For PFFV fluxes both flux vectors vanish in the F_0 slot, the only period containing ξ, so ξ drops out of W and acts only through K_{cs}.

All results of this section hold in the frame fixed by Eqs. (57)–(62) and are conditional on the source papers' own flux-quantization assumptions. Those papers allow odd integer flux quanta, since the toroidal-orientifold restrictions of Frey and Polchinski [100] are not known to apply to Calabi–Yau orientifolds, and compensate the resulting half-integer flux tadpole with one stuck half D3-brane. We adopt both conventions, with no position taken on the open quantization question.

5.2 The DKMM critical point

Every error bound below is a ball radius in the sense of Section 2.3. We continue the periods from the large-complex-structure (MUM) point $s = 0$ along the monomial curve

$$z_1 \;=\; s^4\,, \qquad z_2 \;=\; s^3\,. \tag{63}$$

Restricting the two-parameter Picard–Fuchs ideal of Candelas, Font, Katz and Morrison [101] to Eq. (63) as a D-module, we obtain an operator L_s of order 6 and degree 63 in s. It annihilates all $1 + 2 + 2 + 1 = 6$ members of the restricted Frobenius basis at $s = 0$ (graded by powers of $\log s$), as many as its order, and hence every restricted period and, for any flux, W of Eq. (58) on the curve. The nearest apparent singularity of L_s on the physical ray, $s = 0.017938$, lies at only 1.33 times the vacuum coordinate $s_{\mathrm{vac}} = 0.013468$, placing s_{vac} at almost the full radius at which the MUM-point series can be tail-bounded (Section S3.3 of the Supplementary Material lists the singularities). Therefore we continue along two homotopic paths, the real axis and a complex detour, in several legs that avoid the apparent singularities, bounding each remainder with (15); the two resulting balls overlap (Table 6).

The critical point itself lies slightly off the curve (Section S3.3 of the Supplementary Material). Since the restriction has no order drop ($r = 6 = 2(h + 1)$ for $h = 2$ complex-structure moduli), the transverse derivatives of the periods are $\mathbb{Q}(s)$-linear combinations of on-curve data, so continuing off the curve needs no new differential equation. To enclose the critical point, let $z^* = (s^{*4}, s^{*3})$ be the point of the curve (63) at a fixed rational s^* close to s_{vac} (given in Section S3.3), and define the *transverse coordinates*

$$w_a \;:=\; z_a/z_a^* - 1\,, \qquad a = 1, 2\,, \tag{64}$$

which vanish at z^* (and are not the moduli U^a). After one short continuation leg off the curve, we apply the Krawczyk test (16) to the real and imaginary parts of the three left sides of Eq. (60) as functions of the six real coordinates $x = (\mathrm{Re}\,w_1, \mathrm{Im}\,w_1, \mathrm{Re}\,w_2, \mathrm{Im}\,w_2, \mathrm{Re}\,\tau, \mathrm{Im}\,\tau)$. With the Jacobian ball-evaluated over a box X_0 of radius 10^{-10} about an approximate zero $\tilde{x}$, the Krawczyk image is strictly interior to X_0, so the $DW = 0$ point in X_0 exists and is unique, and six contraction iterations shrink all box radii below 4.9×10^{-149}:

$$\mathcal{K}(\tilde{x}, X_0) \;\subset\; \mathrm{int}\, X_0\,, \qquad X_{k+1} \;=\; \mathcal{K}(\tilde{x}_k, X_k) \cap X_k\,, \qquad \mathrm{rad}\, X_0 \;=\; 10^{-10}\,, \qquad \mathrm{rad}\, X_6 \;\sim\; 5 \times 10^{-149}\,. \tag{65}$$

