Supplementary material for "Exact methods for string flux vacua"

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Conventions

This supplement to "Exact methods for string flux vacua" contains the material that the main text cites by section number: the proofs of the existence and isolation criteria for K3×K3 vacua, the root-free determinant thresholds, the enumeration of vacua in ten flux sectors and a no-go theorem at the AESZ34 attractor point (Section S1); a conjecture suggested by that enumeration (Section S2); notes on the KKLT constructions and the numerical data for the benchmark computation (Section S3); the tests of the implementation and the contents of the distributed material (Section S4); the lattice data behind Lemma 3.3 and Theorems 3.4 and 3.5, with the theta-series argument for that lemma (Section S5); the arithmetic data of the charge-22 flux (Section S6); and the data for the Hulek–Verrill sector (Section S7). Section, equation, theorem and table numbers without the prefix S refer to the main text. The hypotheses (H1)–(H4), (H-RANK) and (O) and the stabilization predicate (FS) are those defined in the main text and are not restated here. The reference list at the end of this supplement is numbered independently of the one in the main text.

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S1 Proofs and further enumeration results

We first prove the vacuum-existence criterion (23) of Section 3.1 of the main text and the isolation criterion stated with it (Section S1.1). The same subsection explains why the enumeration of Section 3 at support rank $r \geq 4$ is a finite search. We then give the root-free determinant thresholds behind the lower end of the bracket $[128, 160]$ of Section 3.3 and the status of the open definite stratum (Section S1.2), followed by the classification of flux vacua as on or off the enhanced-symmetry loci in ten sectors (Section S1.3). Next we show that in the lattice-type sectors the first off-locus budget is determined by minimizing an integer-valued quadratic form over support pairs and flux matrices, before any vacuum equation is solved (Section S1.4). Finally, we prove a no-go theorem for the known algebraic cycles at the AESZ34 attractor point (Section S1.5).

S1.1 Proofs of the existence and isolation criteria

The vacuum locus V_N of Section 3.1 of the main text is nonempty if and only if the flux satisfies (23), and its points are isolated exactly when every eigenspace of S and of $\tilde{S}$ is definite; we prove the two statements in turn. Write $V = \Gamma \otimes \mathbb{R}$ and $\tilde{V} = \tilde{\Gamma} \otimes \mathbb{R}$, both of signature $(3, 19)$, and recall that $g : \tilde{V} \to V$ and $\tilde{g} : V \to \tilde{V}$ are adjoint, $(g\tilde{v}, v) = (\tilde{v}, \tilde{g}v)$. Hence $S = g\tilde{g}$ and $\tilde{S} = \tilde{g}g$ are self-adjoint. Only the two real quadratic spaces enter, and integrality of N plays no role.

Suppose first that $(\Sigma, \tilde{\Sigma}) \in V_N$. Taking adjoints in (21) gives $\tilde{g}(\Sigma) \subseteq \tilde{\Sigma}$ and $\tilde{g}(\Sigma^{\perp}) \subseteq \tilde{\Sigma}^{\perp}$. Thus S preserves the decomposition $V = \Sigma \oplus \Sigma^{\perp}$ into a positive-definite and a negative-definite subspace. A self-adjoint operator on a definite space is diagonalizable with real eigenvalues, and therefore S is diagonalizable over $\mathbb{R}$. If $Su = \lambda u$ with u in either summand, then $\lambda\,(u, u) = (u, g\tilde{g}u) = (\tilde{g}u, \tilde{g}u)$ with $\tilde{g}u$ in the summand of $\tilde{V}$ of the same sign, and hence $\lambda \geq 0$.

For the kernel clause, $\ker \tilde{g} \subseteq \ker S$ holds for every flux. If $Su = 0$, split $u = u_+ + u_-$ along $\Sigma \oplus \Sigma^{\perp}$. Since S preserves both summands, $Su_+ = Su_- = 0$. Then $(\tilde{g}u_+, \tilde{g}u_+) = (u_+, Su_+) = 0$ with $\tilde{g}u_+ \in \tilde{\Sigma}$, and a definite space has no nonzero null vector, so $\tilde{g}u_+ = 0$. In the same way $\tilde{g}u_- = 0$. Hence $\ker S = \ker \tilde{g}$, and the argument with the two factors exchanged gives the statements for $\tilde{S}$ and g.

Suppose conversely that (23) holds. Eigenspaces $E_\lambda(S)$ for distinct eigenvalues of the self-adjoint operator S are mutually orthogonal. Thus $V = \bigoplus_\lambda E_\lambda(S)$ is an orthogonal decomposition with the form nondegenerate on each summand, and likewise for $\tilde{S}$ on $\tilde{V}$. For $\lambda > 0$ the map $\tilde{g}$ carries $E_\lambda(S)$ isomorphically onto $E_\lambda(\tilde{S})$ and multiplies the form by λ, because $\tilde{S}\tilde{g} = \tilde{g}S$, $g\tilde{g} = \lambda$ on $E_\lambda(S)$ and

$(\tilde{g}u, \tilde{g}u') = \lambda\,(u, u')$. In particular $E_\lambda(S)$ and $E_\lambda(\tilde{S})$ have the same signature. We choose maximal positive-definite subspaces $P_\lambda \subseteq E_\lambda(S)$ for each $\lambda > 0$, $P_0 \subseteq \ker S$ and $\tilde{P}_0 \subseteq \ker \tilde{S}$, and set

$$\Sigma \;=\; \bigoplus_{\lambda>0} P_\lambda \,\oplus\, P_0\,, \qquad \tilde{\Sigma} \;=\; \bigoplus_{\lambda>0} \tilde{g}(P_\lambda) \,\oplus\, \tilde{P}_0\,. \tag{S1.1}$$

Both are positive definite. Since the eigenspace decompositions are orthogonal, the dimensions of Σ and $\tilde{\Sigma}$ equal the positive indices of V and $\tilde{V}$, and both are therefore positive 3-planes.

It remains to check (21) for the pair (S1.1). For $\lambda > 0$ we have $g\tilde{g}(P_\lambda) = P_\lambda \subseteq \Sigma$, and g maps the orthogonal complement of $\tilde{g}(P_\lambda)$ in $E_\lambda(\tilde{S})$ into the orthogonal complement of P_λ in $E_\lambda(S)$, which lies in $\Sigma^\perp$. On $\ker \tilde{S}$ the kernel clause gives $g = 0$, since $\ker \tilde{S} = \ker g$. Hence $g(\tilde{\Sigma}) \subseteq \Sigma$ and $g(\tilde{\Sigma}^\perp) \subseteq \Sigma^\perp$, which is (21). Without the kernel clause the last step fails, because $g(\tilde{P}_0)$ may then contain a nonzero vector, which is null and cannot lie in Σ. This failure is the zero-eigenvalue exception of [1] recalled in Section 3.1.

Isolation. Suppose that $V_N \neq \emptyset$. Then the points of V_N are isolated if and only if every eigenspace of S and of $\tilde{S}$, kernels included, is definite. In that case Σ is the sum of the positive-definite eigenspaces of S, $\tilde{\Sigma}$ is the sum of the positive-definite eigenspaces of $\tilde{S}$, and V_N consists of this single point.

To see this, let $(\Sigma, \tilde{\Sigma}) \in V_N$. By the first part of the proof above, S preserves Σ and $\Sigma^\perp$ and is diagonalizable on each. Hence $\Sigma = \bigoplus_\lambda \big(\Sigma \cap E_\lambda(S)\big)$ and $\Sigma^\perp = \bigoplus_\lambda \big(\Sigma^\perp \cap E_\lambda(S)\big)$. Each eigenspace therefore splits as $E_\lambda(S) = \big(\Sigma \cap E_\lambda(S)\big) \oplus \big(\Sigma^\perp \cap E_\lambda(S)\big)$ into a positive-definite and a negative-definite summand, and $\Sigma \cap E_\lambda(S)$ is a maximal positive-definite subspace of $E_\lambda(S)$. The same holds for $\tilde{\Sigma}$ and $\tilde{S}$. For $\lambda > 0$ we have, moreover, $\tilde{\Sigma} \cap E_\lambda(\tilde{S}) = \tilde{g}\big(\Sigma \cap E_\lambda(S)\big)$: the right side lies in the left because $\tilde{g}$ maps Σ into $\tilde{\Sigma}$ and $E_\lambda(S)$ isomorphically onto $E_\lambda(\tilde{S})$, and the two sides have the same dimension because $E_\lambda(S)$ and $E_\lambda(\tilde{S})$ have the same signature. Hence every point of V_N is of the form (S1.1). If every eigenspace of S and of $\tilde{S}$ is definite, a maximal positive-definite subspace of an eigenspace is the whole eigenspace or zero according to its sign, so the subspaces P_λ, P_0 and $\tilde{P}_0$ in (S1.1) are forced. Then Σ is the sum of the positive-definite eigenspaces of S, $\tilde{\Sigma}$ that of $\tilde{S}$, and V_N is this one point, which is isolated.

Suppose instead that some eigenspace $E_\lambda(S)$ is indefinite; an indefinite eigenspace of $\tilde{S}$ is treated in the same way with the two factors exchanged. By the splitting above, $E_\lambda(S)$ contains vectors $u_+ \in \Sigma$ and $u_- \in \Sigma^\perp$ with $(u_+, u_+) = 1$ and $(u_-, u_-) = -1$, which are orthogonal because $\Sigma \perp \Sigma^\perp$. For real s we set $u(s) = u_+ \cosh s + u_- \sinh s$, a vector with $(u(s), u(s)) = 1$, and replace the line through u_+ in Σ by the line through $u(s)$,

$$\Sigma(s) \;=\; \big(\Sigma \cap u_+^\perp\big) \,\oplus\, \mathbb{R}\, u(s)\,. \tag{S1.2}$$

This is again a positive 3-plane, since $u(s)$ has positive norm and is orthogonal to the positive-definite plane $\Sigma \cap u_+^\perp$, and $\Sigma(s) \neq \Sigma$ for $s \neq 0$. The family (S1.2) is a hyperbolic rotation inside $E_\lambda(S)$ that mixes a Σ-direction with a $\Sigma^\perp$-direction and changes nothing in the other eigenspaces.

If $\lambda > 0$ we match the rotation on the second factor through $\tilde{g}$: let $\tilde{\Sigma}(s)$ agree with $\tilde{\Sigma}$ in every eigenspace of $\tilde{S}$ other than $E_\lambda(\tilde{S})$ and set $\tilde{\Sigma}(s) \cap E_\lambda(\tilde{S}) = \tilde{g}\big(\Sigma(s) \cap E_\lambda(S)\big)$. Since $\tilde{g}$ multiplies the form by λ on $E_\lambda(S)$ and $g\tilde{g} = \lambda$ there, $\tilde{\Sigma}(s)$ is a positive 3-plane with $\tilde{g}(\Sigma(s)) \subseteq \tilde{\Sigma}(s)$ and $g(\tilde{\Sigma}(s)) \subseteq \Sigma(s)$. If $\lambda = 0$, then $u_\pm \in \ker S = \ker \tilde{g}$ by the kernel clause, and we keep $\tilde{\Sigma}(s) = \tilde{\Sigma}$; the two factors then move independently. Indeed $\tilde{g}(\Sigma(s)) = \tilde{g}\big(\Sigma \cap u_+^\perp\big) \subseteq \tilde{\Sigma}$, and $g(\tilde{\Sigma}) \subseteq \Sigma \cap (\ker \tilde{g})^\perp \subseteq \Sigma(s)$ because $(g\tilde{v}, u) = (\tilde{v}, \tilde{g}u) = 0$ for every $u \in \ker \tilde{g}$. In both cases $g(\tilde{\Sigma}(s)) \subseteq \Sigma(s)$ and $\tilde{g}(\Sigma(s)) \subseteq \tilde{\Sigma}(s)$, and the second inclusion gives $g(\tilde{\Sigma}(s)^\perp) \subseteq \Sigma(s)^\perp$ by adjointness. Hence the pair $(\Sigma(s), \tilde{\Sigma}(s))$ satisfies (21) for every s. This curve in V_N passes through the given point, which is therefore not isolated. This completes the proof of the isolation criterion, which is the statement quoted below (23) and used in the proof of Theorem 3.1.

Termination of the enumeration at rank $r \geq 4$. The argument behind the termination statement of Section 3 of the main text is short. Since the eigenvalues of S_0 are positive and sum to at most 48, its characteristic polynomial ranges over a finite set and $|\det A|, |\det \tilde{A}|, |\det B| \leq 48^r$ by (27). Eigenspace-definiteness makes A definite on each eigenspace of S_0, so for a suitable integer polynomial p, with values at the eigenvalues bounded in terms of the discriminant of the minimal polynomial of S_0, the form $p(S_0)^{\mathsf{T}} A$ is positive definite with determinant at most an explicit multiple of 48^r; Hermite reduction of this auxiliary form [48] bounds the entries of A in the reduced basis, the same holds for $\tilde{A}$, and the identity $B^{\mathsf{T}} A B = \tilde{A}\,\tilde{B} B$ then bounds the entries of B in that pair of bases. Every FS-capable tuple of rank r is thus equivalent under (28) to one whose entries are bounded by an explicit function of r, and the enumeration is a finite search. For coherent tuples without eigenspace-definiteness no such bound exists: the class set is infinite already at rank two. The trace bound alone excludes no stratum, since we find coherent tuples of trace at most 34 on each of the 79 strata of nonempty support in the 4×20 grid of Figure 1 of the main text.

S1.2 Root-free thresholds and the open definite stratum

Both the root test of Lemma 3.3 and the floors of Section S1.4 use the smallest determinant $d^*(c)$ of a root-free even positive-definite lattice of rank c. A root-free even lattice has minimum at least 4. Hermite's inequality (12) therefore forbids determinants below $(4/\gamma_c)^c$. The Hermite constant γ_c is known exactly through rank 8, by Gauss, Korkine–Zolotareff and Blichfeldt [18, 12], and beyond rank 8 in ranks 9 and 24 [19, 20]. For $c \leq 8$ the bound $(4/\gamma_c)^c$ is attained, and the threshold has the closed form

$$d^*(c) \;=\; \frac{4^c}{\gamma_c^c} \;=\; 32,\ 64,\ 128,\ 192,\ 256 \qquad (c = 3, \ldots, 7)\,. \tag{S1.3}$$

Each value in Eq. (S1.3) is attained by the rescaled root lattice $\sqrt{2}\,X_c$ with $X_c = A_3, D_4, D_5, E_6, E_7$, of minimum exactly 4 and kissing number $12, 24, 40, 72, 126$. The rank-2 value $d^*(2) = 12$, attained by $\sqrt{2}A_2$, enters the floors of Section S1.4.

The supports on the definite stratum and the complements $K = A^{\perp}$ of rank-3 supports both have rank 19. At this rank γ_{19} is not known exactly, and $d^*(19)$, the minimal determinant of an even positive-definite rank-19 lattice with minimum at least 4, has to be bounded from each side separately. From above, $d^*(19) \leq 128$ with no hypothesis: the laminated lattice Λ_{19} of Conway and Sloane [49], [12, Ch. 6] is an even lattice with $\det = 128$, minimum 4, kissing number 10668 and $|\mathrm{Aut}| = 23592960$ [50] (Section S5.1 gives a Gram matrix for it in an LLL-reduced basis). It is the orthogonal complement of $\sqrt{2}D_5$ in the Leech lattice. Conway and Sloane prove that every 19-dimensional section of the Leech lattice has determinant at least 128 [49, 51]. In the next paragraph we extend this inequality, modulo one genus, to every even positive-definite rank-19 lattice of minimum at least 4, whether or not it is a section of the Leech lattice. We have not decided whether Λ_{19} is the only root-free class at determinant 128, and nothing in this section uses uniqueness. The lattice Λ_{19} is not admissible in the sense of Section 3.2 of the main text: its discriminant group $L^*/L \cong \mathbb{Z}_8 \times \mathbb{Z}_2^4$ needs five generators ($l = 5$ in the notation of (32)). Its discriminant form is that of $\sqrt{2}D_5$ with the sign reversed, $q_{\Lambda_{19}} \cong -q_{\sqrt{2}D_5}$, since a primitive sublattice of a unimodular lattice and its orthogonal complement have discriminant forms that differ by a sign. Thus Λ_{19} bounds $d^*(19)$ from above, but in Section 3 it can be neither the complement K of a rank-3 support nor a rank-19 support. Either one is glued to a lattice of rank at most 3 inside the unimodular lattice $\Gamma \cong H^2(\mathrm{K3}, \mathbb{Z})$ and therefore has a discriminant group on at most three generators.

From below, we proved that in every genus but one of even positive-definite rank-19 lattices with determinant at most 126, every class contains a root. The exception is one genus of determinant 64, whose discriminant group $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ needs five generators. Since this genus has no rank-5 Niemeier complement, the Niemeier-complement argument of Section 2.2 does not apply to it; we have not completed its direct enumeration either, so $d^*(19) \geq 128$ holds modulo exactly that genus. The equality $d^*(19) = 128$ carries exactly that one condition, because a root-free class in the outstanding genus would have determinant 64 and lower the threshold.

Lemma 3.3 and Theorem 3.4 do not depend on the outstanding genus, because its discriminant group needs five generators, whereas an admissible lattice is glued to a support of rank at most 3 and its discriminant group needs at most three. Moreover, for each in-budget rank-3 entry the rank-19 complement of the support of smaller determinant has determinant at most 64 (Eqs. (29) and (30) of Section 3.2), below the *admissible* root-free bracket $[128, 160]$ of Section 3.3. Theorem 3.4 rests on this bound together with the requirement that the smoothness clause of (FS) hold on both factors. Besides W_{512}, W'_{512} and W_{1024}, Section 3.3 of the main text lists seven further root-free admissible lattices, at determinants 160, 180, 196, 208, 216, 220 and 224. Those at determinants 160, 180 and 196 are the coinvariant lattices $\Lambda_{M_{20}}$, Λ_{A_6}, $\Lambda_{L_2(7)}$ of symplectic automorphism groups of K3 surfaces [29, 30, 26, 31, 27], orthogonal complements of the invariant sublattices and hence admissible; those at 208, 216 and 220 are sections listed in Martinet's online tables (208 also in [28]) and are not coinvariant, and the one at 224 we did not find in the literature. The coinvariant lattices $\Lambda_{T_{192}}$ and Λ_{M_9}, not among the seven, also have determinants in this range.

The rest of the definite stratum remains open, and the reduction of Section 3.3 constrains it only as follows. For a pair of root-free admissible supports $T, \tilde{T}$ with glue determinant c (Section 3.1), the determinant chain gives $\det S_0 = c^2 d_T d_{\tilde{T}}$, and an in-budget pair is confined to the *corridor* $c^2\, d_T\, d_{\tilde{T}} \leq 42{,}304{,}460$, the set of triples $(c, d_T, d_{\tilde{T}})$ of glue determinant and support determinants that satisfy this inequality. If one cell is counted for each such triple (the two determinants taken as an unordered pair), 92,440,702 cells remain unexamined. We could exclude the stratum by enumerating the admissible root-free lattices through determinant 510 and proving either a corridor-wide strict-excess condition of the required strength or a lower bound $c \geq 13$ on the glue determinant. Here a strict-excess condition asserts that equality in the arithmetic–geometric mean bound on the charge q of Eq. (35) fails on every corridor cell by at least a prescribed margin. We have proven such a theorem with a constructible margin. However, that margin is some six hundred orders of magnitude smaller than required, so the theorem establishes the inequality's form but not the size the condition needs.

S1.3 Vacua and enhanced-symmetry loci across ten flux sectors

On the rank-≤ 3 stratum of K3×K3, all candidate vacua within the tadpole bound lie on enhanced-symmetry loci (Section 3 of the main text). The fraction of flux vacua that lie on enhanced-symmetry loci below a cap on the flux charge has been tabulated for toroidal orientifolds in [52, 53], and the number of vacua on such loci was counted, in closed form and asymptotically in the charge, in [7, 54]. To test whether flux vacua in other geometries also lie on such loci, we enumerated the fluxes in ten sectors. In each of the ten sectors we proved that the finite list of fluxes is complete, and we fixed in advance a catalog of enhanced-symmetry loci and a distance threshold. We then classified every flux as on-locus (ON) or off-locus (OFF) by proving that its distance from the catalog lies below or above the threshold. We call a sector enumerated and classified in this way a census. Within a sector we group the fluxes by the value B of their flux charge, which in this section we also call the *budget* of the flux, and a *bin* is the set of fluxes at one value of B. OFF means off every locus in the catalog, a claim relative to the registered catalog and nothing larger. No verdict rests on sampling or on a model of an unexplored tail.

The ten sectors span seven geometry families. Two are K3×K3-type lattice sectors on orbifolds of the Kanno–Watari class [39], labeled G2-a and G2-b. A third, G2-c, is enumerated on both of its $c_2/2$ quantization branches, integral (INT) and coset (COSET), which we keep separate because the physical branch is not known. Three small sectors use A_{17}, $A_7{+}A_7$, and $D_4{+}D_4$ frames (G3-1, G3-2, G3-3). The last three sectors require period computations. Sector G4 consists of the budgets $B = 1, \ldots, 16$ of the one-parameter Calabi–Yau threefold family AESZ34, named after its entry in the tables of Almkvist, van Enckevort, van Straten and Zudilin [42]; its Picard–Fuchs operator is Eq. (S1.10) below. Sector G5 is the $B = 1$ bin of a second one-parameter threefold family, and G6 consists of the budgets $B = 1, \ldots, 30$ on the S_6-invariant diagonal slice of the Hulek–Verrill fourfold of Section 4. For G4 the catalog of loci consists of the three conifold points $\phi = 1/25$, $1/9$ and 1 of the family [47], which are the zeros of the leading coefficient S_4 of the operator (S1.10); distances are measured in the parameter ϕ. For G6 the catalog is the set of eleven special points of Section 4.3, with distances measured in φ.

In all we classified 1,195,197 flux rows across ten sectors of seven geometry families. This total refers to ordered rows and includes only the eight sectors enumerated row by row,

$$48{,}476 \;+\; 3{,}541 \;+\; 64 \;+\; 6{,}353 \;+\; 1{,}078{,}375 \;+\; 58{,}388 \;=\; 1{,}195{,}197 \tag{S1.4}$$

in the order G2-a, G2-b, the three G3 frames together, G2-c INT, G2-c COSET, and G6. In the two threefold sectors, G4 and G5, a flux is a pair of three-form flux vectors on which $SL(2,\mathbb{Z})$ acts, and we count $SL(2,\mathbb{Z})$-orbit classes instead of rows and do not pool them into Eq. (S1.4).

G4 has 49,544 candidate classes, distinguished by a complete $SL(2,\mathbb{Z})$ invariant (the column Hermite normal form of the flux pair together with a determinant sign). Of these, 150 are certified vacua over budgets 1–16, all with $W \neq 0$, and there is no $W = 0$ vacuum located under the census net (not a certified absence). G5 has 3,800,356 $SL(2,\mathbb{Z})$ classes at $B = 1$ with 8,037,776 members, among which we located 76 candidate vacuum loci, 38 of them vacua with $W \neq 0$ and 38 proven to have no vacuum. In this sector verdicts exist only at $B = 1$, and bins 2–16 carry no verdicts of any kind. Every row of the G6 slice is classified, and the slice has 28 vacua, all with $W \neq 0$, on the flux cut and box R of Eq. (47) only. The distance bounds in the two threefold sectors carry one pending certification layer (supremum envelopes), and the G4 and G5 entries are conditional on it.

We proved each flux set complete, the lattice sets by a determinant-chain bound followed by Fincke–Pohst enumeration and the fourfold slice by exhaustive enumeration of its box. For the two threefold sectors completeness refers to the $SL(2,\mathbb{Z})$-orbit classes of fluxes, each represented once by its Hermite normal form. To test whether off-locus fluxes are detected, we drew fluxes at random from the candidate lists among those marked off-locus by a floating-point computation and classified them by the exact method, which gave OFF for every draw (14 of 14 cases). The two G6 draws are excluded rows with no vacuum, so the informative count is 12 of 12, and the test is one-sided: ON-side blindness remains untested.

Across all ten sectors of Table S1, $F < 1$ in every tested bin, that is, no bin has all its vacua on-locus. For the low-count bins the table's last column gives the corresponding upper bounds on the on-locus fraction. The off-locus lists hold 12,376 rows in G2-a and 1,614 in G2-b, and the COSET branch has 523,014 off-locus orbit classes. G4, G5, and the G6 slice have off-locus vacua from their first populated budgets ($B = 1$, 1, and 4). The on-locus fraction exceeds one half in the top window of G2-a (0.5958) and is about two fifths in that of G2-b (0.4033). Thus only in these two sectors does an appreciable fraction of the vacua lie on the loci, whereas the other eight sectors have $F = 0$ or F close to zero.

We designed the enumeration of the ten sectors to test a conjectured forcing law $B^*(n) = \alpha n$, with n the number of complex-structure moduli and $\alpha \in [0.409, 0.439]$, where B^* is the threshold budget of a sector, below which every nonempty bin is fully on-locus. The test was to be a growth-law

sector	on-locus vacua / vacua in cumulative windows (ascending B)	top-window F	$p_U(95\%)$
G2-a (member class)	107/122, 1499/2043, 18243/30619	0.5958	0.602806 [h]
G2-b (member class)	7/10, 94/175, 1091/2705	0.4033	0.419099
G2-c INT	7/23, 64/436, 531/6353	0.0836	0.089515
G2-c COSET [c]	524/6330, 4246/85456, 32347/1078375	0.0300	0.031175 [h]
G3-1	0/6, 0/12, 0/42	0	0.068843
G3-2	0/4, 0/12, 0/16	0	0.170750
G3-3 [n]	0/6, 0/6	0	0.393038
G4 ($B = 1..16$)	$0/N_{\text{vac}}$ every window (top 0/150)	0	0.019773
G5 ($B = 1$ only)	0/38	0	0.075808
G6 diagonal slice ($B = 1..30$)	$0/N_{\text{vac}}$ every bin (top 0/28)	0	0.101466

Table S1: No tested bin in any of the ten sectors has all of its vacua on an enhanced-symmetry locus. For each sector the second column lists, in cumulative windows in ascending flux charge B, the on-locus fraction F as the number of on-locus vacua over the number of vacua (flux rows, for the lattice sectors) at the distance threshold fixed for that sector. Each window runs from the lowest budget of the sector through a higher one and the last window is the whole enumerated range, whose denominator, for a lattice sector, is the sector's row count entering Eq. (S1.4) or, for G2-a and G2-b, its member-class count; single-bin counts are the successive differences. The third column is F in the top window. The last column, $p_U(95\%)$, is a one-sided 95% upper bound on the top-window on-locus fraction, Clopper–Pearson exact except for the entries marked [h], which are Hoeffding bounds used only for the $F < 1$ claim; the upper bound lies far above F for the low-count windows (0.39 on the six-candidate G3-3 bins) and close to it for the populous ones (0.60 at the top window of G2-a). A budget B is one value of the flux charge, and ON and OFF are relative to the catalog of loci and the threshold fixed for the sector, as defined in the text. [c]Conditional on the $c_2/2$ quantization branch; both branches show the same result. [n]The two G3-3 bins fall below the minimum population required for a bin to count (six candidates against a minimum of eight): $F = 0$ measured, no verdict composed, and they enter no tally. G2-a and G2-b fractions are over the member classes of their flux sets (30,619 of 48,476 and 2,705 of 3,541 rows), that is, over the rows whose supports satisfy the embedding condition; the remaining rows were classified as well but do not enter F. In the G4 and G6 rows N_{vac} is the number of vacua in each window or bin. The G4 and G5 rows are subject to the condition on their distance bounds stated in the text.

fit over the sectors with a sufficiently populated threshold budget. Since no sector has such a threshold, the results of the enumeration are instead the $F < 1$ statement above and the identity of Section S1.4, and the registered fit produced nothing.

S1.4 First off-locus budgets and lattice floors

For each sector X of Table S1 we define its *wall*, or *first-off budget*, as the smallest budget at which an off-locus vacuum appears. The *absence wall* $B_{\text{abs}}(X)$, one unit lower, is the largest budget up to which no off-locus vacuum occurs. For the four lattice-type sectors G2-a, G2-b and the two branches of G2-c, Theorem S1.1 below gives the wall by integer arithmetic, before any vacuum equation is solved.

A *row* of a lattice-type sector is a support pair with even Gram matrices A_1, A_2, determinants d_1, d_2 and frame scales s_1, s_2 (positive numbers fixed by the sector, one for each frame), together with an integer flux matrix $M \in \mathbb{Z}^{2\times 2}$, $\det M \neq 0$. Its budget is

$$N \;=\; \frac{s_1 s_2}{2}\,\mathrm{tr}\big(A_1 M A_2 M^{\mathsf{T}}\big)\,. \tag{S1.5}$$

The arithmetic–geometric mean inequality bounds the trace from below for every row,

$$\operatorname{tr}\bigl(A_1 M A_2 M^{\mathsf T}\bigr) \;\geq\; 2\sqrt{d_1 d_2}\,\bigl|\det M\bigr| \;\geq\; 2\sqrt{d_1 d_2}\,, \tag{S1.6}$$

because $A_1 M A_2 M^{\mathsf T}$ is similar to the positive-definite matrix $A_1^{1/2} M A_2 M^{\mathsf T} A_1^{1/2}$, whose two eigenvalues have product $d_1 d_2 (\det M)^2$. The form (S1.5), together with the parities of A_1, A_2 and M that the sector imposes, confines N to an arithmetic progression $\Pi_X \subset \mathbb{Z}$.

A support pair is *eligible* if the sector's exclusions (root-freedom where it applies) do not bar it from having an off-locus vacuum. The *eligibility floor* $F(X)$ (distinct from the fraction F above) is the minimum of the exact budget N over eligible pairs (A_1, A_2) and integer matrices M with $\det M \neq 0$. When an eligible pair exists the minimum is attained, because N takes positive integer values. By (S1.6), $F(X)$ is at least $s_1 s_2 \sqrt{d_1 d_2}$ for the eligible pair of smallest $d_1 d_2$, rounded up to the first value in Π_X; this rounded bound need not itself be a value of N. On the integral branch of G2-c, $\Pi_X = 4\mathbb{Z}$ and the smallest eligible pair has determinants $(3, 4)$. Hence $N \geq 4\sqrt{12} = 13.85\ldots$, and $N \geq 16$ on every eligible pair; the pair $(3, 4)$ attains 16 at $|\det M| = 1$ by an enumerated row, so $F = 16$ there.

Theorem S1.1 (wall equals floor on the lattice class)**.** *Let X be a lattice-type sector of the census satisfying: (W1) the vacua enumerated are exactly the integer points of the box of flux quanta; (W2) on- or off-locus status is a function of the support pair alone; (W3) the budget is the exact form* (S1.5) *and lies in Π_X at every point of the box. If X has an eligible support pair and $F(X)$ is within the enumerated budget range, then*

$$\text{first-off}(X) \;=\; F(X)\,, \qquad B_{\text{abs}}(X) \;=\; F(X) - 1\,, \tag{S1.7}$$

and every minimizing row (A_1, A_2, M) of $F(X)$ is itself an enumerated off-locus vacuum.

For the proof, note first that under (W2) a support pair is eligible precisely when its rows are off-locus. An off-locus vacuum in the box therefore lies on an eligible pair and, by (W3), has budget N at some M with $\det M \neq 0$, hence at least $F(X)$. Conversely, a minimizing row of $F(X)$ has eligible supports, $\det M \neq 0$ and budget $F(X)$ inside the enumerated range. By (W1) it is therefore an enumerated vacuum and by (W2) it is off-locus. On the coset branch of G2-c, whose box holds only the flux matrices not divisible by 2, the minimizing row still lies in the box, since $M = 2V$ gives $N(M) = 4N(V) > N(V)$ and no minimizer is divisible by 2. By (W3), $F(X) \in \Pi_X$, and every budget below $F(X)$, some of them possibly empty by parity, carries no off-locus vacuum, so that $B_{\text{abs}}(X) = F(X) - 1$. The inequality (S1.6) enters only in making the search for the minimum finite.

W1 is a hypothesis at class level: it is verified row by row on every censused sector, is not proven for all conceivable lattice censuses, and fails wherever vacuum positions depend on transcendental periods. On the four lattice-type sectors the identity (S1.7) holds,

$$\text{first-off} \;=\; F \;=\; (7,\,10,\,16,\,4)\,, \qquad B_{\text{abs}} \;=\; (6,\,9,\,15,\,3) \;=\; F - 1\,, \tag{S1.8}$$

in the order (G2-a, G2-b, G2-c integral, G2-c coset). The G2-a and G2-b walls refer to the member classes of Table S1. The coset entry is conditional on the open $c_2/2$ parity, but each branch separately has wall equal to floor (16 and 4), and the identity thus holds on either branch. Since the theorem forces first-off $= F$ throughout the class, the four equalities in Eq. (S1.8) are a consistency test of the theorem at full census strength rather than findings.

The four values of F are exact minima, each attained at $|\det M| = 1$ on the eligible pair of smallest $d_1 d_2$. We write (a, b, c) for the Gram matrix $\left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$. The attaining rows are $(1, 1, 2), (1, 1, 2)$

with determinants $(d_1, d_2) = (7, 7)$, frame scales $(s_1, s_2) = (1, 1)$ and $M = \left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, giving $N = 7$; $(1, 1, 2), (1, 1, 1)$ with $(7, 3)$, $(1, 2)$ and $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, giving $N = 10$; and $(1, 1, 1), (1, 0, 1)$ with $(3, 4)$, $(2, 2)$ and $M = \left(\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, giving $N = 16$ on the integral branch of G2-c. On the coset branch the flux is $M/2$ for the same M, which is not divisible by 2, and its budget is one quarter of the integral value, $N = 4$. In each sector the minimum coincides with the bound (S1.6) rounded up into Π_X; that coincidence is a property of these four sectors, not a consequence of (W1)–(W3).

Outside the class the floor is a lower bound only, and the wall is a measurement. Here the floor of a sector is still the smallest budget of a nonzero flux whose support is eligible, as inside the class; the general form of this definition is Eq. (S2.1) of Section S2. G4 has floor 1, with its first vacuum and first off-locus vacuum both at budget 1, attained by the flux with $SL(2, \mathbb{Z})$-orbit label $((1, 0, -6, 3), (0, 1, 4, -1), 0, 1; -1)$, the complete invariant described in Section S1.3 above (column Hermite normal form with determinant sign). This flux has a $W \neq 0$ vacuum at $z = (0.0260250\ldots, 0.0162211\ldots)$, the real and imaginary parts of ϕ, whose distance from the catalog, attained at the conifold point $\phi = 1/25$, is proven to lie in $[0.0214095, 0.0214122]$. In G5 only the bin $B = 1$ is classified, and the floor, the budget of the first vacuum and the first off-locus budget are likewise all equal to 1. In both sectors the surplus of Section 4.3 (the budget of the first vacuum minus the floor) is therefore zero, as is the first off-locus budget minus the floor, subject to the condition on the G4 and G5 distance bounds stated above (Section S1.3).

A surplus measured outside the lattice class need not be an effect of the loci, since nothing yet distinguishes dynamics delaying escape from the loci from dynamics delaying vacuum existence entirely. We therefore give floor, first vacuum and first off-locus vacuum separately. The one positive surplus occurs in the S_6-invariant Hulek–Verrill slice, where the floor is 1 and the first vacuum on the flux cut and box R has budget 4 (Proposition 4.1); outside R the sector has vacua at budgets 3 and 2. For K3×K3 itself we classified no rows, and it is not an instance of Theorem S1.1. In Section S2 we compare its proven charge bounds and known fully stabilizing fluxes (Section 3) with the law conjectured there.

S1.5 Algebraic cycles at the AESZ34 attractor point

For the threefold family AESZ34 [42] (sector G4) we also prove that none of the algebraic cycles known on the family realizes the rational splitting of the middle cohomology at its rank-two attractor point (Theorem S1.2). The fibers of this one-parameter family are free quotients of the Hulek–Verrill threefolds $\bar{X}_a$, the smooth projective models constructed in [40] of the hypersurfaces in $\mathbb{P}^4$ obtained by clearing denominators in

$$(X_1+X_2+X_3+X_4+X_5)\Big(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}+\frac{1}{X_5}\Big) \;=\; t\,, \qquad a = (1{:}1{:}1{:}1{:}1{:}t)\,, \quad t = \frac{1}{\phi}\,, \tag{S1.9}$$

with ϕ the parameter of [47]; the pencil is the A_4 case of the root-lattice pencils of [55]. The group $G = \langle \sigma, \iota \rangle \cong \mathbb{Z}/10$ is generated by the cyclic shift $\sigma : X_n \mapsto X_{n+1}$ of order 5 and the inversion $\iota : X_n \mapsto 1/X_n$ of order 2. It acts on $\bar{X}_a$, freely when $t \notin \{0, 1, 9, 25\}$; at each excluded value the member of the pencil is singular or the G-action on it is not free, and the point $\phi = 0$ of maximal unipotent monodromy is $t = \infty$. There are two free quotients, $\bar{X}_a/G$ with Hodge numbers $(h^{1,1}, h^{2,1}) = (5, 1)$ and $\bar{X}_a/\langle\sigma\rangle$ with $(h^{1,1}, h^{2,1}) = (9, 1)$. Both have $b_3 = 4$, while $\bar{X}_a$ itself has $h^{1,1} = 45$, $h^{2,1} = 5$ and $b_3 = 12$. We write X_ϕ for either quotient; the statements below hold for both.

The family has a rank-two attractor point at $\phi = -1/7$ [47], that is, at $t = -7$, where the middle cohomology $H^3(X_\phi, \mathbb{Q})$ is reported to split rationally into two planes, one of which, denoted W here and H_{flux} in [56], satisfies $W \otimes \mathbb{C} = H^{2,1} \oplus H^{1,2}$. In this subsection the letter W denotes this plane, and the superpotential of the main text enters only in the phrase "$W = 0$ vacuum". The relation between rank-two attractor points and flux vacua with vanishing superpotential is discussed in [57, Sec. 3.8] and [47]. By the criterion of [56], the rational splitting at a rank-two attractor point (with $h^{2,1} = 1$) is equivalent to a supersymmetric flux vacuum with vanishing superpotential: a rank-two flux lattice inside W gives the vacuum, with axio-dilaton fixed to a ratio of two periods of W. The splitting is verified to a thousand digits in [47] but not proven, and an exact vacuum at the attractor point requires it to hold over $\mathbb{Q}$. The integral fluxes are then read off from $W \cap H^3(X_\phi, \mathbb{Z})$, and an algebraic cycle realizing W would prove the splitting.

The known candidates for an algebraic cycle realizing W are the twenty ruled-surface cycles $\{\alpha^{ij}, \beta^{ij}\}$ of Hulek and Verrill [40] on the threefold $\bar{X}_a$. For each pair $i < j \leq 5$ the hyperplane section $X_i + X_j = 0$ of $\bar{X}_a$ is a surface birational to $E \times \mathbb{P}^1$. Here E is the plane cubic $(X_k + X_l + X_m)(X_k^{-1} + X_l^{-1} + X_m^{-1}) = t$ in the three remaining coordinates, which varies over the family and at $\phi = -1/7$ is the conductor-14 elliptic curve with label `14.a4` in the L-functions and Modular Forms Database (LMFDB). The cycles $\alpha^{ij}, \beta^{ij} \in H_3(\bar{X}_a, \mathbb{Z})$ are the pushforwards of the two generators of $H_1(E, \mathbb{Z})$, times the ruling, through the ten surfaces $X_i + X_j = 0$. Their Poincaré dual classes are the Gysin images of $H^1(E)(-1)$, of pure Hodge type $(2,1) + (1,2)$, and on one surface $\alpha^{ij} \cdot \beta^{ij} = -2$ while $\alpha^{ij} \cdot \alpha^{ij} = \beta^{ij} \cdot \beta^{ij} = 0$. We call $H^{2,1}(\bar{X}_a) \oplus H^{1,2}(\bar{X}_a)$ the Hodge stratum of these classes.

Theorem S1.2 (attractor no-go)**.** *Let $a = (1{:}1{:}1{:}1{:}1{:}t)$ with $t \notin \{0, 1, 9, 25\}$, any smooth fiber of the pencil ($t = 1/\phi$), the attractor fiber $t = -7$ included; let $\bar{X}_a$ be the smooth Hulek–Verrill model, $G = \langle \sigma, \iota \rangle \cong \mathbb{Z}/10$ its free symmetry group, and $S_t \subset H_3(\bar{X}_a, \mathbb{Q})$ the span of the twenty ruled-surface cycles. Then S_t is G-stable and $S_t^G = 0$, already $S_t^{\mathbb{Z}/5} = 0$; every class in S_t pushes forward to zero in $H_3(X_\phi, \mathbb{Q})$, where X_ϕ is either free quotient, $\bar{X}_a/G$ or $\bar{X}_a/\langle\sigma\rangle$ with $\langle\sigma\rangle \cong \mathbb{Z}/5$, each with $h^{2,1} = 1$; and S_t, of dimension 8 with nondegenerate intersection form, is exactly the non-invariant part of its Hodge stratum, so the flux space W receives no cycle class from any Hulek–Verrill ruled surface.*

We give two proofs. The first, which extends the orbit-sum computation of [58, Prop. 4.6] on the span of the twenty Hulek–Verrill cycles [40, Cor. 5.11], is a direct computation followed by a Hodge-theoretic rigidity step. The 20×20 intersection matrix of the classes α^{ij}, β^{ij} is computed in Corollary 5.11 of [40]. It is skew, of rank 8, and independent of t. The involution ι fixes all twenty classes, and σ permutes them without signs along the two orbits $\{12, 23, 34, 45, 15\}$ and $\{13, 24, 35, 14, 25\}$ of index pairs. Indeed, σ carries the surface $X_i + X_j = 0$ to $X_{i+1} + X_{j+1} = 0$, and the automorphisms of E that σ and ι induce (a cyclic permutation of the three coordinates of E, and the Cremona involution $X_k \mapsto 1/X_k$) have no fixed points on the smooth fibers. They are therefore translations of the genus-one curve E and act trivially on $H_1(E, \mathbb{Z})$. Hence S_t is G-stable, G acts on it through $\langle\sigma\rangle \cong \mathbb{Z}/5$, and the $\langle\sigma\rangle$-invariants of S_t are spanned by the four orbit sums of the α^{ij} and of the β^{ij} over the two orbits.

Each of the four orbit sums pairs to zero with all twenty generators, and their 4×4 Gram matrix vanishes identically. For example, the sum of the α^{ij} over the first orbit pairs with β^{12} to $(-2) + 1 + 0 + 0 + 1 = 0$, the terms in the order of the orbit. This orthogonality is the statement of [58, Prop. 4.6]. With any other choice of signs in the action of σ compatible with $\sigma^5 = 1$ the orbit sums would not lie in the radical; the first proof therefore depends on this sign determination, which the second proof avoids.

We now use Hodge theory to show that the four orbit sums, which lie in the radical of the intersection form on S_t, vanish in $H_3(\bar{X}_a, \mathbb{Q})$. The span of the two orbit sums over one orbit is the image of the σ-averaged Gysin map $H^1(E, \mathbb{Q})(-1) \to H^3(\bar{X}_a, \mathbb{Q})$, a morphism of rational Hodge structures. The rational Hodge structure $H^1(E, \mathbb{Q})$ is irreducible (a rational sub-structure is stable under complex conjugation and has even rank). The image is therefore either zero or of rank two and type $(2,1)+(1,2)$. A rational Hodge substructure of type $(2,1)+(1,2)$ is nondegenerately polarized, whereas the image pairs to zero with S_t and hence with itself. The four orbit sums therefore vanish in $H_3(\bar{X}_a,\mathbb{Q})$, which gives $S_t^{\mathbb{Z}/5}=0$ and a fortiori $S_t^G=0$.

For the pushforward we write $q:\bar{X}_a\to X_\phi$ for the quotient by $H=G$ or $H=\langle\sigma\rangle$. Since H acts freely, $q^*q_*=\sum_{h\in H}h$ on $H_3(\bar{X}_a,\mathbb{Q})$ and q^* is injective. Thus $q_*c=0$ exactly when $\sum_h h\,c=0$, and for $c\in S_t$ this sum is an H-invariant element of S_t and vanishes. For the dimension we first note that the rank of the intersection form gives $\dim S_t\ge 8$. On the other hand $S_t\otimes\mathbb{C}$ is a G-stable subspace of the Hodge stratum $H^{2,1}(\bar{X}_a)\oplus H^{1,2}(\bar{X}_a)$, whose G-invariant part is the corresponding stratum of the quotient, and $S_t^G=0$. Hence $\dim S_t\le 2\cdot 5-2\cdot 1=8$. We conclude that $\dim S_t=8$, that the form on S_t is nondegenerate, and that $H^3(\bar{X}_a,\mathbb{Q})=H^3(\bar{X}_a,\mathbb{Q})^G\oplus S_t$ orthogonally, with ranks $4+8=12$.

The second proof is variational and does not use the signs in the action of σ. Its input is the Picard–Fuchs operator of the family [47],

$$\begin{aligned}
&\mathcal{L} \;=\; S_4\,\theta^4+S_3\,\theta^3+S_2\,\theta^2+S_1\,\theta+S_0\,, \qquad \theta=\phi\,\frac{d}{d\phi}\,,\\
&S_4=(\phi-1)(9\phi-1)(25\phi-1)\,, \qquad S_3=2\phi\,(675\phi^2-518\phi+35)\,,\\
&S_2=\phi\,(2925\phi^2-1580\phi+63)\,, \qquad S_1=4\phi\,(675\phi^2-272\phi+7)\,, \qquad S_0=5\phi\,(180\phi^2-57\phi+1)\,,
\end{aligned} \tag{S1.10}$$

whose solutions are the periods of the holomorphic three-form. The local system of solutions is thus dual to $H^3(X_\phi)$ [47]. The singular points of $\mathcal{L}$ are $\phi=0,1/25,1/9,1,\infty$, and at $\phi=\infty$ the indicial polynomial is $(s-1)^2(s-2)^2$. At $\phi=0$ the indicial polynomial is $-s^4$ and all four local exponents vanish. The power-series solution there is unique up to scale, with coefficients $a_n=\sum_{i+j+k+l+m=n}\big(n!/(i!\,j!\,k!\,l!\,m!)\big)^2=1,5,45,545,7885,127905,\ldots$. A unipotent local monodromy has one Jordan block for each independent single-valued solution. The local monodromy T at $\phi=0$ is therefore a single Jordan block of size four, and $T-1$ has rank 3; this is the point of maximally unipotent monodromy.

We now let $V=R^3\pi_*\mathbb{Q}$ be the rational variation of Hodge structure of the family over $\mathbb{P}^1\setminus\{0,1/25,1/9,1,\infty\}$, pulled back to any finite cover. It has rank $b_3=4$, weight 3 and Hodge numbers $(1,1,1,1)$, and it is polarized by the intersection form. The ten surfaces, the curves E on them and the G-action are defined algebraically and uniformly over the pencil (S1.9). The H-averaged Gysin images of $H^1(E)(-1)$, which lie in $H^3(\bar{X}_a,\mathbb{Q})^H=H^3(X_\phi,\mathbb{Q})$, therefore span a sub-variation U of V with Hodge types in $\{(2,1),(1,2)\}$. We claim that no nonzero such sub-variation exists.

Suppose $U\neq 0$. It has rank two, since $h^{2,1}(X_\phi)=1$ and a rational piece is stable under complex conjugation. By the Hodge–Riemann relations the polarization is nondegenerate on U, and $V=U\oplus U^\perp$ as variations. The quotient $V/U\cong U^\perp$ is then a polarized variation of rank two and type $(3,0)+(0,3)$. On it $F^2=F^3$, and Griffiths transversality, $\nabla F^3\subset F^2\otimes\Omega^1=F^3\otimes\Omega^1$, implies that the Hodge line F^3 is preserved by the flat connection. It is therefore flat, as is its complex conjugate. Every monodromy transformation of the quotient preserves these two lines and the Hodge form, which is definite on each, and acts on them by scalars of modulus one. The monodromy of the quotient thus lies in a compact group and is diagonalizable, and its only unipotent element is the identity. However, the maximally unipotent monodromy at $\phi=0$ is a single Jordan block of size four, which induces on every rank-two quotient by a stable plane a unipotent transformation other than the identity. Explicitly, U is stable under T, and since $(T-1)V$ has dimension $3>2=\dim U$ there is a vector $v\in V$ with $(T-1)v\notin U$; the transformation induced on V/U is then unipotent and different from the identity. This contradiction proves the claim, and with it $S_t^H=0$ for both H and every smooth t.

No such sub-variation exists over any finite cover of the pencil either, since there the local monodromy at $\phi = 0$ is a power of the Jordan block above and is again a single unipotent block of size four. Hence a cycle realizing W, if one exists, is rigid at $\phi = -1/7$: no correspondence from a family of curves C, flat over a finite cover of the pencil, induces a nonzero map $H^1(C)(-1) \to H^3(X_\phi)$ at any fiber. Such a correspondence would give a morphism of variations whose image is a sub-variation with Hodge types in $\{(2,1),(1,2)\}$, and we have shown that there is none.