quantity	ball midpoint	radius
$\mathrm{Im}\,\tau_{\mathrm{vac}} = 1/g_s$	$6.855457256\ldots$	6.8×10^{-149}
$\mathrm{Re}\,\tau_{\mathrm{vac}}$	0	1.64×10^{-150}
g_s (rounded)	0.14587	
$\mathrm{Im}\,U^1$	$2.742176980\ldots$	4.08×10^{-149}
$\mathrm{Im}\,U^2$	$2.056632773\ldots$	8.40×10^{-150}
$\mathrm{Re}\,w_1$	$7.944427495\ldots \times 10^{-5}$	4.8522×10^{-149}
$\mathrm{Re}\,w_2$	$7.944617279\ldots \times 10^{-5}$	3.6392×10^{-149}
$\lvert\mathrm{Im}\,w_1\rvert$, $\lvert\mathrm{Im}\,w_2\rvert$	0	$< 5 \times 10^{-150}$
$e^{-K_\tau} = 2\,\mathrm{Im}\,\tau_{\mathrm{vac}}$	$13.71091451\ldots$	1.35×10^{-148}
$e^{-K_{\mathrm{cs}}}$ (gauge $X^0 = 1$)	$484.6462670\ldots$	5.41×10^{-28}
$e^{-(K_{\mathrm{cs}}+K_\tau)/2}$ (gauge $X^0 = 1$)	$81.51652308\ldots$	5.1×10^{-29}
$\lvert W_0\rvert_{\mathrm{vac}}$, real-axis path (Eq. (66))	$2.037106093\ldots \times 10^{-8}$	4.4×10^{-148}
$\lvert W_0\rvert_{\mathrm{vac}}$, complex detour	$2.037106093\ldots \times 10^{-8}$	7.52×10^{-138}

Table 6: The F-term critical point of the DKMM superpotential exists and is unique in the box of Eq. (65), and the dressed value $|W_0|$ there is enclosed with the radii shown. Each entry is the midpoint of a ball proven to contain the true value, written as a prefix of at most ten digits, together with the radius of that ball (for $\mathrm{Im}\,w_{1,2}$ an upper bound on the modulus; g_s is $1/\mathrm{Im}\,\tau_{\mathrm{vac}}$ rounded to five digits). All quantities refer to the frame of Eq. (57) with the fluxes of Eq. (61) and the $\zeta(3)\chi$ term of Eq. (62) kept; U^1, U^2 are the complex-structure moduli Eq. (58), w_1, w_2 the transverse coordinates of Eq. (64), and $|W_0|_{\mathrm{vac}}$ the dressed value of Eq. (59). The two rows for $|W_0|_{\mathrm{vac}}$ come from continuing the periods along the real s-axis and along a complex detour around the nearest apparent singularity of L_s; the two balls for $|W_0|_{\mathrm{vac}}$ overlap and have 129 leading digits in common at the vacuum. The values of $e^{-K_{\mathrm{cs}}}$ and $e^{-(K_{\mathrm{cs}}+K_\tau)/2}$ depend on the normalization of Π and are given in the gauge $X^0 = 1$; $|W_0|_{\mathrm{vac}}$ does not. Longer digit strings are listed in Section S3.3 of the Supplementary Material.

The final box brackets the reality locus $\mathrm{Re}\,\tau = \mathrm{Im}\,w_1 = \mathrm{Im}\,w_2 = 0$, with $|\mathrm{Re}\,\tau|$, $|\mathrm{Im}\,w_1|$, $|\mathrm{Im}\,w_2| < 5 \times 10^{-150}$.

At the critical point, with all instanton degrees entering through the Picard–Fuchs ideal, we obtain for the dressed $|W_0|$ of Eq. (59) and the axio-dilaton

$$|W_0|_{\mathrm{vac}} \;=\; 2.037106093\ldots \times 10^{-8} \;\pm\; 4.4 \times 10^{-148}\,, \tag{66}$$

$$\tau_{\mathrm{vac}} \;=\; i \cdot 6.855457256\ldots \;\pm\; 6.8 \times 10^{-149}\,, \tag{67}$$

the radii being those of the real-axis continuation. The first radius corresponds to about 140 certified digits for $|W_0|$. We find no published certified-arithmetic computation of a string-vacuum superpotential. Equation (67) gives $g_s = 0.14587$, and Table 6 collects the moduli and Kähler factors. The ball of Eq. (66) also gives the error of truncating the Gopakumar–Vafa (GV) instanton sum at degree $(4, 4)$, an error almost ten orders of magnitude below the bound implied by an assumed growth rate of the invariants (Section S3.3).