The theorem excludes only the known cycles: some other G-invariant algebraic cycle could still realize the splitting (the Hodge conjecture predicts one), but we know of no candidate in the literature. Candelas, de la Ossa, Elmi and van Straten write that "it is not clear to us what exactly we should be looking for" [47]. Dummigan [58] attacks modularity of the fiber through mod-5 monodromy and Iwasawa theory rather than a cycle. The argument at present rests on an unproved reducibility of the 5-adic Galois representation, and even the Galois statement would not by itself give the decomposition in singular cohomology. Without a proof of the splitting, the exact $W = 0$ vacuum at the attractor point, and the transcendence statement for its axio-dilaton that would follow, remain unproven.

Rank-two attractor points of one-parameter families have been searched for through the factorization of Frobenius polynomials in [47, App. B] and [59, Sec. 1.2]. To look for further attractor points we combined such finite-field fingerprints, that is, the requirement that the Frobenius polynomial factorize modulo many primes as it does at a rank-two attractor point [47], with an exact S-integrality test in refined-conductor form. Each of the eight known attractor points passes the integrality test (8 of 8), and the four attractor points recovered on the candidate lists described next are among these eight. We applied both to 107,581 AESZ34 candidates (107,270 from an S-unit list and 311 from a refined-conductor list) and to the fingerprint lists of the hypergeometric families AESZ4 and AESZ11. On the AESZ34 lists only two candidates pass both tests, the known attractor points $\phi = -1/7$ and $\phi = 33 \pm 8\sqrt{17}$, a rediscovery rate of 1.9×10^{-5} against an allowed false-positive rate of 6×10^{-5} per candidate. On the two hypergeometric lists the only candidates that pass are again the known attractor points, $z = -1/5832$ and $z = -1/432$ respectively. We thus found exactly the four known attractor points and no new attractor point on those lists.

S2 A conjectured escape-price law

This section states a conjectured formula for the smallest flux charge at which an enumerated flux sector has a vacuum off every enhanced-symmetry locus of a fixed catalog (Conjecture S2.1 in Section S2.1) and compares that formula with the sectors enumerated in Section S1.3 and with the Hulek–Verrill sector of Section 4 of the main text (Section S2.2). Sections 4.3 and 6 of the main text point to this section, and the comparison draws on Theorem 3.4 and Proposition 4.1 there and on Theorem S1.1 of Section S1.4.

The enumerations of Section S1 and Section 4 suggest a pattern that this paper records as a conjecture, for completeness; no result in the body depends on it. Let X be a flux census (Section S1.3), an enumerated flux sector with lattice Λ_X, a budget $N_{\text{flux}}(G)$ for each flux $G \in \Lambda_X$, and a fixed catalog of enhanced-symmetry loci. Its wall, first-off(X), is the smallest budget at which an off-locus vacuum appears (Section S1.4), where off-locus means off every locus of the catalog. In the discussion section of [3] and in [60] it is posed as an open question whether a flux that stabilizes moduli away from every symmetry locus must generically have many nonvanishing quanta. For an enumerated flux sector, with the symmetry loci restricted to its fixed catalog, we conjecture below a formula for the wall just defined. We first state the law together with the admissibility hypothesis without which it would be empty (Section S2.1), and then compare the law with each enumerated sector (Section S2.2).

S2.1 The law and its admissibility hypothesis

As in Section S1.4, we call a flux support eligible unless an exclusion bars it from having an off-locus vacuum. An exclusion is either arithmetic (root-freedom, with the thresholds of Section S1.2) or, in the amended law stated below, a *domain exclusion*, that is, a proven statement about the periods that bars some supports from having a vacuum in the enumerated domain. The unamended law admits arithmetic exclusions only. We define the eligibility floor of a sublattice $\Lambda' \subseteq \Lambda_X$ as the minimum

$$F(\Lambda') \;=\; \min\bigl\{\, N_{\text{flux}}(G) \;:\; G \in \Lambda' \setminus \{0\},\ \text{the support of } G \text{ eligible} \,\bigr\}\,. \tag{S2.1}$$

On the lattice-type sectors, where the eligible supports are pairs of rank-two lattices, the fluxes entering Eq. (S2.1) are the rows (A_1, A_2, M) with $\det M \neq 0$ of Section S1.4, and $F(\Lambda_X) = F(X)$ by definition. Without the eligibility condition, $F(\Lambda_X)$ is the minimal flux charge of the sector; for a Landau–Ginzburg model this minimum was computed exactly, as a shortest-vector problem, in [9, Sec. 4].

We write $\Lambda_{\text{sel}}(X) \subseteq \Lambda_X$ for the *selected sublattice*, cut out by the integral linear conditions that the vacuum equations impose on the flux, conditions that are determined before any wall is known. On the four lattice-type sectors the equations impose no condition and $\Lambda_{\text{sel}}(X) = \Lambda_X$. On the S_6-invariant Hulek–Verrill sector the fluxes obey the cut of Eq. (47), and the modulus φ of the diagonal slice lies in the box R defined there. Every flux in the cut that has an F-flat point in R satisfies $g_2 = g_3 = 0$, Eq. (53), and Λ_{sel} is that codimension-2 cone. No other selection occurs in our enumerations, and we have no general formula for Λ_{sel}.

Conjecture S2.1 (escape price, amended form)**.** *For every flux census X, the smallest budget at which a certified off-locus vacuum appears equals the eligibility floor of the selected sublattice,*

$$\text{first-off}(X) \;=\; F\bigl(\Lambda_{\text{sel}}(X)\bigr)\,, \tag{S2.2}$$

with F evaluated over the eligible set minus the registered certified exclusions. Hypothesis (admissibility of exclusions): every exclusion entering F is registered and certificate-backed before the wall is measured; a floor adjusted by any exclusion chosen after the measurement does not instantiate the law.

Without the admissibility hypothesis the conjecture is empty, since any measured wall is the floor of some exclusion set chosen after the fact.

S2.2 The enumerated sectors against the law

We compare the law first with the Hulek–Verrill sector, which motivated the amendment. On the Hulek–Verrill cut the arithmetic floor of the full lattice is 1 (Section 4.1), and the selected cone has arithmetic floor 1, since $N_{\text{flux}} = g_1(g_1 + g_5)$ there. On the box R the first vacuum occurs at budget 4, with no solution of any kind below it. Thus 4 is a vacuum-existence threshold there, and on this sector we cannot yet distinguish a law governing the first off-locus vacuum from one governing the first vacuum of any kind. On the diagonal slice outside R the threshold is at most 2, since vacua at budgets 3 and 2 exist just beyond the upper edge of the box (Section 4.2).

Proposition 4.1, that no on-cone flux of budget at most 3 has an F-flat point in R, is a domain exclusion, and with it $F(\Lambda_{\text{sel}}) = 4$. Arithmetic exclusions alone give 1, so the unamended law fails on this box. The Hulek–Verrill slice is consistent in the amended form only, its exclusion certified after the wall was measured. We proved that exclusion to explain why the enumeration found no vacuum in R below budget 4, so this sector supplied the amendment rather than passing a test of it. Every Hulek–Verrill statement here inherits the conditions and the operator hypothesis (O) collected at the end of Section 4.

Three outcomes would falsify the law: an off-locus vacuum below the computed floor, a wall above the floor that no admissible set of exclusions accounts for, or a canonicity failure, in which two enumerations of one geometry require different selected sublattices Λ_{sel}. Each sector for which we have both a floor and a wall agrees with the law, within the budget range and under the hypotheses of its enumeration. The three G3 frames of Table S1 have no computed floor and are not compared.

For the four lattice-type sectors, G2-a, G2-b and the integral and coset branches of G2-c, Theorem S1.1 forces the wall to equal the floor under (W1)–(W3), conditionally on the $c_2/2$ parity for the coset branch. The walls are $(7, 10, 16, 4)$ in that order (Eq. (S1.8)), and these values are also the eligibility floors $F(\Lambda_X)$ of Eq. (S2.1). Hence their agreement tests that theorem and is no independent evidence for the conjecture.

The one-parameter threefold sectors G4 (all budgets $B = 1, \ldots, 16$) and G5 ($B = 1$ only) have floor 1 and first off-locus vacuum at budget 1. No theorem forces this equality, and every G5 statement is confined to $B = 1$. On K3×K3, Theorem 3.4 gives an absence wall of 24 on the rank-≤ 3 stratum under (H1)–(H4), and charge 25 is attained by a fully stabilizing rank-17 flux outside that stratum [1]. On the definite rank-19 stratum it is open whether $N_{\mathrm{flux}} = 24$ is attained (Eq. (39)), and on the unenumerated indefinite strata the fully stabilizing flux of charge 22 and rank 22 of Section 3.4 of the main text lies inside the budget. We therefore count K3×K3 as a case of floor-consistency rather than as an instance of a proof.

Four further enumerations would test the law: the budgets 2–16 of G5, the full six-parameter Hulek–Verrill flux box, a second fourfold with transcendental periods, and the two open K3×K3 cases. Any of them, with a domain exclusion fixed in advance, would also be the first test of the amendment to the law. It remains a conjecture, and none of the results in this paper proves it.

S3 Further notes on the KKLT constructions

The material below supports Section 5 of the main text: the enclosures (66) and (67) of the benchmark vacuum in Section 5.2, the conditional value (69) of Section 5.4, and the treatment of the thirty de Sitter candidates in Section 5.5. It also records two diagnostics of the benchmark computation that the main text only summarizes, the offset of the critical point from the curve on which the periods are continued and the error of a truncated instanton sum, and one discrepancy in a quoted conifold datum of the de Sitter candidates.

In this section we give conditional values of the flux superpotential for five of the de Sitter candidates of Section 5.5 (Section S3.1), list five corrections to values printed in the KKLT literature (Section S3.2), and collect the numerical data for the benchmark computation of Section 5.2 and for the value of Eq. (69) (Section S3.3). Section 5 of the main text and this section draw on thirty-six constructions from this literature (the DKMM benchmark, five AdS vacua from later constructions with the same mechanism, and thirty de Sitter candidates). For two of them we evaluate the superpotential with proven error bounds (Eqs. (66) and (69), the second at fixed inputs), and for five more we confirm the repository value of $|W_0|$ conditionally below. For the remaining twenty-nine of the thirty-six we do not evaluate the superpotential from the periods. Four are AdS vacua (among them the one with published $|W_0| = 4.07556 \times 10^{-21}$) that lack a restricted Picard–Fuchs operator of the required form. The rest are de Sitter candidates for which we evaluate only the flux invariant of Eq. (70): twenty-four with a repository basis, and the thirtieth candidate of Section 5.5, treated below.

Both the invariant $K^T N^{-1} K$ of Eq. (70) and the flux polynomial of Section S3.1 use the integer intersection numbers κ_{abc} of the Calabi–Yau threefold of each construction, which we recompute from the published polytope data. With them we reproduce the Euler number of each of the thirty-five AdS and de Sitter constructions other than the benchmark. For the one vacuum whose κ_{abc} are given in full (the second $(5, 113)$ AdS vacuum of [45]) we also reproduce all 35 values of κ_{abc} printed there. For the thirtieth de Sitter candidate, `manwe`, the only integral basis of prime toric divisors leaves 64 of 107 repository GV rows outside every sampled Mori cone, and we could not refer its repository data to a fixed effective basis (Section 5.5). We take κ_{abc} and the change to the conifold-adapted basis of [16] from two public copies of the geometry, the `manwe` test data of `pfvs` [10] and the `manwe` model file of JAXVacua [11], at the versions cited. The two copies agree entry by entry in their intersection numbers, basis change and conifold curve. In the conifold-adapted basis the second Chern class of the threefold agrees with the one printed in [16], and the fluxes of that paper reproduce exactly the flat direction $p = (0, -8, 0, -2, 4, 5, 5, 4)/40$, the conifold flux $M_{\mathrm{cf}} = 16$, $\lambda = 26/5$ (the value denoted K' there) and $-M \cdot K = 162$, and satisfy Eq. (71). The same fluxes give $\det N = 51118080$ and $K^T N^{-1} K = 143/20$ from either copy.

Section 5.5 of the main text states that we reproduce the published conifold data M_{cf}, K_{cf} and λ of Eq. (71). The one exception is the `aule` family (22 candidates), where we find $\lambda = 17/6$ against the value 17/5 (denoted K' there) quoted in [16]; the repository's own $|z_{\mathrm{cf}}|$ favors 17/6, as does the `aule` test seed of JAXVacua [11].

S3.1 Conditional superpotential values for five de Sitter candidates

For five of the thirty candidates of Section 5.5 we evaluate the flux superpotential in ball arithmetic, using the undressed convention of Section 5.4 in which $|W_0|$ denotes $|W|$ itself, without the factor $e^{(K_{\mathrm{cs}}+K_\tau)/2}$ of Eq. (59). The five are `p8-d`, `aule`, `orome`, `p8-g` and `p9-a`. The inputs are the complex-structure moduli and the axio-dilaton stored as floating-point numbers in the repository [17] of [16] (henceforth the stored moduli). Three of the five (`p8-d`, `aule`, `p8-g`) share one superpotential, evaluated at three distinct points.

In its large-complex-structure form the superpotential is the flux polynomial plus the instanton sum over Gopakumar–Vafa invariants plus the conifold term, and its absolute value at the stored moduli is

$$\begin{aligned} |W| \;=\; \sqrt{2/\pi}\, \Big|\, \tfrac{1}{2}\, N_{ab}\, U^a U^b \;-\; \tau\, K_a U^a \;-\; \frac{1}{4\pi^2} \sum_{q \neq q_{\mathrm{cf}}} n_q\, (q \cdot M)\, \mathrm{Li}_2\big(e^{2\pi i\, q \cdot U}\big) \\ -\; \frac{n_{\mathrm{cf}}\, M_{\mathrm{cf}}}{4\pi^2} \Big[\mathrm{Li}_2\big(e^{2\pi i\, z_{\mathrm{cf}}}\big) - \zeta(2)\Big] \Big| . \end{aligned} \tag{S3.1}$$

Here $U = (U^a)$ are the complex-structure moduli and τ is the axio-dilaton, both at their stored values, and $N_{ab} = \kappa_{abc} M^c$ and K_a are formed from the candidate's flux vectors M and K as in Eq. (70). The sum runs over the finite set of curve classes q retained in the truncation described below, with Gopakumar–Vafa invariants n_q, and omits the conifold class q_{cf} of Eq. (71), which contributes the last term instead. In the last term n_{cf} is the invariant of the conifold class, $M_{\mathrm{cf}} = q_{\mathrm{cf}} \cdot M$ its flux number and $z_{\mathrm{cf}} = q_{\mathrm{cf}} \cdot U$ its modulus; this term vanishes at the conifold point $z_{\mathrm{cf}} = 0$, since $\mathrm{Li}_2(1) = \zeta(2)$. We recompute the Gopakumar–Vafa invariants from toric data, through a fixed range of degrees (470 curve classes, against the 92 or 98 that the repository stores). For `orome` and `p8-g` the invariants are instead those of a related geometry, the *host geometry* of the fifth assumption below, carried over to the candidate by a unimodular change of basis. For each of the five we also reconstructed the first-order system (17) on a one-parameter subfamily, as in Section 2.3, and

candidate	$\lvert W_0\rvert$ at the stored moduli (center $\pm$ tail estimate)	GV allowance	repository value (dev.)	truncated sum
`p8-d`	$0.05344676 \pm 2.2 \times 10^{-18}$	9.0×10^{-4}	0.0534595 (-2.4×10^{-4})	$+2.1 \times 10^{-14}$
`aule`	$0.05384526 \pm 1.5 \times 10^{-18}$	7.9×10^{-4}	0.0538565 (-2.1×10^{-4})	-3.0×10^{-14}
`orome`	$0.03039217 \pm 1.4 \times 10^{-16}$	5.4×10^{-5}	0.0303929 (-2.5×10^{-5})	$+1.3 \times 10^{-13}$
`p8-g`	$0.05348733 \pm 2.1 \times 10^{-18}$	8.9×10^{-4}	0.0534999 (-2.4×10^{-4})	-2.9×10^{-14}
`p9-a`	$0.03031824 \pm 2.1 \times 10^{-16}$	6.3×10^{-5}	0.0303191 (-2.9×10^{-5})	$+8.3 \times 10^{-14}$

Table S2: The five conditional $|W_0|$ values for de Sitter candidates of [16]. For each candidate the repository value lies inside the enclosure of $|W_0|$ at the moduli stored in the repository, once that enclosure is widened by the allowance for Gopakumar–Vafa (GV) invariants beyond the degrees included. Here $|W_0|$ denotes the undressed $|W|$ of Eq. (S3.1), without the factor $e^{(K_{\rm cs}+K_\tau)/2}$ of Eq. (59). Columns: the enclosure of $|W_0|$ at the stored moduli, written as center $\pm$ tail estimate, where the quantity after $\pm$ is an estimate of the omitted instanton tail (the measured decay of the included terms times a safety factor of 10^4), not a proven bound; the GV allowance as a fraction of $|W_0|$, an estimate that dominates the uncertainty; the repository value, with its relative deviation from the center in parentheses; the relative deviation from the repository value when the instanton sum is truncated to the repository's own set of GV invariants. Every row is conditional on the four assumptions of the text (the stored moduli, the possible normalization offset, the GV allowance, the flux-quantization convention). For `orome` and `p8-g` the change of basis taken from a host geometry is a fifth. Nothing here certifies a de Sitter vacuum.

checked that it annihilates independently computed period series on that subfamily; the values below do not use it.

Table S2 lists the results. For each candidate we widen the interval enclosure of $|W_0|$ by an allowance for the Gopakumar–Vafa invariants beyond the degrees included, an estimate of 5.4×10^{-5} to 9.0×10^{-4} relative, taken as five times the relative change of $|W_0|$ between two ranges of included degrees. The radius printed after each center in Table S2 (at most 2.2×10^{-18}, or 2.1×10^{-16} for `orome` and `p9-a`) is a different and much smaller quantity. It is obtained by extrapolating the measured decay of the instanton terms that are included, with a safety factor of 10^4, and is therefore an estimate rather than a bound, negligible against the allowance. For all five candidates we find the repository value inside the widened interval, and we refer to this inclusion as a confirmation of that value. When we truncate the instanton sum to the repository's own set of invariants, our evaluation reproduces the repository value to 10^{-13}–10^{-14} relative.

The confirmations rest on four assumptions. The first concerns the stored moduli: for three of the five (`p8-d`, `p8-g`, `p9-a`) no individual $|W_0|$ is given in [16], and the repository is the only source of a comparison value. Moreover, its floating-point format limits any comparison to $\sim 10^{-13}$ relative. The second concerns the normalization of W_0: the normalization of the repository values may differ from that of Section 5.1 by a residual offset, which we measure and find consistent with zero, but it is not shown to vanish. The third is that the Gopakumar–Vafa allowance, which dominates the uncertainty, covers the omitted degrees. The fourth is the source papers' flux-quantization convention adopted in Section 5.1 (odd flux quanta compensated by a half D3-brane). For `orome` and `p8-g` a fifth assumption enters: the integral basis change $\mathcal{U}$ (with $\det\mathcal{U} = -1$) that relates the repository's flux and intersection data to our period basis was obtained for the host geometry, and we assume that the same $\mathcal{U}$ applies to the candidate. The five confirmations certify the period/superpotential layer at the repository pins and nothing beyond it.

S3.2 Corrections to published values

For interval computations we count a published value as reproduced when our ball, widened by half a unit in the value's last quoted digit, contains it. None of the five defects below propagates to its paper's headline $|W_0|$, but all five matter to a reader reassembling the constructions from the published data.

1. In the DKMM letter [4] the racetrack constant in the text after eq. (18) (equation numbers of arXiv v2) is off by a factor of 2π in magnitude. The printed form and the form implied by the letter's own eqs. (10), (13) and (18) are

$$c_{\text{printed}} \;=\; -\sqrt{2/\pi}\,\frac{8640}{(2\pi i)^3}\,, \qquad c_{\text{implied}} \;=\; \sqrt{2/\pi}\,\frac{8640}{(2\pi i)^2}\,, \tag{S3.2}$$

   whose ratio $c_{\text{printed}}/c_{\text{implied}}$ is exactly $-1/(2\pi i)$. At the DKMM vacuum the two-term racetrack with c_{implied} gives 2.0373×10^{-8}, the letter's value to the accuracy of the two-term approximation (8×10^{-5} relative). With c_{printed} it gives $3.242\times10^{-9} = (\text{correct value})/2\pi$, which matches no published value. Broeckel et al. [5] and Carta, Mininno and Shukla (CMS) [6] give the correct form, and the letter's value of $|W_0|$ follows from it. The printed constant is therefore a misprint, and the effective racetrack should be assembled from c_{implied}. The misprinted constant is reprinted in the worked example of the TASI lectures [24], whose general racetrack formula carries the correct power of 2π.

2. The paper [45] gives the flux tadpole of its second $(5, 113)$ vacuum as $-\frac{1}{2}M\cdot K = 83/2$. Its printed flux vectors $M = (0, 2, 4, 13, -8)$ and $K = (0, -14, 9, -1, 10)$, which agree with its ancillary files, give $M\cdot K = -28+36-13-80 = -85$, so $-\frac{1}{2}M\cdot K = 85/2 = 42.5$. The D3-brane count at this vacuum requires this value: with $Q_{D3} = 60$ the text has "17 mobile D3-branes and a single 'half' D3-brane", and $60-42.5 = 17.5$, whereas the printed $83/2$ would give $60-41.5 = 18.5$. For comparison, the paper's $203/2$ tadpole is printed correctly: its fluxes give $M\cdot K = -203$, and the text's "eight mobile D3-branes" plus a half D3-brane matches $110-101.5 = 8.5$. Thus $83/2$ is a misprint for $85/2$, and the half-integer reflects the source papers' convention of odd flux quanta compensated by a half D3-brane.

3. Two printed values of τ have convergence residues in the final digit. The DKMM letter's eq. (19) reads $\langle\tau\rangle = 6.856\,i$, whereas the vacuum of Eq. (67) has $\operatorname{Im}\tau = 6.855457256\ldots$; the letter's value is reprinted in [24]. Every defensible variant of the letter's equations gives a value in 6.85497–6.85546, and none reproduces 6.8560. CMS report $g_s^{-1} = 6.85505$, whereas their own system without the ξ term, fully converged, gives 6.85507. This 2×10^{-5} residue is twenty times below the 3.9×10^{-4} shift in $\operatorname{Im}\tau$ that the ξ term produces (Section 5.3). Neither residue affects the printed digits of $|W_0|$, because $e^{(K_{\text{cs}}+K_\tau)/2}|W|$ is stationary at the supersymmetric point, so that a displacement of τ of this size changes $|W_0|$ by 10^{-7} relative or less.

4. In both the arXiv and journal versions of [16], the sixth component of the `tulkas` M-vector is printed as the string "`-1-`", with a digit lost. The value is -10: the repository [17] lists $(18, 11, -19, -12, -2, -10, 13, 5)$. The paper's own tadpole identity $-M\cdot K = 162$ with its printed K forces $-132+3\,m_6 = -162$, i.e. $m_6 = -10$. The (M, K) pairs of the paper's five worked candidates, recovered from the repository, reproduce $-M\cdot K = 162/162/162/102/102$. We therefore take the flux data in Section 5.5 from the repository rather than the printed table, except for `manwe`, whose printed fluxes we use in the basis described at the beginning of this section.

5. The same paper gives the eighth component of the `tulkas` p-vector as $-8/44$, where the repository has $+8/44$. Only $+8$ satisfies $K \cdot p = 0$ in Eq. (71); with the printed sign $K \cdot p$ equals $4/11$.

S3.3 Numerical data for the benchmark computations

For the computation of Section 5.2 we give here the data of the restricted operator (order, degree, indicial polynomial, singular points), the point of evaluation, the Krawczyk box with its contraction sequence (Table S3), and the enclosed values at the critical point in both normalizations of the periods (Table S4). For the value of Eq. (69) we describe how the periods of the vacuum 5-81-3213 are fixed as combinations of closed-form series and checked, and we give the inputs, the full ball that Eq. (69) truncates (Eq. (S3.6)), and the dependence of the value on the normalization of the periods. All quantities refer to the conventions of Section 5.1: W is the contraction of Eq. (57) and $|W_0|$ the dressed value of Eq. (59). In this subsection a value written with $\pm$ is a ball, an interval in each of its real and imaginary parts that provably contains the true value. A digit string ending in dots is a truncation of the true value, correct in every digit shown. We label each quantity that depends on the normalization of the period vector Π by that normalization.

On the curve (63) the restricted Picard–Fuchs operator L_s has order 6 and degree 63 in s, and with $\theta = s\,d/ds$ its indicial polynomial at $s = 0$ is $627750\,\theta^4(\theta-3)^2$. We derive L_s both from the restricted period series and from a D-module restriction of the Candelas–Font–Katz–Morrison ideal [46]. The closed-form Γ-series at the large-complex-structure point are fixed only up to the four sign twists $z_a \to \sigma_a z_a$, $\sigma \in \{\pm 1\}^2$, of the two coordinates and one global sign. Of these eight choices exactly one, with $\sigma = (1,1)$, reproduces all 24 Gopakumar–Vafa invariants through degree $(4,4)$, and we use it throughout. Besides the base locus $|s| = 1/3$ and the conifold locus $|s| = 0.1954$, all 45 remaining roots of the leading coefficient of L_s are apparent singularities, and the nearest on the physical ray, at $s = 0.017938$, limits the length of each continuation step. At the MUM point the tail bound (15) allows radii only up to $\rho \simeq 0.0138$, where the absolute-value majorant envelope of the recurrence reaches 1. Since $s_{\mathrm{vac}}/\rho = 0.976$, a single series summed at s_{vac} cannot be usefully tail-bounded, and for this reason we continue the periods in several legs in Section 5.2. The path along the real axis (route R) has 16 legs and the complex detour around the nearest apparent singularity (route C) 54, and on the curve the two final balls share 148 digits at 150-digit working precision.

The ball of Eq. (66) also gives the error of truncating the Gopakumar–Vafa (GV) instanton sum. Evaluated in floating point at the same inputs, the degree-$(4,4)$ truncation deviates from the certified value by 4.489×10^{-19} relative at the curve point and 4.4895×10^{-19} at the vacuum. That deviation is almost ten orders of magnitude below the 3.5×10^{-9} bound implied by an assumed growth rate of the GV invariants.

We evaluate the on-curve quantities of Section 5.2 at the point

$$\begin{aligned} s^* &= 0.0134681831826400193897483719453\,, \\ \tau^* &= 6.8554572563187259046361432042141669\,i \end{aligned} \tag{S3.3}$$

of the curve, where s^* approximates the vacuum coordinate s_{vac} and τ^* the axio-dilaton. Both enter the computation as the exact rational numbers written (terminating decimals). In particular, every quantity evaluated at (s^*, τ^*) is a function value at a known point. The transverse coordinates $w_a = z_a/z_a^* - 1$ $(a = 1, 2)$ of Eq. (64) are measured from the curve point $z^* = (s^{*4}, s^{*3})$; they vanish at z^* and are not the moduli U^a. The fluxes are $F = (7, 3, -24, 0, -16, 50)$ and $H = (0, 3, -4, 0, 0, 0)$, contracted with the period vector in the component order $(F_0, F_1, F_2, X^0, X^1, X^2)$ of Section 5.1, so

coordinate of x	center	final radius
$\mathrm{Re}\, w_1$	$7.944427495308857534704976 \times 10^{-5}$	4.8522×10^{-149}
$\mathrm{Im}\, w_1$	0	4.1127×10^{-150}
$\mathrm{Re}\, w_2$	$7.944617279031411801840431 \times 10^{-5}$	3.6392×10^{-149}
$\mathrm{Im}\, w_2$	0	3.0846×10^{-150}
$\mathrm{Re}\, \tau$	0	1.6362×10^{-150}
$\mathrm{Im}\, \tau$	$6.85545725631872590391568610466817762941 \ldots$	1.9304×10^{-149}

Table S3: The Krawczyk box X of Eq. (65) that encloses the DKMM critical point: the F-term map F has exactly one zero in X, and in every coordinate that zero lies within the final radius of the center (the midpoint of the final box, which agrees with $\tilde{x}$ to all digits shown). Coordinates: $w_a = z_a/z_a^* - 1$ ($a = 1, 2$) are the transverse displacements from the curve point $z^* = (s^{*4}, s^{*3})$ of Eq. (S3.3) (they are not the moduli U^a), and τ is the axio-dilaton. The centers of $\mathrm{Re}\, w_1$ and $\mathrm{Re}\, w_2$ are given to 25 digits and that of $\mathrm{Im}\, \tau$ is truncated (Table S4 gives more digits); the other three centers are exactly zero, and the final radii therefore bound $|\mathrm{Im}\, w_1|$, $|\mathrm{Im}\, w_2|$ and $|\mathrm{Re}\, \tau|$ at the critical point, all below 5×10^{-150}. The first trial radius was 10^{-10} in every coordinate, and the contraction sequence is given in the text.

that $U^1 = X^1/X^0$ and $U^2 = X^2/X^0$ are the ratios of the fifth and sixth components of Π to the fourth.

The curve point (S3.3) approximates the vacuum, and on the curve the periods are enclosed by the continuation of Section 5.2. However, the dressed value $e^{(K_{\mathrm{cs}}+K_\tau)/2}|W|$ on the curve agrees with $|W_0|$ at the true critical point to only four digits (a relative shift of $+2.30482 \times 10^{-4}$), because the exponentially small racetrack balance changes at $\mathcal{O}(1)$ under the transverse offset of the critical point from the curve. At the vacuum the offset of the moduli U from the flat locus is $U - \tau p = (-5.9223, -4.4038) \times 10^{-6}\, i$ within 2×10^{-18}, U is displaced from the curve along $(1, 1)$, and the racetrack balance itself is $B = H \cdot \Sigma \cdot \Pi = -1.51794 \times 10^{-7}\, i$. Continuing off the curve needs no new differential equation. Since the restriction has no order drop ($r = 6 = 2(h+1)$ for $h = 2$ complex-structure moduli), the transverse derivatives of the periods are $\mathbb{Q}(s)$-linear combinations of on-curve data, whose coefficients we obtain by solving a 6×6 linear system.

Table S3 gives the Krawczyk box of Eq. (65). In the rest of this subsection and in Table S3 we write F for the map of the test (16) and no longer for the flux vector. The six components of this map are the real and imaginary parts of the three F-term conditions $D_{U^1}W = D_{U^2}W = D_\tau W = 0$ of Eq. (60), written in the coordinates (w_1, w_2, τ), as functions of the six real coordinates $x = (\mathrm{Re}\, w_1, \mathrm{Im}\, w_1, \mathrm{Re}\, w_2, \mathrm{Im}\, w_2, \mathrm{Re}\, \tau, \mathrm{Im}\, \tau)$. We take as center $\tilde{x}$ a Newton-refined approximate zero (with $|F(\tilde{x})| = 5.147 \times 10^{-155}$), and as Y an approximate inverse of the Jacobian at $\tilde{x}$. With all six radii equal to 10^{-10} the Krawczyk image lies strictly inside the box, and hence F has exactly one zero there.

We then apply the contraction $X_{k+1} = \mathcal{K}(\tilde{x}_k, X_k) \cap X_k$ of Eq. (65) seven times and obtain largest coordinate radii 1.968×10^{-14}, 2.242×10^{-19}, 2.913×10^{-29}, 4.914×10^{-49}, 1.399×10^{-88}, 4.852×10^{-149} and 4.852×10^{-149}. The first six steps contract the box, with the number of correct digits roughly doubling at each step (as in a Newton iteration) until the 150-digit working precision is reached. The seventh no longer shrinks the box, which ends the iteration. Over the final box $|F| \leq 7.4 \times 10^{-147}$ in every component.

Table S4 lists the enclosed values at the critical point from both continuation routes R and C. The values given in Eqs. (66) and (67) are the route-R balls for the dressed $|W_0|_{\mathrm{vac}}$ and for τ_{vac}, with their radii rounded upward. The undressed quantities, such as $e^{-K_{\mathrm{cs}}}$ and $|W|$ itself, depend

quantity	enclosed value	radius
$\lvert W_0\rvert_{\text{vac}}$ (dressed), route R	$2.037106093311183419122861559331984678 7\ldots \times 10^{-8}$	4.36×10^{-148}
$\lvert W_0\rvert_{\text{vac}}$ (dressed), route C	the same center to the digits shown	7.52×10^{-138}
$\operatorname{Im}\tau_{\text{vac}}$	$6.855457256318725903915686104668177629 41\ldots$	6.76×10^{-149}
$\operatorname{Re}\tau_{\text{vac}}$	0	1.64×10^{-150}
$g_s = 1/\operatorname{Im}\tau_{\text{vac}}$	$0.1458691904290253677612689270154392769834$	1.2×10^{-41}
$e^{-K_\tau} = 2\operatorname{Im}\tau_{\text{vac}}$	$13.710914512637451807831372209336355258 8\ldots$	1.35×10^{-148}
$\operatorname{Im}U^1$	$2.742176980208520992207200530211784728\ldots$	4.08×10^{-149}
$\operatorname{Im}U^2$	$2.056632773104733929729445465370154943\ldots$	8.40×10^{-150}
$z_{1,\text{vac}} = z_1^*(1+w_1)$	$3.2905656472914497081950 2\ldots \times 10^{-8}$	
$z_{2,\text{vac}} = z_2^*(1+w_2)$	$2.4432142100482615437560 2\ldots \times 10^{-6}$	
$\varpi_0(z_{\text{vac}})$	$1.000001974354395882066789985394778 40$	3.23×10^{-36}
$e^{-K_{\text{cs}}}$ $(X^0 = \varpi_0(z_{\text{vac}}))$	$484.6481807497504420984387257 39$	1.80×10^{-28}
$e^{-K_{\text{cs}}}$ $(X^0 = 1)$	$484.6462670208857734256380136 25$	5.41×10^{-28}
$e^{-K} = e^{-K_{\text{cs}}}e^{-K_\tau}$ $(X^0 = 1)$	$6644.943535992228397027113774 65$	7.1×10^{-27}
$e^{-(K_{\text{cs}}+K_\tau)/2}$ $(X^0 = 1)$	$81.51652308576604395694906834 18$	5.1×10^{-29}
$e^{(K_{\text{cs}}+K_\tau)/2}$ $(X^0 = 1)$	$0.012267451580925123713515497937 91$	4.9×10^{-33}

Table S4: Enclosed values at the DKMM critical point in the conventions of Section 5.1: each true value lies within the stated radius of the center, and strings ending in dots are truncations. In route R the periods are continued along the real s axis and in route C along a complex detour around the nearest apparent singularity of L_s (Section 5.2); the two balls for $|W_0|_{\text{vac}}$ overlap and share their leading 129 digits at the vacuum, and the radii in Eqs. (66)–(67) are the route-R radii rounded upward. In Section 5.2 g_s is given as 0.14587. Here $z_{a,\text{vac}}$ are the complex-structure coordinates of the critical point, with $z^* = (s^{*4}, s^{*3})$ from Eq. (S3.3) and w_a from Table S3, and ϖ_0 is the fundamental period (S3.4). Entries below the rule depend on the normalization of Π and are labeled by it in parentheses: $X^0 = 1$ is the gauge of Eq. (57), and $X^0 = \varpi_0(z_{\text{vac}})$ is the normalization of the transported periods. They carry about 31 digits because $e^{-K_{\text{cs}}}$ is enclosed only to 30 digits; the entries above the rule are enclosed to within the radii shown.

on the normalization of the period vector Π. Our transported Π has as its X^0 component the fundamental period at the large-complex-structure point,

$$\varpi_0(z_1, z_2) = \sum_{n_1 \geq 3n_2 \geq 0} \frac{(6n_1)!}{(n_2!)^3\,(2n_1)!\,(3n_1)!\,(n_1 - 3n_2)!}\; z_1^{n_1} z_2^{n_2} \tag{S3.4}$$

in the coordinates of [46], whereas the gauge of Eq. (57) sets $X^0 = 1$. Hence passing to that gauge divides Π by $\varpi_0(z_{\text{vac}})$, $e^{-K_{\text{cs}}}$ by $|\varpi_0(z_{\text{vac}})|^2$ and $|W|$ by $|\varpi_0(z_{\text{vac}})|$, while the dressed quantities do not change.

At the vacuum the complex-structure coordinates are $z_{a,\text{vac}} = z_a^*(1+w_a)$, and we find $|\varpi_0(z_{\text{vac}})| - 1 = 1.974354\ldots \times 10^{-6}$, with leading term $60\,z_{1,\text{vac}}$ (Table S4 gives the ball). We give $e^{-K_{\text{cs}}}$ in both normalizations and the remaining Kähler factors in the gauge $X^0 = 1$. These factors carry about 31 digits, far fewer than the other entries of Table S4, because our enclosure of $e^{-K_{\text{cs}}}$ at the vacuum, which enters each of them, has only 30 correct digits. DKMM print $\langle\tau\rangle = 6.856\,i$, $\langle U_1\rangle = 2.742\,i$, $\langle U_2\rangle = 2.057\,i$ and $|W_0| = 2.037 \times 10^{-8}$ [4]. All of these digits agree with the enclosures of Table S4 except the last digit of $\operatorname{Im}\tau$ (the residue of item 3 in Section S3.2).

The value of Eq. (69) for the vacuum 5-81-3213 of Section 5.4 is obtained differently, because no scalar operator is available on its curve $(z_1, \ldots, z_5) = (s^9, s^{10}, s^{10}, s^{10}, s^{10})$. With $h = 5$ complex-structure moduli the restricted Picard–Fuchs operator would have order 13, more than the number

$2h+2=12$ of flux-frame periods, and degree greater than 4,052 in s; we did not derive it. Instead we write the twelve periods as exact linear combinations of the closed-form Γ-series of the mirror and their derivatives with respect to the Frobenius indices, each series truncated at order s^{120}. The coefficient of s^m in every such series is bounded in absolute value by 2^m, so the series converge for $|s|<1/2$. At $s_{\text{vac}}=-1/180$ the remainders beyond order s^{120} are therefore bounded by geometric series and carried as balls, and the logarithms are taken on the branch $\log s=\ln|s|+i\pi$.

The combination giving each period is fixed by matching the large-complex-structure expansion of the flux-frame periods, whose instanton part uses the Gopakumar–Vafa invariants of the ancillary files of [45] (557 curve classes, complete for the classes that contribute through order s^{55}). The matching conditions through order s^{50} form a linear system for 112 coefficients, of rank 80. Its 32-dimensional kernel consists of combinations of the truncated series that vanish identically through order s^{120}, so the periods are determined uniquely to that order. The rank does not grow when matching conditions of higher order are added, and the solution reproduces the expansion at orders s^{51} to s^{55}, which were not used to fix it. As a check that does not enter the value, we reconstructed the 17×17 first-order system (17) of the restricted D-module exactly, from its reductions modulo eight primes by the Chinese remainder theorem and rational reconstruction (123,824 rational coefficients). It agrees with an independent reduction modulo a further prime, and it annihilates the separately computed series of the holomorphic period exactly through s^{3000}; the logarithmic periods are not tested against it. Their only checks, both noted above, are that the rank of the matching system does not grow and that the solution reproduces the expansion at orders s^{51} to s^{55}.

The quantity $|W_0|$ of Eq. (69) is the undressed $|W|$, with W the contraction of Eq. (57) for the flux vectors of this vacuum, evaluated under the conditions stated in Section 5.4 at the fixed inputs

$$s_{\text{vac}} \;=\; -1/180\,, \qquad \tau_* \;=\; 12+19.83563404997038668317023708 91\,i\,, \tag{S3.5}$$

the second being the racetrack solution for τ. Equation (69) of the main text agrees with all six digits of the ancillary value of [45] for this vacuum. Replacing τ_* by the value $1/g_s=19.83564187$ of the ancillary files of [45] would shift $|W_0|$ by 8.2×10^{-5} relative, away from those six digits. Written in full, the ball that Eq. (69) truncates is

$$\begin{aligned}|W| \;=\; &2.037782247569860854044097636492779 7\\ &\quad 294098195325121369695249464740504\times10^{-23} \;\pm\; 7.72\times10^{-91}\end{aligned} \tag{S3.6}$$

where the digit string continues on the second line. The relative radius 3.1×10^{-68} given with Eq. (69) is that of the ball before its center is written in decimal, and the radius in Eq. (S3.6) also covers the rounding of the last written digit. The periods entering $|W|$ here are normalized with $X^0=\varpi_0(s)$, the fundamental period on the curve, whose expansion $\varpi_0(s)=\sum_m a_m s^m$ has $a_0=1$ and next nonzero coefficient $a_{30}=24$. Hence

$$\varpi_0(-1/180)-1 \;=\; 5.27273674516\times10^{-67} \;\pm\; 2.1\times10^{-79}\,, \tag{S3.7}$$

equal to $24/180^{30}$ at leading order. Passing to the gauge $X^0=1$ multiplies $|W|$ by $1-5.27\times10^{-67}$ and changes no written digit of Eq. (69). However, the shift is 13.9 times the relative radius of the ball (S3.6), and in Section 5.4 we therefore give a relative uncertainty below 10^{-66} for that gauge.

S4 Implementation and validation

We test the implementation used in Sections 3–5 of the main text with a suite that reruns each component on stored inputs and compares the results with stored reference outputs. In this section we describe that suite, the control fluxes inserted into the enumeration of Section 3.2, and three further tests of the lattice and period computations. We then list, by kind, the stored data on which the proofs in the main text rest, and indicate which of these data are printed in the other sections of this supplement. Section 2.4 of the main text summarizes these tests and a further comparison, given in Section S4.1, of the enumeration with earlier flux tables on attractive K3 pairs. All integer and rational data are exact, the rational solutions of the linear programs included.

S4.1 Validation of the implementation

The validation suite first recomputes the hash of each stored file and does not run the comparisons if any file has changed. When we corrupted one stored file as a test, the suite named the altered file, ran no comparison and reported a failure. On the version of the code used for this paper the suite reports 37 of 37 checks green. Of these, 32 are regression tests, and the others compare the file hashes and confirm that no test in the source code is left out of the suite. The distributed copy (Section S4.2) omits an internal job-launch script with its test and the inputs of some other tests, which its suite reports as skipped by name rather than as passed.

The suite also contains cases designed to fail. A validation suite that has never failed shows only that its tests have never been triggered. Hence each such case is built to trigger one safeguard in the code: if a change to the code disabled that safeguard, the case would pass and the suite would report a failure. For the period computations the cases are a single perturbed coefficient in an exact connection matrix, a flipped or swapped logarithmic branch in a tower of Frobenius solutions, and a tampered table of stored reference values. The remaining cases are five tampered orbit keys (the labels under which equivalent fluxes are merged), an error injected mid-stream in a flux-enumeration sweep, and a deliberately widened enclosure radius.

Independently of the suite, we insert into each run of the enumeration a set of control fluxes whose correct classification is known, at hash-determined positions that the person running the enumeration cannot predict. We recovered 5,740 of 5,740 such controls among the 855,192 entries of the full sweep described below. Each run of the sweep also stores its inputs and outputs along with a hash of its code.

The full sweep is larger than the list of candidates of Table 2. The entries of the sweep are all the coherent tuples $(A, \tilde{A}, B)$ that the enumeration of Section 3.1 generates within the trace budget at ranks one and two in every signature and at rank three in the definite signatures $(3,0)$ and $(0,3)$. Each ordered pair of supports is counted separately. To these tuples are added the two check fluxes of Table 2 (the rank-6 flux of signature $(3,3)$ and the charge-25 flux). Among these entries, the 412,531 FS-capable candidates of signature $(3,0)$ and the rank-6 check flux make up the 412,532 entries of Table 2.

We also compare the enumeration with that of Braun, Fraiman, Graña, Lüst and Parra de Freitas for purely transcendental fluxes on attractive K3 pairs (Section 3.1 of the main text), which uses a gauge-rank criterion rather than (FS). Restricted to their class and their criterion, our enumeration reproduces the findings of [2], namely the 34 lattice types, the charges in the range $N_{\text{flux}} = 25$–30 at which no solution occurs, and all 18 pairs listed there at $N_{\text{flux}} = 27$, 28 and 30.

Two further tests lie outside the suite and concern the lattice computations. First, four items were computed twice in independent implementations (PARI/GP [61] among them): the complement-class lists for the 54 genera of Lemma 3.3, the invariants of the three rank-19 lattices of Theorem 3.5, the five bounds of Table 3 and the charge-22 flux of Section 3.4. Second, at rank two we test our reduction of definite binary forms against known class numbers. Proper equivalence of forms under $SL(2,\mathbb{Z})$ and isometry of lattices under $GL(2,\mathbb{Z})$ give different counts, and the reduction keeps the two notions distinct. For instance, the even binary lattices of determinants 23, 47 and

71 fall into 2, 3 and 4 isometry classes respectively, against form class numbers $h = 3$, 5, 7. The reduction reproduces both counts and the classical class numbers, $h(-23) = 3$ among them. The mass identity (10) of the main text gives a further completeness test. For example, on a test genus of determinant 92 the Kneser neighbor method gives six classes with mass exactly 53/768, equal to the Smith–Minkowski–Siegel value, which proves the list complete.

A third test concerns the truncation error of the period computations. In Section 2.3 of the main text we control the tail of a series solution with the proven tail bound (15), whereas the empirical rule that bounds the tail by four times the last computed window has two exact counterexamples. One counterexample is the lacunary series $1/(1-z^2)$, which contains only even powers of z. At an odd truncation order its last computed window vanishes and the window rule bounds the tail by zero, while the true tail at one such order we tested is 2.8×10^{-25}. The other is the series solution of an operator whose nearest singularities are a complex-conjugate pair; the coefficients of that series are modulated by a cosine, and the window rule underestimates the tail at all 180 tested truncations.

S4.2 Contents of the distributed material

We store the interval proofs of Sections 4 and 5 as sets of files, each set with a list of its file hashes. For Section 5 we accept a value only when computations at two different working precisions agree, a consistency check that lies outside any proof. For both sections those certificates carry their inputs, radii and results, but neither a hash of the code that produced them nor a timestamp. Separately, the conventions of the KKLT benchmark computation of Section 5 are collected in a dated table whose numeric pins are floating-point reproductions of published numbers rather than certified balls. It lists each sign and normalization that could change $|W_0|$, the treatment of the $\zeta(3)\chi$ term, and the published values we compare against. Our interval computation follows its conventions, and the table is not itself hash-committed.

The data on which the proofs in the main text rest are available at `https://bootloops.ai/files/kklt/` (see Code and data availability in the main text), each set with stand-alone checkers. The implementation and the validation suite of Section S4.1 are available at the same address, as a source tree and as a single archive with a list of file hashes. These data are of three kinds, and we print each kind only in part. The first kind consists of the Farkas vectors that prove Lemma 3.3. For every admissible genus we store the exact rational Farkas vectors of the Niemeier route, one for each complement class and each Coxeter number, and, for the genera also treated by the theta-series route, the rational infeasibility vectors of those linear programs. The second kind consists of the data for the five bounds of Table 3: for each bound we store the vectors y_α, the weights w_α and the matrices Y_p and Λ_p that enter (37).

In Section S5 we print the statements that the vectors of both kinds prove and the lattice data to which they refer, but not the vectors themselves. The printed data are the Gram matrices and gluings, the admissible genera with their rank-5 complement classes, the linear system of one genus set up as an example, and the exact values of the five bounds. Testing a stored vector against these data requires only exact rational arithmetic and does not use the code that produced the vector.

The third kind consists of the interval proofs of Sections 4 and 5 described above. In particular, for Section 4 we store the table of outward-rounded interval hulls of the five fields A_a on the 3,072 sub-boxes of R (Section 4.2) and the first-order bound on the 48 blocks tiling R (the bound that excludes the ray $u = \pm(4, 0, 30, 0, 1)$). In Section S7 we print the worst margins of that table and the box-wide signs of the field components, together with the enclosures of the vacua, but not the table itself, which is also not part of the material available online.

For Section 5 the stored files include the Krawczyk box with its six coordinate radii, which

Section S3 also prints. In each interval proof the stored inputs and radii fix the map F and the box X of the test (16). Thus the containment required by the test can be recomputed from the stored data in ball arithmetic (we use the Arb library [23]) without the code that produced the proof.

S5 Lattice data for Section 3

In this section we collect the lattice data on which Section 3.3, Lemma 3.3 and Theorem 3.5 of the main text rest, in a form that a reader can test with integer and rational arithmetic alone. Section S5.1 gives the Gram matrices of W_{512}, W'_{512} and W_{1024} in the bases in which the flux matrices M of Section 3.3 are written, and the Gram matrix of the determinant-128 lattice Λ_{19} of Section S1.2 in an LLL-reduced basis. It then gives the gluings of the first three lattices into $\Gamma_{3,19} = 3U \oplus 2E_8(-1)$, the lattice Γ of Section 3.1, each in a basis of the overlattice that extends the basis of $W(-1)$. In Section S5.2 we list the 54 admissible genera of Lemma 3.3 with their rank-5 complement classes, state the linear system whose infeasibility proves the lemma, and set up this system for one genus as an example. The same section describes the second route of Section 2.2, through the theta series of a genus and linear programming, and shows why the genus average of the number of roots does not by itself force a root. In Section S5.3 we derive the inequality from which the five bounds of Table 3 follow and give these bounds as fractions. That section also lists the 38 further bounds used in the proof of Theorem 3.5 and describes the local solvability data for Eq. (39).