5.3 The $\zeta(3)\chi$ term and the published values

For the fluxes of Eq. (61) DKMM report $|W_0| = 2.037 \times 10^{-8}$. Broeckel et al. [21] repeat the construction and obtain 2.048×10^{-8}, and Carta, Mininno and Shukla (CMS) [22] give 2.0482×10^{-8} together with six-digit moduli vevs. Solving the three F-term equations (60) in floating point with the constant ξ of Eq. (62) kept in $e^{-K_{\mathrm{cs}}}$ (V1) or dropped (V2), we reproduce all three values.

variant	$\zeta(3)\chi$ term	$\operatorname{Im}\tau$	$\operatorname{Im}U^1$	$\operatorname{Im}U^2$	$\lvert W_0\rvert$
V1 (DKMM frame)	$\chi_A = -540$	6.85545726	2.74217698	2.05663277	$2.03710609 \times 10^{-8}$
V2 (Broeckel/CMS)	dropped	6.85506720	2.74202095	2.05651575	$2.04819630 \times 10^{-8}$
V3 (sign of χ flipped)	$\chi_A = +540$	6.85466913	2.74186172	2.05639633	$2.05947161 \times 10^{-8}$

Table 7: Keeping or dropping a single term, the $\zeta(3)\chi$ term of Eq. (62) in $e^{-K_{\mathrm{cs}}}$, in a floating-point solution of the F-term equations reproduces every published value of the DKMM superpotential: V1 matches all four DKMM digits of $|W_0|$, and V2 matches Broeckel's 2.048×10^{-8} and the digits given by CMS (2.0482×10^{-8}, $g_s^{-1} = 6.85505$, $U = (2.74202, 2.05652)$) except the final digit of g_s^{-1} (Section S3.2 of the Supplementary Material). V3, with the sign of the term flipped, matches no published value. The columns give the variant, the treatment of the $\zeta(3)\chi$ term (through the Euler number χ_A entering Eq. (62)), the moduli $\operatorname{Im}\tau = 1/g_s$ and $\operatorname{Im}U^{1,2}$ at the solution of the F-term equations (60) for the fluxes of Eq. (61), and the dressed $|W_0|$ of Eq. (59) in units of M_{pl}^3. The solutions are computed at 50-digit working precision (F-term residuals $\leq 2 \times 10^{-48}$); the grid is floating-point arithmetic that locks the convention and certifies nothing. GV depth is irrelevant at the accuracy of the published values: degree ≤ 2 versus the full $(4, 4)$ window moves $|W_0|$ by 1.3×10^{-9} relative. Flux sign $(M, K) \to (-M, -K)$ leaves V2 unchanged, so the flipped flux vectors of Broeckel et al. do not affect the comparison.

Table 7 also lists the solution with the sign of χ flipped (V3), and Figure 2 shows the three to scale. Dropping the term shifts $e^{-K_{\mathrm{cs}}}$ by $4|\operatorname{Im}\xi| = 5.234$ out of 479.33, i.e. 1.092%, and hence $|W_0|$ by 0.544%. Thus the ratio of the V2 and V1 values matches the published ratios,

$$\frac{\mathrm{V2}}{\mathrm{V1}} \;=\; 1.0054441\,, \qquad \frac{2.048}{2.037} \;=\; 1.00540\,, \qquad \frac{2.0482}{2.0371} \;=\; 1.00545\,. \tag{68}$$

Both groups state that they drop the term: Broeckel et al. write "the term involving ξ is subdominant," and CMS "we have neglected the Euler characteristic dependent constant pieces denoted as $\tilde{p}_0$." The other candidate causes (GV depth, normalizations, flux signs, vev rounding) give offsets of the wrong size by orders of magnitude.

The computations of all three groups are therefore internally consistent (except for two final-digit τ discrepancies, Section S3.2 of the Supplementary Material). The gap between the published values is carried entirely by one term whose omission the later papers themselves state. The ball of Eq. (66) lies inside the rounding band of DKMM's 2.037×10^{-8}, and restoring the term at the GV depth of Broeckel et al. and CMS gives 2.0371×10^{-8}. Since 0.544% of $|W_0|$ is 111 units in the last digit of CMS's five significant figures, any comparison with the benchmark at that accuracy depends on whether the term is kept.