All Gram matrices below are integer and symmetric. Evenness can be read from the diagonal entries, and the determinant computed from the matrix in integer arithmetic. We obtain the minimum and the kissing number by enumerating the vectors of norm at most 4 (Fincke–Pohst), and a lattice is root-free when no vector of norm 2 occurs among them. A sublattice spanned by the rows of an integer matrix X is primitive when the maximal minors of X have greatest common divisor 1. A gluing is given by the Gram matrix of the overlattice together with integer bases of the two sublattices inside it. To verify it one checks that the overlattice has even diagonal and determinant ± 1, that the two bases are primitive and mutually orthogonal, and that their Gram matrices are those of the two lattices glued. For three families of auxiliary data we state only the property used in the proofs; the data themselves are not tabulated. These are the 1,764 rational Farkas vectors of Section S5.2, the rational weight vectors of the five bounds of Table S9 and of the 38 bounds of Table S10, and the 156 integer matrices of the last paragraph of Section S5.3. All three families are available in full at the address given in Section S4.2, with a checker that tests them in exact rational arithmetic.

S5.1 Gram matrices and gluings of the rank-19 lattices

Equations (S5.1)–(S5.3) give the Gram matrices of the three lattices of Eq. (34) in the bases in which we use them. The flux matrices M of Eq. (35) and the column indices of Section S5.3 refer to these bases, whose vectors $e_0, \dots, e_{18}$ we number from zero. In each matrix every diagonal entry equals the minimum 4 and every off-diagonal entry has absolute value at most 2. We obtained each of the three lattices by Kneser 3-neighbor steps from the lattice $E_8 \oplus E_8 \oplus \mathrm{diag}(2, 2, c')$ of its genus, with $c' = 128$ for the two lattices of determinant 512 and $c' = 256$ for W_{1024}.

$$W_{512} = \begin{pmatrix}
4 & 2 & 2 & 1 & 2 & 0 & 2 & 2 & -2 & 2 & -2 & -1 & -1 & -2 & 2 & 1 & 2 & 2 & -1 \\
2 & 4 & 2 & -1 & 2 & -1 & 2 & 1 & -2 & 0 & 0 & -1 & 1 & 0 & 1 & 2 & 1 & 1 & 0 \\
2 & 2 & 4 & 1 & 1 & 1 & 2 & 2 & -2 & 0 & 0 & -1 & 0 & -1 & 2 & 0 & 2 & 2 & -1 \\
1 & -1 & 1 & 4 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & -1 & -2 & 2 & -1 & 0 & 1 & -2 \\
2 & 2 & 1 & 1 & 4 & 1 & 1 & 2 & -1 & 2 & -1 & -1 & 1 & -1 & 1 & 0 & 1 & 0 & -1 \\
0 & -1 & 1 & 2 & 1 & 4 & 1 & 2 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -2 \\
2 & 2 & 2 & 1 & 1 & 1 & 4 & 1 & -2 & 1 & 0 & 1 & 1 & -1 & 1 & 2 & 0 & 1 & -2 \\
2 & 1 & 2 & 2 & 2 & 2 & 1 & 4 & 0 & 1 & -1 & -1 & 0 & -1 & 1 & -1 & 1 & 2 & -1 \\
-2 & -2 & -2 & 1 & -1 & 0 & -2 & 0 & 4 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & -2 & -1 & 1 \\
2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 4 & -1 & 1 & 0 & -2 & 1 & 0 & 1 & 0 & -1 \\
-2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & -1 & 4 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & 4 & 1 & 0 & 0 & 1 & -2 & -1 & -1 \\
-1 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 4 & 2 & -1 & 1 & -1 & -2 & 0 \\
-2 & 0 & -1 & -2 & -1 & 0 & -1 & -1 & 1 & -2 & 1 & 0 & 2 & 4 & -2 & 1 & -1 & -2 & 1 \\
2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & -1 & -2 & 4 & 0 & 1 & 1 & -1 \\
1 & 2 & 0 & -1 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 4 & -1 & 0 & 0 \\
2 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & -1 & -2 & -1 & -1 & 1 & -1 & 4 & 1 & 0 \\
2 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & -1 & 0 & -1 & -1 & -2 & -2 & 1 & 0 & 1 & 4 & 0 \\
-1 & 0 & -1 & -2 & -1 & -2 & -2 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 4
\end{pmatrix} \tag{S5.1}$$

$$W'_{512} = \begin{pmatrix}
4 & -2 & -2 & -2 & -1 & -1 & -1 & -1 & 2 & -2 & -1 & 1 & -1 & -2 & -2 & 2 & -1 & 2 & -1 \\
-2 & 4 & 2 & 2 & 2 & 1 & -1 & 1 & -1 & 2 & 2 & 1 & -1 & 0 & 0 & -2 & 2 & -2 & -1 \\
-2 & 2 & 4 & 2 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -2 & -1 \\
-2 & 2 & 2 & 4 & 0 & 2 & -1 & 2 & 0 & 1 & 2 & -1 & -1 & 1 & 1 & 0 & 1 & -2 & 0 \\
-1 & 2 & 1 & 0 & 4 & -1 & -1 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & -1 & -1 & 2 & 0 & -1 \\
-1 & 1 & 2 & 2 & -1 & 4 & 1 & 2 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -2 & -1 \\
-1 & -1 & 0 & -1 & -1 & 1 & 4 & -1 & -2 & 0 & -2 & -1 & 1 & 1 & 1 & -1 & -2 & 0 & 0 \\
-1 & 1 & 2 & 2 & 1 & 2 & -1 & 4 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -2 & -1 \\
2 & -1 & 0 & 0 & 0 & -1 & -2 & 1 & 4 & -1 & 0 & 0 & -1 & -2 & -1 & 1 & 0 & 1 & 0 \\
-2 & 2 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 4 & 2 & 0 & -1 & 1 & 1 & -2 & 1 & -1 & 1 \\
-1 & 2 & 0 & 2 & 1 & 0 & -2 & 1 & 0 & 2 & 4 & 1 & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\
1 & 1 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 1 & 4 & 0 & -1 & -2 & 0 & 1 & 1 & -1 \\
-1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 4 & 2 & 1 & 0 & -1 & 1 & 1 \\
-2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -2 & 1 & 0 & -1 & 2 & 4 & 1 & 0 & -1 & -1 & 1 \\
-2 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & -2 & 1 & 1 & 4 & -1 & 0 & 0 & 2 \\
2 & -2 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 1 & -1 \\
-1 & 2 & 1 & 1 & 2 & -1 & -2 & 1 & 0 & 1 & 2 & 1 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\
2 & -2 & -2 & -2 & 0 & -2 & 0 & -2 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 4 & 1 \\
-1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 1 & 2 & -1 & 0 & 1 & 4
\end{pmatrix} \tag{S5.2}$$

$$W_{1024} = \begin{pmatrix}
4 & -1 & -1 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -2 & 2 & 2 & -1 & -2 & 2 & -2 & 2 & 2 \\
-1 & 4 & 2 & 0 & -2 & 1 & 1 & -2 & -1 & -1 & 2 & 0 & -1 & 2 & 2 & 0 & 2 & -2 & 0 \\
-1 & 2 & 4 & 1 & -1 & 0 & 0 & -2 & -2 & 0 & 2 & 1 & 0 & 2 & 0 & -1 & 1 & -2 & 1 \\
2 & 0 & 1 & 4 & 2 & 2 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & -1 & -2 & 1 & -1 & 1 & 2 \\
2 & -2 & -1 & 2 & 4 & 0 & 1 & 2 & 1 & 1 & -1 & 0 & 1 & -2 & -2 & 0 & -2 & 1 & 1 \\
2 & 1 & 0 & 2 & 0 & 4 & 1 & 0 & 2 & -2 & 0 & 2 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 2 & 1 & 1 & 4 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 2 & 0 & 1 & 1 \\
2 & -2 & -2 & 0 & 2 & 0 & 1 & 4 & 2 & -1 & -2 & 0 & 1 & -2 & -1 & 0 & -2 & 1 & 1 \\
2 & -1 & -2 & 1 & 1 & 2 & 1 & 2 & 4 & -2 & -2 & 1 & 1 & -1 & 0 & 1 & -1 & 2 & 1 \\
-1 & -1 & 0 & 0 & 1 & -2 & -1 & -1 & -2 & 4 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 \\
-2 & 2 & 2 & 0 & -1 & 0 & -1 & -2 & -2 & 0 & 4 & 0 & 0 & 1 & 1 & -2 & 2 & -2 & -1 \\
2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 4 & 2 & 1 & -1 & 1 & -1 & 1 & 1 \\
2 & -1 & 0 & 2 & 1 & 2 & 0 & 1 & 1 & -1 & 0 & 2 & 4 & -1 & -2 & 0 & -1 & 1 & 2 \\
-1 & 2 & 2 & -1 & -2 & 0 & -1 & -2 & -1 & -1 & 1 & 1 & -1 & 4 & 2 & 0 & 1 & -2 & 0 \\
-2 & 2 & 0 & -2 & -2 & 0 & -1 & -1 & 0 & -1 & 1 & -1 & -2 & 2 & 4 & -1 & 2 & -2 & -1 \\
2 & 0 & -1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & -1 & 4 & -1 & 2 & 0 \\
-2 & 2 & 1 & -1 & -2 & 0 & 0 & -2 & -1 & -1 & 2 & -1 & -1 & 1 & 2 & -1 & 4 & -1 & -1 \\
2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & -2 & 1 & 1 & -2 & -2 & 2 & -1 & 4 & 0 \\
2 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 2 & 0 & -1 & 0 & -1 & 0 & 4
\end{pmatrix} \tag{S5.3}$$

The determinants of these three matrices are 512, 512 and 1024.

We collect the invariants of these three lattices and of Λ_{19} in Table S5. The two determinant-512 lattices share their determinant, minimum and discriminant form and are not isometric; their kissing numbers already differ. For each of the three, the discriminant form is isometric to that of the ternary lattice $T = \mathrm{diag}(2, 2, c')$ with $c' = 128$, 128, 256. This isometry of forms is Nikulin's

lattice	det	min	kissing number	$\lvert\mathrm{Aut}\rvert$	L^*/L	discriminant form
W_{512}	512	4	5200	2	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{128}$	$\cong q_{\,\mathrm{diag}(2,2,128)}$
W'_{512}	512	4	5204	2	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{128}$	$\cong q_{\,\mathrm{diag}(2,2,128)}$
W_{1024}	1024	4	3664	2	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{256}$	$\cong q_{\,\mathrm{diag}(2,2,256)}$
Λ_{19}	128	4	10668	23592960	$\mathbb{Z}_8 \times \mathbb{Z}_2^4$	$\cong -q_{\sqrt{2}D_5}$

Table S5: The three lattices of Theorem 3.5 are root-free, admissible, and have no automorphisms other than ± 1, while Λ_{19}, of smaller determinant, is root-free but not admissible. For each even positive-definite rank-19 lattice of Section S5.1, given by the Gram matrices (S5.1)–(S5.4), the table lists: the determinant; the minimum, obtained by enumerating the vectors of norm at most 4 (so none has a root); the kissing number, counting $\pm v$ separately; the order of the automorphism group; the discriminant group L^*/L; and the isometry class of the discriminant form. For the first three rows the form is that of the rank-3 lattice $T = \mathrm{diag}(2, 2, c')$, which is the admissibility condition of Section 3.2 and Nikulin's condition (11) for a primitive embedding of $W(-1)$ in $\Gamma_{3,19}$ with complement T; the group of Λ_{19} needs five generators, more than the three that a rank-3 partner allows.

condition (11) for $W(-1) \oplus T$ to extend to an even unimodular lattice of signature $(3, 19)$, and that lattice is $\Gamma_{3,19}$ by uniqueness.

The lattice Λ_{19} of Section S1.2, the laminated lattice of Conway and Sloane in dimension 19 [49], is the orthogonal complement of a copy of $\sqrt{2}D_5$ in the Leech lattice [12, Ch. 6]. Its discriminant form is therefore minus that of $\sqrt{2}D_5$, on the group $\mathbb{Z}_8 \times \mathbb{Z}_2^4$ of order 128, which needs five generators; hence Λ_{19} is not the orthogonal complement of a rank-3 lattice in $\Gamma_{3,19}$ and is not admissible. In Eq. (S5.4) we give its Gram matrix in an LLL-reduced basis, again with all diagonal entries equal to the minimum 4.

$$\Lambda_{19} = \begin{pmatrix}
4 & 2 & 2 & -2 & 2 & -2 & -2 & 2 & 0 & 0 & -2 & -2 & 1 & 0 & -1 & -1 & -1 & 1 & -2 \\
2 & 4 & 2 & -2 & 2 & 0 & 0 & 2 & 1 & -1 & -2 & -2 & 2 & -1 & -2 & 1 & -2 & 1 & -1 \\
2 & 2 & 4 & 0 & 2 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -2 & 1 & -1 & 2 & -2 \\
-2 & -2 & 0 & 4 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & 2 & -1 & 1 & 1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 0 & 4 & 0 & 0 & 2 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 & 2 & -1 \\
-2 & 0 & 0 & 0 & 0 & 4 & 2 & -1 & 1 & -2 & 1 & 0 & 0 & -1 & 0 & 2 & 1 & 1 & 1 \\
-2 & 0 & 0 & 0 & 0 & 2 & 4 & -1 & 1 & -2 & 1 & 0 & 1 & -1 & -1 & 2 & 1 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 & -1 & -1 & 4 & 1 & -1 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & -1 \\
0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 4 & -2 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & -1 & 1 & -1 & -2 & -2 & -1 & -2 & 4 & 1 & 0 & -1 & 2 & 0 & -2 & -1 & -2 & 1 \\
-2 & -2 & -1 & 2 & -1 & 1 & 1 & -2 & 0 & 1 & 4 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 \\
-2 & -2 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & -1 & 1 & 1 & 1 & 0 & 1 \\
1 & 2 & 2 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -2 & 1 & -2 & 1 & -1 \\
0 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 0 & 2 & 2 & -1 & -1 & 4 & 0 & -2 & 0 & -2 & 1 \\
-1 & -2 & -2 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 4 & -1 & 1 & 0 & 1 \\
-1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & -2 & 0 & 1 & 1 & -2 & -1 & 4 & 1 & 2 & 0 \\
-1 & -2 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & 1 & -2 & 0 & 1 & 1 & 4 & 1 & 0 \\
1 & 1 & 2 & 0 & 2 & 1 & 0 & 2 & 1 & -2 & -1 & 0 & 1 & -2 & 0 & 2 & 1 & 4 & -1 \\
-2 & -1 & -2 & 1 & -1 & 1 & 0 & -1 & 1 & 1 & 2 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & 4
\end{pmatrix} \tag{S5.4}$$

The determinant of this matrix is 128.

For each $W \in \{W_{512}, W'_{512}, W_{1024}\}$ we now give an even unimodular overlattice $\Lambda \supset W(-1) \oplus T$. Equations (S5.5)–(S5.7) give the Gram matrix G_Λ of Λ in a basis $f_0, \dots, f_{21}$ whose first nineteen vectors are the basis of $W(-1)$. Thus the upper left 19×19 block of G_Λ is $-W$, the embedding matrix of $W(-1)$ is $X = (I_{19}\ 0)$, one of whose maximal minors is 1, and $W(-1)$ is primitive in Λ. Each G_Λ has even diagonal, $\det G_\Lambda = -1$ and signature $(3, 19)$, whence $\Lambda \cong \Gamma_{3,19}$.

Below each G_Λ we list a 3×22 integer matrix Z whose rows span the orthogonal complement of $W(-1)$ in Λ, with $Z G_\Lambda X^\mathsf{T} = 0$ and $Z G_\Lambda Z^\mathsf{T} = \mathrm{diag}(2, 2, c')$ exactly. The glue index [Λ :

$W(-1)\oplus T] = \left|\det\left(\begin{smallmatrix}X\\ Z\end{smallmatrix}\right)\right|$ is the determinant of the last three columns of Z, namely 512, 512 and 1024, and $\det W\cdot\det T = [\Lambda : W(-1)\oplus T]^2$ as unimodularity requires. The glue vectors themselves are not needed for any statement of Section 3 and are not listed. We computed the complement basis Z from G_Λ alone, as an integer basis of the solutions $z\in\mathbb{Z}^{22}$ of $XG_\Lambda z = 0$.

$$
G_\Lambda(W_{512}) = \begin{pmatrix}
-4 & -2 & -2 & -1 & -2 & 0 & -2 & -2 & 2 & -2 & 2 & 1 & 1 & 2 & -2 & -1 & -2 & -2 & 1 & -5 & -9 & -9\\
-2 & -4 & -2 & 1 & -2 & 1 & -2 & -1 & 2 & 0 & 0 & 1 & -1 & 0 & -1 & -2 & -1 & -1 & 0 & -1 & -7 & -8\\
-2 & -2 & -4 & -1 & -1 & -1 & -2 & -2 & 2 & 0 & 0 & 1 & 0 & 1 & -2 & 0 & -2 & -2 & 1 & -4 & -8 & -7\\
-1 & 1 & -1 & -4 & -1 & -2 & -1 & -2 & -1 & -2 & 0 & -1 & 1 & 2 & -2 & 1 & 0 & -1 & 2 & -6 & -5 & -6\\
-2 & -2 & -1 & -1 & -4 & -1 & -1 & -2 & 1 & -2 & 1 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & -3 & -8 & -9\\
0 & 1 & -1 & -2 & -1 & -4 & -1 & -2 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 2 & -3 & -5 & -3\\
-2 & -2 & -2 & -1 & -1 & -1 & -4 & -1 & 2 & -1 & 0 & -1 & -1 & 1 & -1 & -2 & 0 & -1 & 2 & -3 & -8 & -7\\
-2 & -1 & -2 & -2 & -2 & -2 & -1 & -4 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & -2 & 1 & -5 & -8 & -8\\
2 & 2 & 2 & -1 & 1 & 0 & 2 & 0 & -4 & 0 & -1 & -1 & 0 & -1 & 1 & 1 & 2 & 1 & -1 & 1 & 5 & 2\\
-2 & 0 & 0 & -2 & -2 & -1 & -1 & -1 & 0 & -4 & 1 & -1 & 0 & 2 & -1 & 0 & -1 & 0 & 1 & -5 & -6 & -8\\
2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & -4 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 2 & 3 & 1\\
1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 0 & -4 & -1 & 0 & 0 & -1 & 2 & 1 & 1 & -1 & 1 & 0\\
1 & -1 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -4 & -2 & 1 & -1 & 1 & 2 & 0 & 2 & -2 & -3\\
2 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & -1 & 2 & -1 & 0 & -2 & -4 & 2 & -1 & 1 & 2 & -1 & 5 & 3 & 3\\
-2 & -1 & -2 & -2 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 1 & 2 & -4 & 0 & -1 & -1 & 1 & -5 & -5 & -7\\
-1 & -2 & 0 & 1 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -4 & 1 & 0 & 0 & 1 & -3 & -4\\
-2 & -1 & -2 & 0 & -1 & 0 & 0 & -1 & 2 & -1 & 1 & 2 & 1 & 1 & -1 & 1 & -4 & -1 & 0 & -3 & -5 & -4\\
-2 & -1 & -2 & -1 & 0 & 0 & -1 & -2 & 1 & 0 & 1 & 1 & 2 & 2 & -1 & 0 & -1 & -4 & 0 & -4 & -4 & -3\\
1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -4 & 3 & 5 & 3\\
-5 & -1 & -4 & -6 & -3 & -3 & -3 & -5 & 1 & -5 & 2 & -1 & 2 & 5 & -5 & 1 & -3 & -4 & 3 & -14 & -16 & -18\\
-9 & -7 & -8 & -5 & -8 & -5 & -8 & -8 & 5 & -6 & 3 & 1 & -2 & 3 & -5 & -3 & -5 & -4 & 5 & -16 & -34 & -35\\
-9 & -8 & -7 & -6 & -9 & -3 & -7 & -8 & 2 & -8 & 1 & 0 & -3 & 3 & -7 & -4 & -4 & -3 & 3 & -18 & -35 & -44
\end{pmatrix},
$$

$$
Z = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 2 & 0 & 0\\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0\\
-81 & -95 & -31 & -27 & -95 & -56 & -22 & -77 & -124 & -108 & -69 & -35 & -35 & -42 & -101 & -116 & -95 & -12 & -47 & 0 & 0 & 128
\end{pmatrix}. \tag{S5.5}
$$

$$
G_\Lambda(W'_{512}) = \begin{pmatrix}
-4 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & -2 & 2 & 1 & -1 & 1 & 2 & 2 & -2 & 1 & -2 & 1 & 2 & 1 & 3\\
2 & -4 & -2 & -2 & -2 & -1 & 1 & -1 & 1 & -2 & -2 & -1 & 1 & 0 & 0 & 2 & -2 & 2 & 1 & -2 & 0 & -4\\
2 & -2 & -4 & -2 & -1 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 2 & 1 & -2 & -1 & -5\\
2 & -2 & -2 & -4 & 0 & -2 & 1 & -2 & 0 & -1 & -2 & 1 & 1 & -1 & -1 & 0 & -1 & 2 & 0 & -1 & -2 & -4\\
1 & -2 & -1 & 0 & -4 & 1 & 1 & -1 & 0 & -1 & -1 & -2 & -1 & 0 & 1 & 1 & -2 & 0 & 1 & -3 & -1 & -4\\
1 & -1 & -2 & -2 & 1 & -4 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 2 & 1 & 1 & -1 & -4\\
1 & 1 & 0 & 1 & 1 & -1 & -4 & 1 & 2 & 0 & 2 & 1 & -1 & -1 & -1 & 1 & 2 & 0 & 0 & 2 & 2 & 2\\
1 & -1 & -2 & -2 & -1 & -2 & 1 & -4 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 1 & -1 & -2 & -4\\
-2 & 1 & 0 & 0 & 0 & 1 & 2 & -1 & -4 & 1 & 0 & 0 & 1 & 2 & 1 & -1 & 0 & -1 & 0 & -1 & -1 & 1\\
2 & -2 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -4 & -2 & 0 & 1 & -1 & -1 & 2 & -1 & 1 & -1 & -2 & 0 & 0\\
1 & -2 & 0 & -2 & -1 & 0 & 2 & -1 & 0 & -2 & -4 & -1 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & -2 & -2 & -3\\
-1 & -1 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & -1 & -4 & 0 & 1 & 2 & 0 & -1 & -1 & 1 & -2 & -1 & -4\\
1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -4 & -2 & -1 & 0 & 1 & -1 & -1 & -1 & -2 & -2\\
2 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 2 & -1 & 0 & 1 & -2 & -4 & -1 & 0 & 1 & 1 & -1 & 0 & -2 & -2\\
2 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & 2 & -1 & -1 & -4 & 1 & 0 & 0 & -2 & -1 & -1 & 0\\
-2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & -1 & 1 & 2 & -1 & 0\\
1 & -2 & -1 & -1 & -2 & 1 & 2 & -1 & 0 & -1 & -2 & -1 & 1 & 1 & 0 & 1 & -4 & 1 & 0 & -3 & -1 & -3\\
-2 & 2 & 2 & 2 & 0 & 2 & 0 & 2 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -4 & -1 & 0 & 0 & 2\\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & -2 & 1 & 0 & -1 & -4 & -2 & -2 & 1\\
2 & -2 & -2 & -1 & -3 & 1 & 2 & -1 & -1 & -2 & -2 & -2 & -1 & 0 & -1 & 2 & -3 & 0 & -2 & -6 & -4 & -6\\
1 & 0 & -1 & -2 & -1 & -1 & 2 & -2 & -1 & 0 & -2 & -1 & -2 & -2 & -1 & -1 & -1 & 0 & -2 & -4 & -6 & -7\\
3 & -4 & -5 & -4 & -4 & -4 & 2 & -4 & 1 & 0 & -3 & -4 & -2 & -2 & 0 & 0 & -3 & 2 & 1 & -6 & -7 & -16
\end{pmatrix}, \tag{S5.6}
$$

$$
Z = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 2 & 0\\
-58 & -26 & -106 & -43 & -55 & -96 & -22 & -3 & -8 & -2 & -25 & -102 & -53 & -61 & -12 & -61 & -76 & -15 & -56 & 0 & 0 & 128
\end{pmatrix}.
$$

$$G_\Lambda(W_{1024}) = \begin{pmatrix}
-4 & 1 & 1 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & -2 & -2 & 1 & 2 & -2 & 2 & -2 & -2 & -6 & -5 & -8 \\
1 & -4 & -2 & 0 & 2 & -1 & -1 & 2 & 1 & 1 & -2 & 0 & 1 & -2 & -2 & 0 & -2 & 2 & 0 & 1 & 0 & -2 \\
1 & -2 & -4 & -1 & 1 & 0 & 0 & 2 & 2 & 0 & -2 & -1 & 0 & -2 & 0 & 1 & -1 & 2 & -1 & -1 & -1 & -4 \\
-2 & 0 & -1 & -4 & -2 & -2 & -2 & 0 & -1 & 0 & 0 & -1 & -2 & 1 & 2 & -1 & 1 & -1 & -2 & -6 & -5 & -10 \\
-2 & 2 & 1 & -2 & -4 & 0 & -1 & -2 & -1 & -1 & 1 & 0 & -1 & 2 & 2 & 0 & 2 & -1 & -1 & -3 & -2 & -4 \\
-2 & -1 & 0 & -2 & 0 & -4 & -1 & 0 & -2 & 2 & 0 & -2 & -2 & 0 & 0 & -1 & 0 & -1 & -1 & -5 & -5 & -7 \\
-2 & -1 & 0 & -2 & -1 & -1 & -4 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & -1 & -1 & -3 & -3 & -7 \\
-2 & 2 & 2 & 0 & -2 & 0 & -1 & -4 & -2 & 1 & 2 & 0 & -1 & 2 & 1 & 0 & 2 & -1 & -1 & -1 & -2 & -1 \\
-2 & 1 & 2 & -1 & -1 & -2 & -1 & -2 & -4 & 2 & 2 & -1 & -1 & 1 & 0 & -1 & 1 & -2 & -1 & -4 & -4 & -3 \\
1 & 1 & 0 & 0 & -1 & 2 & 1 & 1 & 2 & -4 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 3 & 3 \\
2 & -2 & -2 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & -4 & 0 & 0 & -1 & -1 & 2 & -2 & 2 & 1 & 2 & 1 & 0 \\
-2 & 0 & -1 & -1 & 0 & -2 & 0 & 0 & -1 & 1 & 0 & -4 & -2 & -1 & 1 & -1 & 1 & -1 & -1 & -6 & -5 & -6 \\
-2 & 1 & 0 & -2 & -1 & -2 & 0 & -1 & -1 & 1 & 0 & -2 & -4 & 1 & 2 & 0 & 1 & -1 & -2 & -5 & -5 & -7 \\
1 & -2 & -2 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & -1 & -1 & 1 & -4 & -2 & 0 & -1 & 2 & 0 & 0 & 1 & 1 \\
2 & -2 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & 2 & -2 & -4 & 1 & -2 & 2 & 1 & 4 & 3 & 5 \\
-2 & 0 & 1 & -1 & 0 & -1 & -2 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 1 & -4 & 1 & -2 & 0 & -4 & -2 & -4 \\
2 & -2 & -1 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & -2 & 1 & 1 & -1 & -2 & 1 & -4 & 1 & 1 & 3 & 2 & 1 \\
-2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 2 & -1 & -1 & 2 & 2 & -2 & 1 & -4 & 0 & -4 & -3 & -3 \\
-2 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -2 & 0 & 1 & 0 & 1 & 0 & -4 & -4 & -4 & -8 \\
-6 & 1 & -1 & -6 & -3 & -5 & -3 & -1 & -4 & 2 & 2 & -6 & -5 & 0 & 4 & -4 & 3 & -4 & -4 & -16 & -13 & -20 \\
-5 & 0 & -1 & -5 & -2 & -5 & -3 & -2 & -4 & 3 & 1 & -5 & -5 & 1 & 3 & -2 & 2 & -3 & -4 & -13 & -12 & -18 \\
-8 & -2 & -4 & -10 & -4 & -7 & -7 & -1 & -3 & 3 & 0 & -6 & -7 & 1 & 5 & -4 & 1 & -3 & -8 & -20 & -18 & -36
\end{pmatrix},$$

$$Z = \begin{pmatrix}
0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\
-177 & -39 & -246 & -208 & -73 & -64 & -53 & -212 & -30 & -132 & -200 & -29 & -112 & -43 & -50 & -221 & -227 & -136 & -239 & 0 & 0 & 256
\end{pmatrix}. \tag{S5.7}$$

In each case the last three columns of Z form the matrix $\mathrm{diag}(2, 2, c')$, whose determinant $4c'$ is the glue index.