The grid of Table 7 is floating-point arithmetic, a convention lock rather than a certification, and the proven value of Eq. (66) is independent of it. We can nonetheless decide the sign of χ in exact arithmetic: the change of basis from the Frobenius periods to the integral symplectic frame, solved in fractions over $\mathbb{Q}[v^{\pm 1}, \zeta(3)]$ with $v = 2\pi i$, exists under the flip $\chi_A \to +540$ only with a spurious counterterm $+1080\,\zeta(3) = 2\,\Delta\xi\, v^3$ that the pinned frame does not need. This excludes V3, whereas the keep-versus-drop convention of V1 against V2 remains a floating-point lock. Section S3.2 of the Supplementary Material collects five corrections to the published data of these constructions, none of which changes its paper's $|W_0|$.

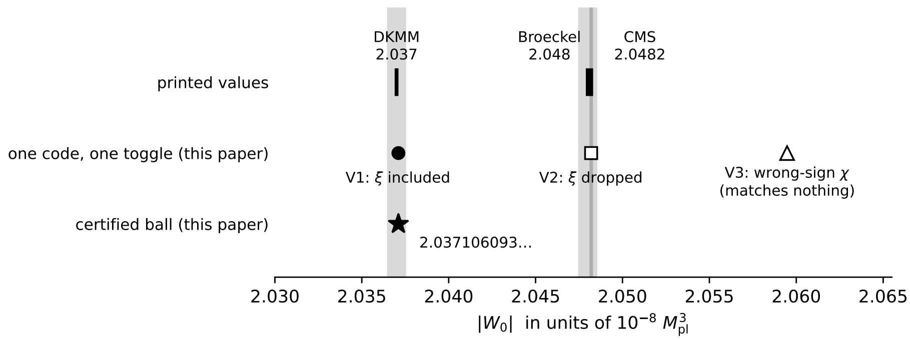

Figure 2: Solving the F-term equations in floating point with the $\zeta(3)\chi$ term kept, dropped or sign-flipped gives DKMM's value, Broeckel's and CMS's value, and no published value, respectively; the proven value lies inside DKMM's rounding interval. Top row: the three published values of $|W_0|$ for the fluxes of Eq. (61), DKMM's 2.037×10^{-8}, Broeckel's 2.048×10^{-8} and CMS's 2.0482×10^{-8}, with their rounding intervals shaded. Middle row: the variants V1 (term kept), V2 (term dropped) and V3 (wrong sign of χ) of Table 7. Bottom row: the proven value of Eq. (66), which includes all instanton degrees; its radius (4.4×10^{-148}) is far below plot resolution. The horizontal axis is the dressed $|W_0|$ of Eq. (59) in units of $10^{-8}\, M_{\rm pl}^3$.

5.4 A second vacuum at fixed inputs

The AdS vacuum labeled 5-81-3213 in [98], with $|W_0| = 2.03778 \times 10^{-23}$ in that paper's ancillary data and $h = 5$ complex-structure moduli, cannot be treated by the scalar-operator method used above. Its Picard–Fuchs operator restricted to the curve below would have order $13 \neq 2h + 2 = 12$ and degree greater than 4,052, and the operator itself remains underived. Instead we evaluate the periods directly on the monomial curve $(z_1, \ldots, z_5) = (s^9, s^{10}, s^{10}, s^{10}, s^{10})$ through the large-complex-structure point. There the twelve flux-frame periods are exact linear combinations of derivatives of the closed-form Γ-series of the mirror, fixed by matching the large-complex-structure expansion to finite order (Section S3.3 of the Supplementary Material). The coefficients of these series grow at most geometrically, giving a radius of convergence $\geq 1/2$, while $|s_{\rm vac}| = 1/180$. We therefore sum the series at $s_{\rm vac}$ with geometric-series bounds on their tails, and no differential equation is transported.

We evaluate $|W_0|$ in the *undressed* convention of [98], in which $|W_0|$ denotes $|W|$ without the factor $e^{(K_{\rm cs}+K_\tau)/2}$ of Eq. (59), at the fixed inputs $s_{\rm vac} = -1/180$ and $\tau_* = 12 + 19.83563404\ldots\, i$ (the racetrack solution for τ). Conditional on the quantization convention adopted above (with $Q_{\rm flux} = 71/2$ as published), on these fixed inputs, and on the finite-order identification of the periods, we obtain

$$|W_0| \;=\; 2.037782247\ldots \times 10^{-23}\,, \qquad \text{relative radius } 3.1 \times 10^{-68}\,, \tag{69}$$