S5.2 The admissible genera and their complements

Recall from Section 3.2 that a genus is admissible when it is even of signature $(0, 19)$, its discriminant form q is defined on a group with $l \leq 3$ generators, and some even positive-definite rank-3 lattice T has $q_T = -q$. Its members are then the orthogonal complements $T^\perp$ of the primitive embeddings $T \hookrightarrow \Gamma_{3,19}$. Lemma 3.3 concerns the admissible genera with determinant in the window $\{24, 26, \ldots, 64\}$, which are thus obtained from the discriminant forms of the even positive-definite ternary lattices in the same determinant range. The reduced ternary forms of determinant at most 64 fall into 90 classes and 67 genera; 13 of these genera have determinant at most 22 and fall under Lemma 3.2, and 54 lie in the window. In Table S6 we list the 54 genera, each specified by a ternary partner T. In Table S7 we add for each rank-19 genus its Siegel mass, between 1.7383×10^{-7} (row 2) and 7.0978×10^{-3} (row 54), and a lower bound on $|\mathrm{Aut}|$ of every class, between 141 and 5752731. We do not enumerate the rank-19 genera themselves class by class. Table S7 also gives the genus average $E(2)$ of the number of roots, used in the next paragraph, which falls from $2067090112512/31513745731 \approx 65.5933$ (row 4) to $127875870/3202291 \approx 39.9326$ (row 53).

Genus averaging alone does not force a root. The root count of a class is $E(2) + a_f(2)$, genus average plus cusp-form coefficient. In all 54 genera the root count of $E_8{+}E_8{+}T$ exceeds $E(2)$ by a relative excess between 6.363 (determinant 24) and 11.099 (determinant 64). The symmetric inequality $|a_f(2)| < E(2)$, which would force a root in every class, is therefore false. A sign-aware bound on the negative part of $a_f(2)$ might cover the window, but that is a conjecture there and is not used here.

We now state the linear system used in the Niemeier-complement argument of Section 2.2 to prove Lemma 3.3. Fix an admissible genus $\mathcal{G}$ with ternary partner T, let $\mathcal{G}'$ be the genus of the positive-definite lattices $L(-1)$, $L \in \mathcal{G}$, and let C be the genus of even positive-definite rank-5 lattices with discriminant form $-q_T$, the complement genus of Section 2.2. A class L of $\mathcal{G}$ is root-free exactly when the positive-definite lattice $L(-1) \in \mathcal{G}'$ has minimum at least 4.

(i) To describe C, note that for every K with $q_K = -q_T$ the lattices T and K glue to an even unimodular positive-definite lattice of rank 8, which is E_8. Hence the classes of C are the orthogonal complements of the primitive embeddings $T \hookrightarrow E_8$, and enumerating these embeddings up to $\mathrm{Aut}(E_8)$ lists C. The list is complete when $\sum_K 1/|\mathrm{Aut}(K)|$ equals the Siegel mass of C as in Eq. (9) (Table S6). Table S8 gives the Gram matrix, $|\mathrm{Aut}(K)|$ and the number n_K of roots of each of the 98 classes.

(ii) Suppose that $L' \in \mathcal{G}'$ is root-free. Since $q_{L'} = q_T = -q_K$, the lattice L' and any $K \in C$ glue to an even unimodular positive-definite lattice N of rank 24 in which both are primitive, a Niemeier lattice [21]. Every K in Table S8 has $n_K \geq 4$ roots, and these are roots of N, so that N is not the Leech lattice. By Venkov's theorem [22], Eq. (13), the root system $R(N)$ then has a single Coxeter number h with $|R(N)| = 24h$ and $\sum_{r\in R(N)} rr^{\mathsf{T}} = 2h\, I_{24}$, and h lies in the set

$$H = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 22, 25, 30, 46\} \tag{S5.8}$$

of the 18 Coxeter numbers of the 23 Niemeier lattices with roots.

(iii) To project onto K, let $k_1, \ldots, k_5$ be the basis of K with Gram matrix G_K, and attach to each root $r \in R(N)$ the integer vector $x(r) = (r \cdot k_1, \ldots, r \cdot k_5)$, the coordinates of its projection $r_K \in K^*$. Pairing Venkov's identity with k_i and k_j gives

$$\sum_{r\in R(N)} x(r)\, x(r)^{\mathsf{T}} = 2h\, G_K\,, \qquad |R(N)| = 24h\,. \tag{S5.9}$$

We sort the roots by their projections. A root with $x(r) = 0$ would lie in L', which has minimum 4, so there is none. A root with $r_K \in K \setminus \{0\}$ is a root of K, because $r - r_K \in L'$ has norm $2 - r_K^2 \leq 0$ and vanishes; these n_K roots contribute the known integer matrix $Q_K = \sum x(r)x(r)^{\mathsf{T}}$, summed over the roots of K, to (S5.9). Every other root has $r_K = v \in K^* \setminus K$ with $0 < t_v := v^2 < 2$. Two roots with the same projection v differ by a vector of L' of norm at least 4. Their components along L', of norm $2 - t_v$, therefore have pairwise inner products at most $-t_v$, and positivity of the Gram matrix of these components bounds the number of roots over v by $\lfloor 2/t_v \rfloor$. Let $\mathcal{S}(K)$ be the finite set of such v modulo sign and $m_v \geq 0$ the number of roots projecting to $\pm v$. For a root-free $L' \in \mathcal{G}'$ glued to K inside a Niemeier lattice of Coxeter number h, the counts m_v then solve

$$\sum_{v\in\mathcal{S}(K)} m_v\, x(v)\, x(v)^{\mathsf{T}} = 2h\, G_K - Q_K\,, \qquad \sum_{v\in\mathcal{S}(K)} m_v = 24h - n_K\,, \qquad 0 \leq m_v \leq 2\lfloor 2/t_v \rfloor\,, \tag{S5.10}$$

fifteen linear equations from the entries of a symmetric 5×5 matrix and one from the count, with integer right sides. This is the system $\mathrm{LP}(K, h)$ of Eq. (13).

Write $b \in \mathbb{Z}^{16}$ for the right sides of (S5.10), a_v for the column of the unknown m_v and $c_v = 2\lfloor 2/t_v \rfloor$ for its cap. Any solution satisfies $y \cdot b = \sum_v m_v\, (y \cdot a_v) \geq -\sum_v c_v \max(0, -y \cdot a_v)$ for every $y \in \mathbb{Q}^{16}$. A rational vector y with

$$y \cdot b + \sum_{v\in\mathcal{S}(K)} c_v \max(0, -y \cdot a_v) < 0 \tag{S5.11}$$

therefore shows that (S5.10) has no solution; we call such a y a Farkas vector for $\mathrm{LP}(K, h)$. If (S5.10) is infeasible for every $h \in H$ and a single class $K \in C$, no root-free class exists in $\mathcal{G}'$, and every class of $\mathcal{G}$ contains a root.

For every $h \in H$ and every one of the 98 classes, 1,764 = 18 × 98 pairs in all, we have found a Farkas vector y satisfying (S5.11). The left sides of (S5.11) at these vectors are rational numbers, the largest being -20. One class per genus suffices for the lemma, so that neither the remaining classes nor the 54 mass identities of Table S6 are needed for the proof. Of the 54 genera, 48 occur as the complement genus of at least one of the 44,463 rank-≤ 3 candidates whose support of smaller determinant has its determinant in the window. The six that occur for no candidate, rows 44, 45, 49, 50, 52 and 54 at determinants 60 and 64, are nevertheless covered by the lemma.

The second route named in Section 2.2 of the main text uses modular forms. The theta series of a positive-definite even lattice L of rank c and its weighted average over the genus $\mathcal{G}$ of L are

$$\theta_L(\tau) \;=\; \sum_{x\in L} q^{(x,x)/2} \;=\; 1+\sum_{n\geq 1} r_L(n)\, q^{\,n}\,, \qquad \frac{1}{\mathrm{mass}(\mathcal{G})}\sum_{[L']\in\mathcal{G}} \frac{\theta_{L'}(\tau)}{|\mathrm{Aut}(L')|} \;=\; E_{\mathcal{G}}(\tau)\,, \tag{S5.12}$$

with $q=e^{2\pi i\tau}$. Here $r_L(n)$ counts the vectors of norm $2n$ ($r_L(1)$ counts roots), and by the Siegel–Weil formula the genus average $E_{\mathcal{G}}$ is an Eisenstein series computable from local data [12]. The difference $\theta_L - E_{\mathcal{G}}$ lies in the finite-dimensional space $\mathcal{S}_{\mathcal{G}}$ of cusp forms of weight $c/2$, with level and character fixed by the genus. Hence the root count of a class is the genus average plus a cusp-form coefficient, and a root-free class $L\in\mathcal{G}$ would satisfy

$$r_L(1) \;=\; 0\,, \qquad r_L(n) \;\geq\; 0 \;\;(n\geq 2)\,, \qquad \theta_L - E_{\mathcal{G}} \;\in\; \mathcal{S}_{\mathcal{G}}\,. \tag{S5.13}$$

In a basis of $\mathcal{S}_{\mathcal{G}}$ the system (S5.13) is a linear feasibility problem with rational data.

We impose (S5.13) at both cusps $i\infty$ and 0, since at the cusp 0 the Fricke involution of the level relates θ_L to the theta series of the rescaled dual lattice L^*, whose coefficients must also be nonnegative. If the problem is infeasible, linear-programming duality gives a Farkas certificate, a vector of rational multipliers combining the constraints into an inequality that cannot hold. We check the multipliers by a finite computation in rational arithmetic, and their existence proves that every class in the genus contains a root.

For 32 of the 54 genera, marked in column ϑ of Table S6, the theta-series system (S5.13) at the level of the genus (column N of the same table), with the sign and integrality conditions on the coefficients imposed at both cusps, is also infeasible with a rational Farkas vector, which proves those cases a second time. For the other 22 genera the Niemeier route is the only proof. At the cusp at 0 the transformed theta series counts the vectors of the dual lattice by their norm scaled by N. The Siegel–Weil formula then implies further equalities at this cusp, since at an index where the genus average of these nonnegative counts vanishes the count vanishes in every class. The marks in column ϑ do not depend on these equalities and refer in all 32 rows to Farkas vectors for the system (S5.13) without them, although for rows 3, 5, 31 and 50 the equalities shorten the search for such a vector.

As an illustration, consider row 1 of Table S6, of determinant 24. Its ternary partner is $T=\mathrm{diag}(2,2,6)$, with discriminant group $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6$ on three generators, and the genus consists of the lattices $T^{\perp}$ for the primitive embeddings $T\hookrightarrow\Gamma_{3,19}$. The primitive embeddings of T into E_8 have a single complement class K, the first row of Table S8, with $|\mathrm{Aut}(K)|=192$. The mass of C is $1/192 = 1/|\mathrm{Aut}(K)|$, and no class is missing. This K has $n_K=14$ roots and $|\mathcal{S}(K)|=66$. For each $h\in H$ the system (S5.10) thus has 66 unknowns and 16 equations, and in all eighteen cases we have found a Farkas vector y satisfying (S5.11). Hence every class of this genus contains a root, and the 3133 candidates with this complement genus are excluded. For this genus the theta-series system at level $N=12$ is infeasible as well.

Table S6: Each of the 54 admissible genera of Lemma 3.3 contains a root in every class, proven by the Niemeier route of Section S5.2 for all 54 and a second time by the theta-series route for the 32 marked in column ϑ. A row is an even genus of signature $(0,19)$, fixed by its determinant and by its discriminant form $q = -q_T$, where T is the even positive-definite ternary lattice whose Gram matrix has rows $(2a,t,s)$, $(t,2b,r)$, $(s,r,2c)$, read from the column $a\ b\ c\ r\ s\ t$ (Brandt–Intrau coefficients); the members of the genus are the complements $T^{\perp}$ of the primitive embeddings $T \hookrightarrow \Gamma_{3,19}$. Columns: A_T, the invariant factors of the discriminant group, 2·2·6 standing for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$; l, its number of generators; h_T, the number of classes in the genus of T, any of which specifies the same row (the table lists one reduced representative); $\#K$ and mass(C), the number of classes and the Siegel mass of the rank-5 complement genus C of discriminant form $-q_T$, whose classes are listed in Table S8 and satisfy $\sum_K 1/|\mathrm{Aut}(K)| = \mathrm{mass}(C)$ in every row; N, the level of the genus; rows, the number of rank-≤ 3 candidates of Section 3.2 whose support of smaller determinant has this complement genus (44,463 in all, none for six genera). The number of Farkas vectors satisfying (S5.11) for a row, one for each class K and each of the 18 Coxeter numbers $h \in H$, is $18 \times \#K$, 1,764 in all.

#	det	A_T	l	T: $a\ b\ c\ r\ s\ t$	h_T	$\#K$	mass(C)	N	rows	ϑ
1	24	2·2·6	3	1 1 3 0 0 0	1	1	1/192	12	3133	✓
2	24	2·2·6	3	1 2 2 −2 0 0	1	2	5/2304	12	2318	✓
3	24	24	1	1 2 2 1 1 1	1	1	1/144	48	5296	✓
4	24	24	1	1 1 4 0 0 −1	1	1	1/240	48	1644	✓
5	26	26	1	1 2 2 −1 0 −1	1	2	7/480	52	7230	✓
6	28	28	1	1 1 5 1 1 1	1	2	5/768	56	1462	✓
7	28	28	1	1 1 4 −1 −1 0	2	1	1/96	56	4315	✓
8	30	30	1	1 1 4 0 −1 0	1	2	13/1440	60	1980	✓
9	30	30	1	1 1 5 0 0 −1	2	1	1/96	60	2787	
10	32	2·2·8	3	2 2 2 2 2 2	1	1	1/768	16	390	✓
11	32	2·4·4	3	1 2 2 0 0 0	1	1	1/192	8	1093	✓
12	32	2·2·8	3	1 1 4 0 0 0	1	1	1/192	16	955	✓
13	34	34	1	1 1 6 1 1 1	2	2	1/40	68	2802	✓
14	36	3·12	2	1 1 6 0 0 −1	1	1	1/192	24	187	✓
15	36	36	1	1 1 5 −1 −1 0	1	2	3/320	72	434	✓
16	36	36	1	1 2 3 −2 −1 0	1	2	3/256	72	947	✓
17	36	3·12	2	2 2 2 1 2 2	1	2	5/1152	24	512	✓
18	38	38	1	1 1 5 0 −1 0	2	2	1/32	76	1868	✓
19	40	2·2·10	3	1 1 5 0 0 0	1	2	13/2304	20	280	✓
20	40	2·2·10	3	1 2 3 −2 0 0	1	2	1/64	20	650	✓
21	40	40	1	1 2 3 −1 0 −1	1	1	1/48	80	899	
22	40	40	1	1 1 7 1 1 1	1	2	1/80	80	96	
23	42	42	1	1 2 3 −1 −1 0	1	2	1/48	84	616	
24	42	42	1	1 1 7 0 0 −1	2	2	5/288	84	404	
25	44	44	1	1 1 6 −1 −1 0	2	2	5/192	88	498	
26	44	44	1	1 2 3 0 −1 0	1	3	61/3840	88	262	
27	46	46	1	1 1 6 0 −1 0	3	2	11/240	92	410	
28	48	48	1	1 2 4 2 1 1	1	2	1/36	96	188	
29	48	48	1	1 1 8 0 0 −1	2	1	1/48	96	103	
30	48	2·2·12	3	1 2 3 0 0 0	1	2	5/384	24	168	✓
31	48	2·2·12	3	2 2 2 0 0 −2	1	1	1/192	24	60	✓
32	48	2·2·12	3	1 1 6 0 0 0	1	2	1/96	24	51	✓
33	50	5·10	2	2 2 2 −1 −1 −1	1	2	13/1440	20	47	✓
34	52	52	1	1 2 4 −2 −1 0	2	2	7/192	104	143	
35	52	52	1	1 1 7 −1 −1 0	2	3	17/768	104	5	
36	54	54	1	1 1 7 0 −1 0	2	3	9/160	108	61	
37	54	3·18	2	2 2 2 1 1 1	1	1	1/96	36	20	✓
38	54	3·18	2	1 1 9 0 0 −1	2	1	1/96	36	11	✓
39	56	2·2·14	3	2 2 3 2 2 2	1	3	25/2304	28	13	✓
40	56	2·2·14	3	1 1 7 0 0 0	2	2	1/32	28	32	✓
41	56	56	1	1 3 3 2 1 1	1	2	1/40	112	9	
42	56	56	1	1 2 4 0 0 −1	2	1	1/24	112	48	
43	58	58	1	1 1 10 1 1 1	3	3	7/96	116	19	
44	60	60	1	1 3 3 1 1 1	1	3	13/768	120	0	
45	60	60	1	1 1 10 0 0 −1	1	3	1/80	120	0	
46	60	60	1	1 2 4 0 −1 0	2	2	5/192	120	7	
47	60	60	1	1 1 8 −1 −1 0	2	2	13/576	120	4	

Table S6 (continued)

#	det	A_T	l	T: $a\ b\ c\ r\ s\ t$	h_T	$\#K$	mass(C)	N	rows	ϑ
48	62	62	1	1 1 8 0 −1 0	3	3	1/12	124	4	
49	64	2·2·16	3	1 3 3 −2 0 0	1	1	1/48	32	0	✓
50	64	2·2·16	3	2 2 3 −2 −2 0	1	1	1/96	32	0	✓
51	64	2·4·8	3	1 2 4 0 0 0	1	2	1/48	16	1	✓
52	64	2·2·16	3	1 1 8 0 0 0	1	1	1/96	32	0	✓
53	64	4·4·4	3	2 2 2 0 0 0	1	1	1/384	8	1	✓
54	64	64	1	1 1 11 1 1 1	2	3	1/15	128	0	

Table S7: Across the rank-19 genera of Table S6 the genus average $E(2)$ of the root number falls from ≈ 65.5933 at determinant 24 to ≈ 39.9326 at determinant 64. Listed by the row number and determinant of Table S6: the Siegel mass $\sum_L 1/|\mathrm{Aut}(L)|$ over the classes L of the genus, as an exact fraction; a lower bound $|\mathrm{Aut}|_{\min}$ on the order of the automorphism group valid for every class of the genus; and the genus average $E(2)$ of the number of roots (the norm-2 coefficient of the Siegel–Weil Eisenstein series of Eq. (S5.12)), exact and to three decimals. We do not enumerate these genera class by class; Lemma 3.3 is proven through the rank-5 complements of Table S8. The $E(2)$ column is the genus average discussed in Section 3.2, which a single class can undershoot.