in agreement with all six digits of the ancillary value, since the relative deviation 1.10×10^{-6} is below half a unit in their last place (2.45×10^{-6}). Here the periods are normalized by the holomorphic period $X^0 = \varpi_0(s)$, $\varpi_0(0) = 1$; in the gauge $X^0 = 1$ no digit of Eq. (69) changes and the relative uncertainty is below 10^{-66}. However, no F-term critical-point certificate exists for this vacuum, and nothing here parallels the Krawczyk box of Eq. (65); the six-digit agreement is sensitive to τ_* (Section S3.3 of the Supplementary Material). The two certified vacua of this section are therefore not equal grade: roughly 140 unconditional digits against roughly 68 conditional ones.

5.5 The thirty de Sitter candidates

The thirty candidate de Sitter vacua of [52] comprise 5 worked out in the paper and 25 given only in the group's repository `kklt_de_sitter_vacua` [53] (release v1.0.0, CYTools data format [102]). Their authors build them as conifold flux vacua, following the conifold extension of the racetrack mechanism [54], and whether they are perturbatively flat was thus never open in the authors' own terms. The candidates' status as de Sitter vacua rests on Kähler-sector stabilization, uplift and effective-field-theory control, which no integer invariant touches.

For 29 of the 30 candidates the repository fluxes M^a, K_a and intersection numbers κ_{abc} refer to a fixed unimodular *effective basis* of divisor classes of the A-side manifold, chosen among the prime toric divisors. For all thirty candidates the flux data satisfy

$$N_{ab} \;=\; \kappa_{abc} M^c\,, \qquad \det N \;\neq\; 0\,, \qquad K^T N^{-1} K \;\neq\; 0 \tag{70}$$

with N an $n\times n$ matrix, n the number of complex-structure moduli of the candidate. A perturbatively flat flux vacuum in the DKMM sense has $p = N^{-1}K$ with $K \cdot p = 0$ and hence $K^T N^{-1} K = 0$, and the rational number $K^T N^{-1} K$ is invariant under $GL(n, \mathbb{Z})$ changes of the effective basis. Therefore none of the thirty is perturbatively flat in any basis. The flat direction instead solves the conifold variant

$$N\,p - K \;=\; \lambda\, q_{\mathrm{cf}}\,, \qquad q_{\mathrm{cf}} \cdot p \;=\; 0\,, \qquad K \cdot p \;=\; 0\,, \tag{71}$$

with q_{cf} the conifold curve class and λ rational. We reproduce the published conifold data M_{cf}, K_{cf} and λ, with one discrepancy in a quoted λ noted in Section S3 of the Supplementary Material. On the 29 candidates with a repository basis the invariant of Eq. (70) takes four values, one per geometry,

$$K^T N^{-1} K \;\in\; \left\{ -\tfrac{289}{4064}\,,\; \tfrac{5887}{15492}\,,\; \tfrac{360}{1529}\,,\; \tfrac{8649}{27230} = \tfrac{93^2}{27230} \right\}, \tag{72}$$

two from the large families of 22 and 5 candidates, whose members share their flux and complex-structure data, and two from single-geometry candidates.

For the thirtieth candidate, `manwe`, the one integral basis of prime toric divisors does not accommodate the repository data (Section S3 of the Supplementary Material), and we take instead the integral conifold-adapted basis distributed with the public codes `pfvs` [28] and JAXVacua [29]. In that basis the fluxes printed in [52] reproduce the published p, M_{cf} and $\lambda = 26/5$ exactly, and give $\det N = 51118080$ and $K^T N^{-1} K = 143/20 \neq 0$. For five of the thirty we confirm the repository values of the superpotential at the stored moduli, within an allowance for the GV invariants beyond the degrees included (Section S3.1 of the Supplementary Material, Table S2).

The results of this section hold with the following qualifications. The certified statements concern the classical flux superpotential alone: its F-term critical point and its value in the frame of Eqs. (57)–(62), for the flux data of the cited constructions. Given the integer flux vectors as data, the Krawczyk containment of Eq. (65) and Eqs. (66)–(67) are unconditional statements about that function in this frame. Equation (69) is conditional on the fixed inputs $(s_{\mathrm{vac}}, \tau_*)$ and on the finite-order identification of the periods, while Eqs. (70)–(72) are exact statements about the published flux integers. Kähler-sector stabilization and uplift, together with the effective-field-theory control questions [103, 104], are untouched, and nothing in this section certifies a de Sitter vacuum.

6 Conclusions

On K3×K3 we proved, on one stratum, the bound indicated by the searches of Bena, Blåbäck, Graña and Lüst [7, 8]. Under (H1)–(H4), every fully stabilizing flux whose support has rank at most three carries $N_{\text{flux}} \geq 25$ against a budget of 24 (Theorem 3.4). Off that stratum the bound fails under the same hypotheses: we constructed a fully stabilizing flux of charge 22 and rank 22 whose vacuum is fixed by an order-66 lattice symmetry. On the definite rank-19 stratum, which remains open, we exhibited three root-free lattices on which the flux charge is at least 24 or 25 (Theorem 3.5), with 24 attained precisely when the Diophantine equation (39) is solvable.

In the S_6-invariant sector of the Hulek–Verrill fourfold, whose lattice admits charge 1, we proved under hypothesis (O) that within a flux cut and a modulus box R the first vacuum appears at charge 4 (Proposition 4.1). Lower-charge vacua lie just outside R; a unit-charge vacuum at a smooth point outside R is neither found nor excluded.

For the DKMM benchmark [20] we proved that exactly one F-term critical point lies in the box of Eq. (65) and enclosed $|W_0|$ to about 140 digits. The half-percent spread of its three published values [20, 21, 22] comes from keeping or dropping the $\zeta(3)\chi$ term of the prepotential; each computation is internally consistent. All thirty de Sitter candidates of [52] have a nonzero flux invariant $K^T N^{-1} K$ (Eq. (70)), consistent with their construction as conifold vacua. Nothing in this paper certifies a de Sitter vacuum, and every statement in it concerns solutions of the classical flux equations.

With lattice arithmetic we decide full stabilization only at generic points of moduli space, those free of orbifold singularities, including smooth loci fixed by discrete symmetries [13]; the charge-22 vacuum lies on such a locus. In Section S1.3 of the Supplementary Material we enumerate vacua in ten flux sectors and find, at every tested charge with vacua, a vacuum off the enhanced-symmetry loci listed there. In four sectors the smallest charge of such a vacuum equals a lower bound computed from the lattice (Theorem S1.1). These enumerations suggest a conjectured escape-price law (Section S2), and nothing in this paper upgrades it. For the one-parameter threefold AESZ34 [95, 105], Theorem S1.2 shows that no known algebraic cycle establishes a $W = 0$ vacuum at the $\phi = -1/7$ attractor point; it closes the known-cycle route and no more.

On the indefinite strata of K3×K3 (ranks 4 through 22) the finite enumeration of Section 3.1 is not yet done: it requires fundamental domains of the infinite unit groups. Solving Eq. (39) and enumerating the one outstanding genus of Section S1.2 of the Supplementary Material, on which the smallest root-free determinant over all rank-19 lattices depends, are finite computations. We have not derived Picard–Fuchs systems for the six-parameter Hulek–Verrill family or for evaluating the de Sitter candidates' superpotentials independently of the repository moduli [53]. The Gauss–Manin systems of Section 2.3 are so far reconstructed and checked rather than derived. The main open problem concerns the many-moduli regime, in which the Tadpole Conjecture is posed and which nothing in this paper reaches. Our two tools need only a flux lattice and a Picard–Fuchs system, but whether they remain practical in that regime is untested.

Code and data availability. The data and machine-checkable proofs for this paper are available at `https://bootloops.ai/files/kklt/`, with a SHA-256 manifest of every file and stand-alone checkers for each set. The lattice data of Section 3 and the Supplementary Material comprise the Gram and gluing matrices of the rank-19 lattices and the 54 admissible genera with their 98 complement classes. They also include the 1,764 rational Farkas vectors of Lemma 3.3, the exact proofs of the bounds of Table 3, and the charge-22 flux with its isometry to Kondō's lattice. The same address holds the data of the Hulek–Verrill sector of Section 4 and the KKLT interval proofs of Section 5. The `terrier` routines and the validation suite of Section S4.1 of the Supplementary Material are available at the same address, as a source tree and as a single archive.

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