#	det	mass of the rank-19 genus	$\lvert\mathrm{Aut}\rvert_{\min}$	$E(2)$	$\approx$
1	24	6058752757810589/11619303595714805760000	1917772	2063076375508/31513745731	65.466
2	24	5626556391995831/32368060016634101760000	5752731	54290658386/829393369	65.458
3	24	82996613120693/158857666347663360000	1914026	28205907968/431695147	65.338
4	24	6058752757810589/11641997548050186240000	1921518	2067090112512/31513745731	65.593
5	26	10765418216602033841/5022443979247724789760000	466535	4440784059922/70611350023	62.891
6	28	40371950752577187377/19372283919955509903360000	479846	13437445937490/221273965267	60.728
7	28	81931910201254361/39161356563335086080000	477975	5948914770680/98343979689	60.491
8	30	428843499994620731/110383384159290654720000	257398	130609342140736/2230568824549	58.554
9	30	1980779078327409053/509796945261987102720000	257372	46400687339472/792518637199	58.548
10	32	2056934555822089/3540256564319354880000	1721133	180825600/3202291	56.468
11	32	8003636403977/2304854534062080000	287976	181534720/3202291	56.689
12	32	8003636403977/2304854534062080000	287976	181537490/3202291	56.690
13	34	42769207164460798451/1784289308416954859520000	41720	941198287607613/17112152560033	55.002
14	36	224173852038991793/22408656934592839680000	99962	44424077238/829393369	53.562
15	36	6058752757810589/302835073139343360000	49984	171504108/3202291	53.557
16	36	296134546947149/14800210341396480000	49978	44415050706/829393369	53.551
17	36	58346619023847179/11619303595714805760000	199143	1644834720/30838501	53.337
18	38	218389565791605690281/3348295986165149859840000	15332	13636733435945620/262135948859769	52.022
19	40	41354789299349159/2397634075306229760000	57978	17620144613348/347470989537	50.710
20	40	3908023586970293537/75525473372146237440000	19326	79289440889982/1563617851771	50.709
21	40	53534569684524569/1032574831259811840000	19289	1084043595264/21419422627	50.610
22	40	205685451945804923/3982788634859274240000	19364	4181311010304/82295676409	50.808
23	42	3031480677446411357/37762736686073118720000	12457	60017218235330/1212909082831	49.482
24	42	418651598900012751293/5215614901526483435520000	12459	107760915341535015/2177557092192547	49.487
25	44	3693181207506354983/30210189348858494976000	8180	356485702069508/7388292236945	48.250
26	44	174742185058932840473/1434983994070778511360000	8213	3386614169660326/69915135857059	48.439
27	46	67490196586097831213/185380707367995310080000	2747	14045821591402368/297034450377269	47.287
28	48	82996613120693/310268879585280000	3739	1534200710/33207319	46.201
29	48	296134546947149/1111269150351360000	3753	38464603515/829393369	46.377
30	48	5626556391995831/2145911479992320000	7491	729402851840/15758474011	46.286
31	48	1557099458757321373/34857910787144417280000	22387	111777480300/2424134287	46.110
32	48	6058752757810589/45387904670760960000	7492	112215823360/2424134287	46.291
33	50	14558585345259918317/113288210058219356160000	7782	10279270560/226660789	45.351
34	52	10765418216602033841/19580678281667543040000	1819	1720735708230350/38765631162627	44.388
35	52	319077282840968825603/582625080299413831680000	1826	39822740676188100/893649813285943	44.562
36	54	20168176988328399/13085466123304960000	649	56522425652/1295085441	43.644
37	54	3070874685465641/11954016044974080000	3893	696968256160/15972720439	43.635
38	54	3070874685465641/11954016044974080000	3893	697180746482/15972720439	43.648
39	56	40371950752577187377/113288210058219356160000	2807	9482141421600/221273965267	42.852
40	56	5981029444691568353/5594479509047869440000	936	37929146695552/885095817201	42.853
41	56	314791023404819387/295021380359946240000	938	74079797248/1725332977	42.937
42	56	81931910201254361/76487024537763840000	934	37854776393728/885095817201	42.769

Table S7 (continued)

#	det	mass of the rank-19 genus	$\lvert\mathrm{Aut}\rvert_{\min}$	$E(2)$	$\approx$
43	58	436963426352045634281З/149017568615042383872000 0	342	957115290147996870/22728034732978627	42.112
44	60	1530127204075918883/1541336191268290560000	1008	1244349246314402/29998328763361	41.481
45	60	15570388712297432б711/15686059854214987776000 0	1008	33597429650488854/809871751540969	41.485
46	60	3763480248822077200 7/3776273668607311872000 0	1004	622174623157201/15057854106781	41.319
47	60	428843499994620731/430344577619066880000	1004	2488698492628804/60225358262823	41.323
48	62	107363329493929699503 29/20089775916990899159040 00	188	174948067244702976/4295655204896707	40.727
49	64	8003636403977/4501669011840000	563	128377344/3202291	40.089
50	64	8003636403977/9003338023680000	1125	16432049295/409893248	40.089
51	64	8003636403977/4501669011840000	563	128377344/3202291	40.089
52	64	8003636403977/9003338023680000	1125	16432550769/409893248	40.090
53	64	2056934555822089/13829127204372480000	6724	127875870/3202291	39.933
54	64	16007272807954/2255230667064375	141	128628081/3202291	40.168

Table S8: For each of the 98 classes K of the rank-5 complement genera C and each of the 18 Coxeter numbers $h \in H$ of Eq. (S5.8), the system (S5.10) built from the Gram matrix listed here is infeasible, which proves Lemma 3.3 for the row of Table S6 to which the class belongs. Columns: row number and determinant of Table S6; the class label; the upper triangle $G_{11}G_{12}G_{13}G_{14}G_{15} \mid G_{22}G_{23}G_{24}G_{25} \mid G_{33}G_{34}G_{35} \mid G_{44}G_{45} \mid G_{55}$ of the Gram matrix G_K of K in the basis $k_1, \ldots, k_5$ of step (iii), one matrix row per group (even, positive definite, $\det G_K = \det T$, discriminant form $-q_T$); the order of $\mathrm{Aut}(K)$; the number n_K of roots of K; and $|\mathcal{S}(K)|$, the number of vectors $v \in K^* \setminus K$ with $0 < v^2 < 2$ modulo sign, which is the number of unknowns m_v in (S5.10). Within each row of Table S6 the values $1/|\mathrm{Aut}(K)|$ sum to $\mathrm{mass}(C)$.

#	det	class	G_K (upper triangle)	$\lvert\mathrm{Aut}(K)\rvert$	n_K	$\lvert\mathcal{S}(K)\rvert$
1	24	K_1	$2\ -1\ -1\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 4$	192	14	66
2	24	K_1	$2\ -1\ -1\ -1\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 6$	2304	24	63
2	24	K_2	$2\ -1\ 0\ 0\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 2$	576	12	70
3	24	K_1	$2\ -1\ 0\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ -1\ -1 \mid 2\ 0 \mid 4$	144	12	71
4	24	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 0 \mid 2\ 0 \mid 6$	240	20	63
5	26	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 1 \mid 2\ 1 \mid 6$	240	20	68
5	26	K_2	$2\ -1\ -1\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 1 \mid 2\ 0 \mid 4$	96	14	71
6	28	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 1 \mid 8$	768	24	71
6	28	K_2	$2\ -1\ -1\ 0\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ -1 \mid 4$	192	14	79
7	28	K_1	$2\ -1\ 0\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ -1 \mid 2\ -1 \mid 4$	96	10	88
8	30	K_1	$2\ -1\ -1\ -1\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 0 \mid 2\ 0 \mid 6$	480	20	78
8	30	K_2	$2\ -1\ 0\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ -1\ 0 \mid 2\ 0 \mid 4$	144	12	83
9	30	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 1 \mid 4\ -1 \mid 4$	96	12	87
10	32	K_1	$2\ 0\ 0\ 0\ -1 \mid 2\ 0\ 0\ -1 \mid 2\ 0\ -1 \mid 2\ -1 \mid 4$	768	8	85
11	32	K_1	$2\ -1\ -1\ 0\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 4$	192	14	78
12	32	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 4\ 0 \mid 4$	192	12	85
13	34	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 0 \mid 2\ 0 \mid 8$	240	20	83
13	34	K_2	$2\ -1\ 0\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ -1 \mid 2\ 0 \mid 4$	48	10	91
14	36	K_1	$2\ -1\ 0\ 0\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ -1 \mid 2\ -1 \mid 4$	192	10	85
15	36	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 1 \mid 2\ 1 \mid 8$	240	20	73
15	36	K_2	$2\ -1\ -1\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ -1 \mid 6$	192	14	75
16	36	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 1 \mid 10$	768	24	72
16	36	K_2	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 1 \mid 4\ 2 \mid 4$	96	12	77
17	36	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 4\ 1 \mid 4$	384	12	85
17	36	K_2	$2\ -1\ 0\ 0\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ -1\ 0 \mid 2\ 0 \mid 4$	576	12	86
18	38	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 1 \mid 4\ 0 \mid 4$	48	12	92
18	38	K_2	$2\ -1\ -1\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 1 \mid 2\ -1 \mid 6$	96	14	91
19	40	K_1	$2\ -1\ -1\ -1\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 10$	2304	24	88
19	40	K_2	$2\ 0\ 0\ 0\ -1 \mid 2\ 0\ 0\ -1 \mid 2\ 0\ -1 \mid 2\ 0 \mid 4$	192	8	103
20	40	K_1	$2\ -1\ -1\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 6$	192	14	89
20	40	K_2	$2\ -1\ 0\ 0\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ 0 \mid 2\ 0 \mid 4$	96	10	90
21	40	K_1	$2\ -1\ 0\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ -1\ -1 \mid 4\ 0 \mid 4$	48	8	100
22	40	K_1	$2\ -1\ -1\ -1\ -1 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 1 \mid 4\ 0 \mid 4$	96	12	94
22	40	K_2	$2\ -1\ -1\ -1\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 1\ 0 \mid 2\ 0 \mid 8$	480	20	88

Table S8 (continued)

#	det	class	G_K (upper triangle)	$\vert\mathrm{Aut}(K)\vert$	n_K	$\vert\mathcal{S}(K)\vert$
23	42	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 1 ∣ 4 1 ∣ 4	96	12	98
23	42	K_2	2 −1 0 0 0 ∣ 2 0 0 0 ∣ 2 0 −1 ∣ 2 0 ∣ 4	96	10	100
24	42	K_1	2 −1 −1 0 −1 ∣ 2 0 0 0 ∣ 2 0 1 ∣ 2 0 ∣ 6	96	14	96
24	42	K_2	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 2 0 ∣ 6	144	12	96
25	44	K_1	2 −1 −1 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 −1 ∣ 6	192	14	100
25	44	K_2	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 4 2 ∣ 4	48	8	103
26	44	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 1 ∣ 12	768	24	80
26	44	K_2	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 2 0 ∣ 10	240	20	84
26	44	K_3	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 −1 ∣ 4	96	12	85
27	46	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 1 ∣ 2 1 ∣ 10	240	20	99
27	46	K_2	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 −1 0 ∣ 4 2 ∣ 4	24	8	110
28	48	K_1	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 4 1 ∣ 4	48	8	106
28	48	K_2	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 −1 0 ∣ 2 0 ∣ 6	144	12	105
29	48	K_1	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 4 −1 ∣ 4	48	12	101
30	48	K_1	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 0 ∣ 4	192	12	101
30	48	K_2	2 0 0 0 −1 ∣ 2 0 0 −1 ∣ 2 0 0 ∣ 2 0 ∣ 4	128	8	104
31	48	K_1	2 0 0 −1 −1 ∣ 2 0 −1 −1 ∣ 2 −1 −1 ∣ 4 1 ∣ 4	192	6	102
32	48	K_1	2 −1 −1 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 6	192	14	101
32	48	K_2	2 −1 0 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 4	192	10	105
33	50	K_1	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 2 0 ∣ 10	480	20	89
33	50	K_2	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 −1 ∣ 4	144	8	94
34	52	K_1	2 −1 −1 0 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 −1 ∣ 8	192	14	112
34	52	K_2	2 0 0 −1 −1 ∣ 2 0 −1 −1 ∣ 2 −1 0 ∣ 4 0 ∣ 4	32	6	118
35	52	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 1 ∣ 14	768	24	97
35	52	K_2	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 4 0 ∣ 4	96	12	110
35	52	K_3	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 0 −1 ∣ 2 −1 ∣ 6	96	10	111
36	54	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 2 0 ∣ 12	240	20	99
36	54	K_2	2 −1 −1 0 −1 ∣ 2 0 0 0 ∣ 2 0 1 ∣ 2 −1 ∣ 8	96	14	101
36	54	K_3	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 −1 0 ∣ 4 0 ∣ 4	24	8	104
37	54	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 1 ∣ 4 −1 ∣ 6	96	12	97
38	54	K_1	2 0 0 −1 −1 ∣ 2 0 −1 0 ∣ 2 0 −1 ∣ 4 −1 ∣ 4	96	6	127
39	56	K_1	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 14	2304	24	89
39	56	K_2	2 −1 −1 0 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 8	192	14	99
39	56	K_3	2 0 0 0 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 4	192	8	99
40	56	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 0 ∣ 6	96	12	108
40	56	K_2	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 2 ∣ 4	48	8	110
41	56	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 4 −1 ∣ 6	48	12	99
41	56	K_2	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 1 ∣ 2 1 ∣ 12	240	20	99
42	56	K_1	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 −1 0 ∣ 4 1 ∣ 4	24	8	113
43	58	K_1	2 −1 −1 0 −1 ∣ 2 0 0 0 ∣ 2 0 1 ∣ 2 0 ∣ 8	96	14	116
43	58	K_2	2 −1 0 −1 0 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 4 0 ∣ 4	24	8	120
43	58	K_3	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 0 −1 ∣ 2 0 ∣ 6	48	10	121
44	60	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 1 ∣ 16	768	24	97
44	60	K_2	2 −1 −1 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 −1 ∣ 4	192	12	106
44	60	K_3	2 −1 0 0 0 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 4 −1 ∣ 4	96	8	110
45	60	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 1 ∣ 6	192	12	108
45	60	K_2	2 −1 −1 −1 0 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 2 0 ∣ 12	480	20	109
45	60	K_3	2 −1 −1 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 −1 ∣ 8	192	14	108
46	60	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 4 −1 −1 ∣ 4 1 ∣ 4	48	6	125
46	60	K_2	2 −1 0 0 0 ∣ 2 0 0 0 ∣ 2 0 −1 ∣ 2 −1 ∣ 6	192	10	124
47	60	K_1	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 −1 −1 ∣ 2 0 ∣ 8	144	12	120
47	60	K_2	2 0 0 −1 −1 ∣ 2 0 −1 −1 ∣ 2 −1 0 ∣ 4 1 ∣ 4	64	6	124
48	62	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 0 1 ∣ 4 0 ∣ 6	48	12	117
48	62	K_2	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 1 ∣ 4 2 ∣ 6	48	12	119
48	62	K_3	2 −1 0 −1 0 ∣ 2 0 0 0 ∣ 2 −1 0 ∣ 4 −1 ∣ 4	24	8	121
49	64	K_1	2 −1 0 −1 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 4 0 ∣ 4	48	8	117
50	64	K_1	2 −1 0 0 −1 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 6	96	10	110
51	64	K_1	2 −1 −1 0 0 ∣ 2 0 0 0 ∣ 2 0 0 ∣ 2 0 ∣ 8	192	14	111
51	64	K_2	2 0 0 −1 −1 ∣ 2 0 −1 −1 ∣ 2 0 0 ∣ 4 0 ∣ 4	64	6	115
52	64	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 4 2 2 ∣ 4 0 ∣ 4	96	6	131
53	64	K_1	2 0 −1 −1 −1 ∣ 2 −1 −1 −1 ∣ 4 0 0 ∣ 4 0 ∣ 4	384	4	123
54	64	K_1	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 2 0 ∣ 14	240	20	105
54	64	K_2	2 −1 −1 −1 −1 ∣ 2 0 0 0 ∣ 2 1 0 ∣ 4 0 ∣ 6	48	12	111

54	64	K_3	$2\ {-1}\ 0\ {-1}\ 0 \mid 2\ 0\ 0\ 0 \mid 2\ 0\ {-1} \mid 4\ {-1} \mid 4$	24	8	113

S5.3 Exact data for Theorem 3.5

Let $W_A, W_B \in \{W_{512}, W'_{512}, W_{1024}\}$ and $M \in M_{19}(\mathbb{Z})$ with $\det M \neq 0$, and let $G = M^\mathsf{T} W_A M$ be the Gram matrix of the columns of M in W_A. Then $q(M) = \operatorname{tr}(W_B G)$ as in Eq. (37), and the bounds use three properties of G:

$$\text{(P1)}\quad y^\mathsf{T} G y \geq 4 \quad \text{for } y \in \mathbb{Z}^{19} \setminus \{0\}\,, \qquad \text{(P2)}\quad \det G = \det W_A\,(\det M)^2 \geq D := 512\,, \qquad \text{(P3)}\quad \det\big(Y^\mathsf{T} G Y\big) \geq 12\,, \tag{S5.14}$$

the last for every 19×2 integer matrix Y of rank 2. Here (P1) is root-freeness of W_A and (P2) restates the determinant of G. Property (P3) is Hermite's bound for rank-2 lattices: at minimum 4 the determinant is at least the squared minimum divided by $\gamma_2^2 = 4/3$, that is $16/(4/3) = 12$. This bound is sharp here because all three lattices contain pairs of minimal vectors with inner product ± 2. The data of a bound are integer vectors y_α with rational weights w_α, and rank-2 integer matrices Y_p with rational $L_p = \left(\begin{smallmatrix} l_{1p} & 0 \\ m_p & l_{2p} \end{smallmatrix}\right)$, $l_{1p}, l_{2p} \geq 0$, and $\Lambda_p = L_p L_p^\mathsf{T} \succeq 0$ of determinant $l_{1p}^2 l_{2p}^2$. The residual is

$$\mathcal{R} \;=\; W_B - \sum_\alpha w_\alpha\, y_\alpha y_\alpha^\mathsf{T} - \sum_p Y_p \Lambda_p Y_p^\mathsf{T}\,. \tag{S5.15}$$

If $\mathcal{R} \succ 0$ and all $w_\alpha \geq 0$, then for every such M

$$\begin{aligned} q(M) &= \operatorname{tr}(\mathcal{R}\, G) + \sum_\alpha w_\alpha\, y_\alpha^\mathsf{T} G y_\alpha + \sum_p \operatorname{tr}\big(\Lambda_p Y_p^\mathsf{T} G Y_p\big) \\ &\geq 19\,(D \det \mathcal{R})^{1/19} + 4 \sum_\alpha w_\alpha + \sum_p 2\, l_{1p} l_{2p} \sqrt{12} \;=:\; V_{\text{true}} \\ &\geq 19\, \frac{t}{1000} + 4 \sum_\alpha w_\alpha + \sum_p 2\, l_{1p} l_{2p}\, \beta \;=:\; V\,, \end{aligned} \tag{S5.16}$$

with the integer t and the rational β given by

$$t = \big\lfloor 1000\,(D \det \mathcal{R})^{1/19} \big\rfloor\,, \qquad \beta = \frac{3464101615129909}{10^{15}}\,, \qquad \beta^2 < 12\,; \tag{S5.17}$$

in particular $(t/1000)^{19} \leq D \det \mathcal{R}$, and both replacements lower the bound. We obtain the first term from the arithmetic–geometric mean inequality on the eigenvalues of $G^{1/2} \mathcal{R} G^{1/2}$ together with (P2), the second from (P1), and the third from the 2×2 inequality $\operatorname{tr}\big(\Lambda_p Y_p^\mathsf{T} G Y_p\big) \geq 2\sqrt{\det \Lambda_p\ \det(Y_p^\mathsf{T} G Y_p)}$ together with (P3). Since q is even, $V > 48$ gives $q \geq 50$ and $V > 46$ gives $q \geq 48$.

All five bounds of Table 3 take $D = 512$, also when $W_B = W_{1024}$, because $\det G \geq 512$ for every choice of W_A. Moreover, writing $q_{A,B}$ for the form (35) of the ordered pair (W_A, W_B), we have $q_{A,B}(M) = q_{B,A}(M^\mathsf{T})$, and the bound formed from $W_B = W_{1024}$ thus covers every ordered pair containing W_{1024}. The weights are dyadic rationals with denominators at most 2^{40}, multiplied by a common factor $1023/1024$ that makes $\mathcal{R}$ strictly positive definite, as one verifies by a rational LDL^T factorization with positive pivots. In Table S9 we give, for each of the five bounds, t, the numbers of vectors y_α and of pairs with nonzero Λ_p, the exact rational V, and decimal prefixes of V and V_{true}; the vectors y_α, Y_p and their weights are not tabulated. The values in Table 3 are truncations of V.

In rows 4 and 5 of Table S9, and in the 38 bounds below, weights on unit vectors $y_\alpha = e_j$ may be negative. A negative weight on e_j is valid on the fluxes whose j-th column is a minimal vector

row	W_B	scope	t	n_y	n_p	V (exact)	V	V_{true}	gives
1	W_{1024}	every M	24	195	0	$\dfrac{13245254530664589}{274877906944000}$	48.18595527709 . . .	48.1872387 . . .	$q \geq 50$
2	W_{512}	every M	10	105	0	$\dfrac{41004589043118677}{879609302220800}$	46.61682060386 . . .	46.6181517 . . .	$q \geq 48$
3	W'_{512}	every M	9	103	0	$\dfrac{819922474416771261}{17592186044416000}$	46.60719664666 . . .	46.6199819 . . .	$q \geq 48$
4	W_{512}	columns minimal	2	485	3600	$\dfrac{3236774333023505560301398680789356351499492673}{633825300114114700748351602688000000000000000}$	51.06729460690 . . .	51.0831432 . . .	$q \geq 50$
5	W'_{512}	columns minimal	2	280	3600	$\dfrac{1558706606751224585043442115963081932936421899}{316912650057057350374175801344000000000000000}$	49.18410819102 . . .	49.1918105 . . .	$q \geq 50$

Table S9: The five bounds of Table 3 as exact rational numbers: each V is the value of Eq. (S5.16) for one choice of data, valid for every integer flux M with $\det M \neq 0$ in the scope named in its row. Since $q(M)$ is even, $V > 48$ gives $q \geq 50$ ($N_{\text{flux}} \geq 25$) and $V > 46$ gives $q \geq 48$ ($N_{\text{flux}} \geq 24$); the statements about N_{flux} hold under (H1)–(H4) as in Theorem 3.5. Rows are numbered as in Table 3; W_B is the lattice from which the residual $\mathcal{R}$ of Eq. (S5.15) is formed, and $D = 512$ throughout, so that row 1 covers every ordered pair containing W_{1024} and rows 2–5 the determinant-512 pairs. Columns: $t = \lfloor 1000\,(D \det \mathcal{R})^{1/19} \rfloor$; n_y, the number of vectors y_α with nonzero weight; n_p, the number of pairs with nonzero Λ_p (rows 1–3 use none); V (exact), the rational value; V and V_{true}, decimal prefixes of V and of the value with exact radicals, $V \leq V_{\text{true}}$; "gives", the resulting bound on q. Scope "columns minimal" means $G_{jj} = 4$ for all j; in these rows the weights on e_j are negative at $j \in \{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$ (row 4) and $j \in \{0, 1, 2, 3, 4, 5, 9, 12\}$ (row 5), indices from 0 in the bases of Eqs. (S5.1)–(S5.2).

of W_A, that is $G_{jj} = 4$, because for them $w_\alpha(G_{jj} - 4) = 0$ identically and (S5.16) holds as stated. Rows 4 and 5 allow negative weights at all nineteen j and bound q on fluxes all of whose columns are minimal; statements about rows follow by applying a bound to M^{T}, whose columns are the rows of M measured in W_B.

A reader can test the conventions on the identity flux $M = I$: on the ordered pair (W_A, W_B) it has $q(I) = \operatorname{tr}(W_B W_A)$, and Eqs. (S5.1)–(S5.3) give

$$q(I) \;=\; \begin{array}{c|ccc} & W_{512} & W'_{512} & W_{1024} \\ \hline W_{512} & 832 & 296 & 320 \\ W'_{512} & 296 & 798 & 350 \\ W_{1024} & 320 & 350 & 934 \end{array} \qquad (\text{rows } W_A,\ \text{columns } W_B), \tag{S5.18}$$

all even, symmetric under exchange of the pair, and far above the tadpole bound $q \leq 48$. The determinant bound (36) alone gives $19\,(512^2)^{1/19} = 36.6386879218\ldots$ on the determinant-512 pairs and $19\,(512 \cdot 1024)^{1/19} = 38$ exactly on the mixed pairs. Row 1 of Table S9 gives $q \geq 50$ on the five ordered pairs containing W_{1024}. On the four determinant-512 ordered pairs, rows 2 and 3 give $q \geq 48$. Rows 4 and 5, applied to M and to M^{T} (for the pair (W_{512}, W'_{512}), row 5 to the columns and row 4 to the rows), give $q \geq 50$ unless M has a non-minimal column and a non-minimal row.

For the clause of Theorem 3.5 on two columns and two rows we use 38 further bounds of the form (S5.16), one for each $W_B \in \{W_{512}, W'_{512}\}$ and each index $J \in \{0, \ldots, 18\}$. In the bound labeled J the column J is released from minimality: negative weights occur on e_j for $j \neq J$ only, so that the bound holds on every flux whose columns other than the J-th are minimal. All 38 bounds, listed in Table S10, exceed 48, the smallest being $48.93761820967\ldots$ for W_{512} (at $J = 0$)

	$W_B = W_{512}$			$W_B = W'_{512}$		
J	V	V_{true}	j with $w < 0$	V	V_{true}	j with $w < 0$
0	$48.937618209\ldots$	$48.945324628\ldots$	$\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$	$48.861752063\ldots$	$48.861859979\ldots$	$\{1, 2, 3, 4, 5, 7, 9, 10, 12\}$
1	$51.007087938\ldots$	$51.014857034\ldots$	$\{0, 3, 4, 5, 6, 7, 8, 13, 17\}$	$48.588627007\ldots$	$48.595864087\ldots$	$\{0, 2, 3, 4, 5, 7, 9, 10, 12\}$
2	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$48.933423074\ldots$	$48.949371386\ldots$	$\{0, 1, 3, 4, 5, 9, 10, 12\}$
3	$50.999868994\ldots$	$51.001027097\ldots$	$\{0, 1, 4, 5, 6, 7, 8, 9, 12, 13\}$	$49.121456183\ldots$	$49.127602423\ldots$	$\{0, 1, 2, 4, 5, 9, 12\}$
4	$51.442907914\ldots$	$51.450147583\ldots$	$\{0, 1, 3, 5, 6, 7, 8, 11, 12, 13\}$	$48.874949789\ldots$	$48.885607159\ldots$	$\{0, 1, 2, 3, 5, 7, 9, 10, 12\}$
5	$51.337892353\ldots$	$51.353158715\ldots$	$\{0, 1, 3, 4, 6, 7, 8, 12, 13\}$	$48.655098154\ldots$	$48.671550321\ldots$	$\{0, 1, 2, 3, 4, 7, 9, 11, 12\}$
6	$51.449596688\ldots$	$51.450788409\ldots$	$\{0, 1, 3, 4, 5, 7, 8, 11, 12, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
7	$51.151687812\ldots$	$51.157814670\ldots$	$\{0, 1, 3, 4, 5, 6, 8, 11, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
8	$51.527630389\ldots$	$51.534728495\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 11, 12, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
9	$51.527394074\ldots$	$51.544594517\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 11, 12, 13\}$	$48.374743789\ldots$	$48.385133031\ldots$	$\{0, 1, 2, 3, 4, 5, 7, 10, 12, 17\}$
10	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
11	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
12	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.145625558\ldots$	$49.161114080\ldots$	$\{0, 1, 2, 3, 4, 5, 9\}$
13	$51.387830166\ldots$	$51.394014410\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 11, 12\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
14	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
15	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
16	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
17	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$
18	$51.067294606\ldots$	$51.083143276\ldots$	$\{0, 1, 3, 4, 5, 6, 7, 8, 9, 13\}$	$49.184108191\ldots$	$49.191810593\ldots$	$\{0, 1, 2, 3, 4, 5, 9, 12\}$

Table S10: All 38 bounds with one released column exceed 48, so on a determinant-512 pair a flux with $q = 48$, that is $N_{\text{flux}} = 24$, has at least two non-minimal columns, and at least two non-minimal rows by the same bounds applied to M^{T}. Each row gives, for $W_B = W_{512}$ and for $W_B = W'_{512}$, the bound of the form (S5.16) with $D = 512$ in which the weight on e_J is nonnegative and the weights on the other unit vectors e_j are free in sign: V is its exact rational value and V_{true} the value with exact radicals, both truncated after nine decimals, and the last column lists the indices j at which the weight is negative (indices from 0 in the bases of Eqs. (S5.1)–(S5.2)). The bound holds on every integer flux M with $\det M \neq 0$ whose columns j in that set are minimal vectors of W_A. A flux whose only non-minimal columns are a and b is outside the scope of both bounds $J = a$ and $J = b$ only if b lies in the set of row a and a in the set of row b; this happens for 55 of the 171 pairs on the left and 40 of 171 on the right. The minima are at $J = 0$ (left) and $J = 9$ (right), Eq. (S5.19).

and $48.37474378941\ldots$ for W'_{512} (at $J = 9$), with exact values

$$\begin{aligned} &\frac{7754475137154665831153667815567204083438658981}{1584563250285286751870879006720000000000000000} \quad (W_{512},\ J = 0), \\ &\frac{3832642062533830703897294827387286461418611577}{792281625142643375935439503360000000000000000} \quad (W'_{512},\ J = 9). \end{aligned} \tag{S5.19}$$

A flux with $q = 48$ on a determinant-512 pair must therefore have at least two non-minimal columns and, by the same bounds applied to M^{T}, at least two non-minimal rows. The negative-weight sets of Table S10 also constrain the positions of the non-minimal columns. If the only non-minimal columns of such a flux are at positions a and b, the bound labeled $J = a$ holds for it unless e_b carries a negative weight in that bound, and likewise the bound labeled $J = b$ unless e_a does. Either bound, where it holds, excludes such a flux, so the position pair (a, b) can occur only if both these weights are negative. This leaves 55 of the 171 position pairs for W_{512} and 40 of 171 for W'_{512}. Finally, inserting $q \leq 48$ into (36) gives $(\det M)^2 \leq (48/19)^{19}/512^2 = 169.3265930014\ldots$, and hence $|\det M| \leq 13$. An annealed minimization over integer flux matrices, roughly 35,000 restarts per pair, never found q below 92 on the determinant-512 pairs, but search minima are upper bounds on the true minimum and prove nothing.

Neither the bounds of Tables S9 and S10 nor congruences at $p \in \{2, 3\}$ exclude a solution of Eq. (39), a tadpole-saturating flux on a determinant-512 pair. We treat local solvability case by case: a case consists of one of the three unordered determinant-512 pairs (W_A, W_B), a prime $p \in \{2, 3\}$

and one of the 26 integers d with $0 < |d| \le 13$, 156 cases in all. The fourth ordered pair follows by transposition, and primes $p \ge 5$ are not treated. In every case we have found an integer matrix $M_0 \in M_{19}(\mathbb{Z})$ with

$$\operatorname{tr}\Big(M_0 W_B M_0^{\mathsf{T}} W_A\Big) \equiv 48\,, \qquad \det M_0 \equiv d\,, \tag{S5.20}$$

where the two congruences hold modulo 2^{42} and 2^{41} respectively for $p = 2$ and both modulo 3^{27} for $p = 3$, and M_0 satisfies the following spanning condition.

Let $P = W_A M_0 W_B$, half the gradient of q at M_0, and let $P' = \det(M_0)\, M_0^{-\mathsf{T}}$ be the cofactor matrix, the gradient of the determinant. The spanning condition is that the 361 pairs (P_{ij}, P'_{ij}) reduced modulo p span $\mathbb{F}_p^2$. Replacing M_0 by $M_0 + p^k\Delta$ with $k \ge 1$ changes q by $2p^k \operatorname{tr}(P^{\mathsf{T}}\Delta)$ modulo $2p^{k+1}$ and det by $p^k \operatorname{tr}(P'^{\mathsf{T}}\Delta)$ modulo p^{k+1}, and leaves P and P' unchanged modulo p. From the congruences (S5.20) and the spanning condition we therefore obtain, by Hensel lifting, a p-adically convergent sequence of integer matrices whose limit solves $q = 48$ and $\det = d$ in $M_{19}(\mathbb{Z}_p)$. The absence of a local obstruction at $p \in \{2, 3\}$, used in Section 3.3, follows from the existence of such an M_0 in each of the 156 cases, and we do not tabulate the matrices M_0 here.

S6 Arithmetic data for the charge-22 flux

The flux constructed in Section 3.4 of the main text is specified by the following data, which consist of integers and algebraic numbers given in closed form; decimal values, where shown, are truncations. In Section S6.1 we give the number field, the elements δ and ε_S in the basis of powers of ζ_{66}, and the Gram matrices of the lattices L and $\Lambda = U \oplus L$ of Eq. (41). In Section S6.2 we give the integer matrices of the maps g and $\tilde{g}$ and the flux matrix N in the convention of Eq. (19). We then expand the characteristic polynomial of $S = g\tilde{g}$, list the twelve distinct eigenvalues with the sign of the intersection form on each eigenspace, and show that the smoothness clause of (FS) and the tadpole condition hold. Sections S6.1 and S6.2 give the data behind Table 4 of the main text and the computations that Section 3.4 summarizes. In Section S6.3 we identify Λ, together with an isometry of order 66, with the second cohomology lattice of the K3 surface of Kondō [37] that carries an automorphism of order 66, by an explicit integer change of basis onto the corresponding entry of the database of [35]. Throughout this section $\zeta = \zeta_{66}$ and $K := \mathbb{Q}(\zeta)$; in Sections 3–5 of the main text the letter K has other meanings. We write $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$, a bar denotes complex conjugation, and $\mathcal{P} = \{1, 5, 7, 13, 17, 19, 23, 25, 29, 31\}$ indexes the ten real places $\sigma_k : \zeta \mapsto e^{2\pi i k/66}$ of the maximal real subfield $\mathbb{Q}(\zeta + \zeta^{-1})$, as in Section 3.4. Matrices act on column vectors of coordinates: the j-th column of the matrix of a map holds the coordinates of the image of the j-th basis vector.

S6.1 The trace form on $\mathbb{Z}[\zeta_{66}]$

The field K has degree $[K : \mathbb{Q}] = 20$, the cyclotomic polynomial Φ_{66} is irreducible over $\mathbb{Q}$, and $\mathbb{Z}[\zeta]$ is the full ring of integers of K, with $\mathbb{Z}$-basis $1, \zeta, \ldots, \zeta^{19}$, the power basis. The discriminant d_K of K satisfies $|d_K| = 3^{10}\, 11^{18} = 328{,}307{,}557{,}444{,}402{,}776{,}721{,}569$. The set $\mathcal{P}$ is a system of representatives of $(\mathbb{Z}/66)^\times/\{\pm 1\}$, one for each pair of complex-conjugate embeddings of K. The element $\delta = (\zeta_3 - \zeta_3^{-1})(\zeta_{11} - \zeta_{11}^{-1})^9$ of Eq. (40), with $\zeta_3 = \zeta^{22}$ and $\zeta_{11} = \zeta^6$, is real, being a product of ten purely imaginary numbers. Its norm $N_{K/\mathbb{Q}}(\delta)$ has absolute value $|d_K|$, and the principal ideal (δ) is the different of $\mathbb{Z}[\zeta]$, because $\delta/\Phi'_{66}(\zeta)$ is a unit. The construction requires a real unit ε of $\mathbb{Z}[\zeta + \zeta^{-1}]$ such that $\varepsilon\delta^{-1}$ is positive at exactly one of the ten real places. We take

$\varepsilon_S = -\,u_7 u_{17} u_{19} u_{23} u_{29}$ with $u_a = (\zeta^a - \zeta^{-a})/(\zeta - \zeta^{-1})$. In the power basis the two elements read

$$\begin{aligned}
\delta = {}& -55\zeta^{19} - 85\zeta^{18} + 270\zeta^{17} - 107\zeta^{16} - 83\zeta^{15} + 54\zeta^{14} - 28\zeta^{13} \\
& - 2\zeta^{12} + 18\zeta^{11} + 20\zeta^{10} + 10\zeta^{9} - 35\zeta^{8} + 150\zeta^{7} - 145\zeta^{6} \\
& - 127\zeta^{5} + 186\zeta^{4} - 17\zeta^{3} - 8\zeta^{2} - 16\zeta - 19\,, \\
\varepsilon_S = {}& 1806\zeta^{19} + 3522\zeta^{18} + 3186\zeta^{17} + 1014\zeta^{16} - 1087\zeta^{15} - 1170\zeta^{14} + 582\zeta^{13} \\
& + 2227\zeta^{12} + 1869\zeta^{11} - 358\zeta^{10} - 2451\zeta^{9} - 4374\zeta^{8} - 4304\zeta^{7} - 2331\zeta^{6} \\
& - 462\zeta^{5} - 566\zeta^{4} - 2505\zeta^{3} - 4320\zeta^{2} - 4096\zeta - 2016\,.
\end{aligned} \tag{S6.1}$$

For k running through $\mathcal{P}$ in increasing order, the signs of $\sigma_k(\delta^{-1})$ are $-$, $+$, $+$, $-$, $-$, $+$, $+$, $-$, $-$, $+$, and $\min_{k\in\mathcal{P}} |\sigma_k(\delta^{-1})| = 0.0012363962\ldots$. The product $\varepsilon_S\delta^{-1}$ is positive at exactly one real place, $k = 1$, and negative at $k = 5, 7, 13, 17, 19, 23, 25, 29, 31$, with $\min_{k\in\mathcal{P}} |\sigma_k(\varepsilon_S\delta^{-1})| = 8.916541579\ldots \times 10^{-6}$. Any other real unit ε for which $\varepsilon\delta^{-1}$ is positive at exactly one real place would serve equally well in the construction, and all data below refer to the choice ε_S.

The lattice L is the group $\mathbb{Z}[\zeta]$ with the symmetric bilinear form

$$b(x, y) \;=\; \operatorname{Tr}\bigl(\varepsilon_S\delta^{-1}\, x\bar{y}\bigr) \;=\; 2\sum_{k\in\mathcal{P}} \sigma_k\bigl(\varepsilon_S\delta^{-1}\bigr)\operatorname{Re}\Bigl(\sigma_k(x)\,\overline{\sigma_k(y)}\Bigr)\,. \tag{S6.2}$$

The second equality follows by grouping the twenty embeddings of K into complex-conjugate pairs, the ten numbers $\sigma_k(\varepsilon_S\delta^{-1})$ being real. Under the isomorphism $L\otimes\mathbb{R} \to \mathbb{C}^{10}$, $x \mapsto (\sigma_k(x))_{k\in\mathcal{P}}$, the space $L\otimes\mathbb{R}$ becomes the b-orthogonal sum of ten real planes P_k, with P_k the k-th factor $\mathbb{C} \cong \mathbb{R}^2$. On P_k the form b is definite, with the sign of $\sigma_k(\varepsilon_S\delta^{-1})$. Each positive place therefore contributes two positive directions of b and each negative place two negative ones, so that the signature is $(2, 18)$. The form is integral on $\mathbb{Z}[\zeta]$ because (δ) is the different, which means that $\operatorname{Tr}(\delta^{-1}z) \in \mathbb{Z}$ for every $z \in \mathbb{Z}[\zeta]$. It is unimodular because $\delta^{-1}\mathbb{Z}[\zeta]$ is the full dual of $\mathbb{Z}[\zeta]$ under the trace pairing and ε_S is a unit.

In the power basis we write $B_{ij} = b(\zeta^i, \zeta^j) = \operatorname{Tr}(\varepsilon_S\delta^{-1}\zeta^{i-j})$, $0 \le i, j \le 19$, for the Gram matrix of L. The entry B_{ij} depends only on $|i-j|$, because $\varepsilon_S\delta^{-1}$ is real, so that B is the symmetric Toeplitz matrix

$$\begin{gathered}
B_{ij} = \tau\bigl(|i-j|\bigr)\,, \qquad \tau(n) = \operatorname{Tr}\bigl(\varepsilon_S\delta^{-1}\zeta^n\bigr)\,, \\
\bigl(\tau(0), \ldots, \tau(19)\bigr) = (12726,\ 12746,\ 12769,\ 12744,\ 12623,\ 12359,\ 11879,\ 11146,\ 10182,\ 9031, \\
7736,\ 6363,\ 5010,\ 3738,\ 2562,\ 1477,\ 480,\ -480,\ -1477,\ -2562)\,,
\end{gathered} \tag{S6.3}$$

and the sequence continues by $\tau(33-n) = -\tau(n)$, a consequence of $\zeta^{33} = -1$. Every diagonal entry equals $\tau(0) = \operatorname{Tr}(\varepsilon_S\delta^{-1}) = 12{,}726$, which makes L even. Moreover $\det B = 1$ and the signature is $(2, 18)$. An even unimodular lattice of indefinite signature is determined by its signature up to isometry [12], and hence $L \cong 2U \oplus 2E_8(-1)$. Multiplication by ζ preserves b, because $\zeta\bar{\zeta} = 1$, and is an isometry of L of order 66 with characteristic polynomial Φ_{66}. We give its matrix M_ζ in Eq. (S6.5) below.

On $\Lambda = U \oplus L$, with U a hyperbolic plane with basis f_1, f_2, we use the basis $(f_1, f_2, 1, \zeta, \ldots, \zeta^{19})$, in which the Gram matrix is

$$d \;=\; d_U \oplus B\,, \qquad d_U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{S6.4}$$

so that Λ is an even lattice of rank 22 with $\det d = -1$ and signature $(3, 19)$. By the same uniqueness theorem $\Lambda \cong 3U \oplus 2E_8(-1)$, the K3 lattice Γ of Section 3.1 [25]. Through this isometry we identify $H^2(Y, \mathbb{Z})$ and $H^2(\tilde{Y}, \mathbb{Z})$ with Λ. Both factors then carry the same basis and the same Gram matrix, so that $\tilde{d} = d$ in the notation of Section 3.1.

S6.2 The flux, its spectrum and the tadpole

The flux acts on L through multiplication by elements of $\mathbb{Z}[\zeta]$, and the matrix of every such multiplication is a polynomial in the matrix M_ζ of multiplication by ζ, which we therefore give first. In the power basis,

$$M_\zeta \;=\; \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1
\end{pmatrix}, \tag{S6.5}$$

whose first nineteen columns encode the shift $\zeta^j \mapsto \zeta^{j+1}$ and whose last column holds the coordinates of ζ^{20} reduced by $\Phi_{66}(\zeta) = 0$. The matrix satisfies $M_\zeta^{\mathsf{T}} B M_\zeta = B$ and $M_\zeta^{33} = -\mathbf{1}$, with $\mathbf{1}$ the identity matrix.

On L the map $\tilde{g}$ is multiplication by $\alpha = 1 + \zeta$ and the map g is multiplication by $\bar{\alpha} = 1 + \zeta^{-1}$. Their matrices are

$$M_{1+\zeta} \;=\; \mathbf{1} + M_\zeta\,, \qquad M_{1+\zeta^{-1}} \;=\; \mathbf{1} + M_\zeta^{-1} \;=\; \mathbf{1} - M_\zeta^{32}\,, \tag{S6.6}$$

the last equality because $\zeta^{-1} = -\zeta^{32}$. The first is Eq. (S6.5) with 1 added to each diagonal entry, and the second, written out in full, is

$$M_{1+\zeta^{-1}} \;=\; \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}, \tag{S6.7}$$

whose first column $(0,0,1,1,0,-1,-1,0,1,1,1,0,-1,-1,0,1,1,0,-1,-1)^{\mathsf{T}}$ lists the coordinates of $1+\zeta^{-1}$. The two maps are adjoint with respect to b: by Eq. (S6.2), $b(\alpha x, y) = \mathrm{Tr}(\varepsilon_S \delta^{-1}\,\alpha x\,\bar{y}) = \mathrm{Tr}(\varepsilon_S \delta^{-1}\,x\,\overline{\bar{\alpha} y}) = b(x, \bar{\alpha} y)$ for all x, y, that is

$$M_{1+\zeta}^{\mathsf{T}}\,B \;=\; B\,M_{1+\zeta^{-1}}\,. \tag{S6.8}$$

Both matrices have determinant $N_{K/\mathbb{Q}}(1+\zeta) = 1$. In particular $1+\zeta$ is a unit, and g and $\tilde{g}$ are invertible on L.

On the hyperbolic plane we take, in the basis f_1, f_2,

$$\begin{aligned}
\tilde{g}_U &= \begin{pmatrix}1&1\\1&2\end{pmatrix}, \qquad g_U = \begin{pmatrix}2&1\\1&1\end{pmatrix}, \qquad \tilde{g}_U^{\mathsf{T}} d_U = d_U\, g_U\,,\\
S|_U &= g_U \tilde{g}_U = \begin{pmatrix}3&4\\2&3\end{pmatrix}, \qquad \tilde{S}|_U = \tilde{g}_U g_U = \begin{pmatrix}3&2\\4&3\end{pmatrix}.
\end{aligned} \tag{S6.9}$$

The block $S|_U$ has trace 6, determinant 1 and characteristic polynomial $x^2 - 6x + 1$, irreducible over $\mathbb{Q}$, with eigenvalues $3 \pm 2\sqrt{2}$. Its eigenvectors are $\sqrt{2} f_1 \pm f_2$, of norm $\pm 2\sqrt{2}$, and those of $\tilde{S}|_U$ are $f_1 \pm \sqrt{2} f_2$, also of norm $\pm 2\sqrt{2}$. We write $\ell_{3\pm 2\sqrt{2}} = \mathbb{R}(\sqrt{2} f_1 \pm f_2)$ and $\tilde{\ell}_{3\pm 2\sqrt{2}} = \mathbb{R}(f_1 \pm \sqrt{2} f_2)$ for these eigenlines in $U \otimes \mathbb{R}$. On the full lattice $\Lambda = U \oplus L$ the flux is then the pair of integral 22×22 matrices

$$\tilde{g} \;=\; \tilde{g}_U \oplus M_{1+\zeta}\,, \qquad g \;=\; g_U \oplus M_{1+\zeta^{-1}} \tag{S6.10}$$

in the basis $(f_1, f_2, 1, \zeta, \ldots, \zeta^{19})$. They are mutually adjoint with respect to d, that is $\tilde{g}^{\mathsf{T}} d = d\, g$, by Eqs. (S6.8) and (S6.9), and $\det g = \det \tilde{g} = 1$.

In the convention of Eq. (19), namely $g = N\tilde{d}$ and $\tilde{g} = N^{\mathsf{T}} d$ [1], and with $\tilde{d} = d$, the flux matrix is $N = g\, d^{-1} = N_U \oplus N_L$ with $N_U = g_U\, d_U^{-1} = g_U\, d_U$ and $N_L = M_{1+\zeta^{-1}} B^{-1}$, an integer matrix because B is unimodular. In the basis $(f_1, f_2, 1, \zeta, \ldots, \zeta^{19})$ on both factors it is

$$N = \left(\begin{array}{cc|cccccccccccccccccccc}
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1072 & -2144 & 544 & 2890 & 749 & -1639 & -1062 & -449 & -339 & 1647 & 2237 & -93 & -575 & -733 & -1695 & -10 & 2711 & 1395 & -1818 & -1616 \\
0 & 0 & 544 & -1600 & -1600 & 1818 & 2023 & -890 & -1085 & 105 & -788 & -308 & 2268 & 528 & -668 & 308 & -812 & -1705 & 1085 & 2490 & -423 & -1818 \\
0 & 0 & 1818 & 2362 & -1600 & -3418 & 0 & 2023 & 928 & 733 & 105 & -2606 & -2126 & 450 & 528 & 1150 & 2126 & -812 & -3523 & -733 & 2490 & 1395 \\
0 & 0 & -323 & 2567 & 1818 & -3095 & -2772 & 1639 & 1690 & -18 & 1072 & -147 & -3448 & -638 & 1025 & -134 & 1450 & 2136 & -2128 & -3523 & 1085 & 2711 \\
0 & 0 & -1639 & -890 & 2023 & 1639 & -1133 & -1133 & -10 & -572 & 321 & 2136 & 327 & -644 & -63 & -953 & -1150 & 1460 & 2136 & -812 & -1705 & -10 \\
0 & 0 & 10 & -1629 & -890 & 2013 & 1629 & -1133 & -1123 & 0 & -572 & 311 & 2126 & 317 & -644 & -53 & -943 & -1150 & 1450 & 2126 & -812 & -1695 \\
0 & 0 & 623 & -439 & -1085 & 305 & 1067 & -10 & -500 & 123 & -339 & -620 & 853 & 338 & -258 & 318 & -53 & -953 & -134 & 1150 & 308 & -733 \\
0 & 0 & -339 & -788 & 105 & 1072 & 321 & -572 & -339 & -216 & -216 & 575 & 884 & 27 & -237 & -258 & -644 & -63 & 1025 & 528 & -668 & -575 \\
0 & 0 & 575 & 236 & -788 & -470 & 497 & 321 & 3 & 236 & -216 & -791 & 0 & 309 & 27 & 338 & 317 & -644 & -638 & 450 & 528 & -93 \\
0 & 0 & 1165 & 2812 & -308 & -3771 & -1312 & 2136 & 1476 & 545 & 575 & -1956 & -3121 & 0 & 884 & 853 & 2126 & 327 & -3448 & -2126 & 2268 & 2237 \\
0 & 0 & -1165 & 1072 & 2268 & -961 & -2283 & 327 & 961 & -312 & 884 & 1165 & -1956 & -791 & 575 & -620 & 311 & 2136 & -147 & -2606 & -308 & 1647 \\
0 & 0 & -575 & -668 & 528 & 1025 & -63 & -644 & -258 & -237 & 27 & 884 & 575 & -216 & -216 & -339 & -572 & 321 & 1072 & 105 & -788 & -339 \\
0 & 0 & 339 & -236 & -668 & 189 & 686 & -63 & -305 & 81 & -237 & -312 & 545 & 236 & -216 & 123 & 0 & -572 & -18 & 733 & 105 & -449 \\
0 & 0 & -623 & -1356 & 308 & 1773 & 489 & -953 & -676 & -305 & -258 & 961 & 1476 & 3 & -339 & -500 & -1123 & -10 & 1690 & 928 & -1085 & -1062 \\
0 & 0 & -10 & -1705 & -812 & 2136 & 1460 & -1150 & -953 & -63 & -644 & 327 & 2136 & 321 & -572 & -10 & -1133 & -1133 & 1639 & 2023 & -890 & -1639 \\
0 & 0 & 1639 & 1629 & -1705 & -2451 & 497 & 1460 & 489 & 686 & -63 & -2283 & -1312 & 497 & 321 & 1067 & 1629 & -1133 & -2772 & 0 & 2023 & 749 \\
0 & 0 & 323 & 3034 & 1085 & -3846 & -2451 & 2136 & 1773 & 189 & 1025 & -961 & -3771 & -470 & 1072 & 305 & 2013 & 1639 & -3095 & -3418 & 1818 & 2890 \\
0 & 0 & -1818 & -423 & 2490 & 1085 & -1705 & -812 & 308 & -668 & 528 & 2268 & -308 & -788 & 105 & -1085 & -890 & 2023 & 1818 & -1600 & -1600 & 544 \\
0 & 0 & -544 & -2362 & -423 & 3034 & 1629 & -1705 & -1356 & -236 & -668 & 1072 & 2812 & 236 & -788 & -439 & -1629 & -890 & 2567 & 2362 & -1600 & -2144 \\
0 & 0 & 1072 & -544 & -1818 & 323 & 1639 & -10 & -623 & 339 & -575 & -1165 & 1165 & 575 & -339 & 623 & 10 & -1639 & -323 & 1818 & 544 & -1072
\end{array}\right) \tag{S6.11}$$

with the vertical rule separating the coordinates on U from those on L. Because N_L involves B^{-1}, the entries of N are much larger than those of g and $\tilde{g}$. In the power basis used here 398 of the 484 entries are nonzero and $\max_{i,j} |N_{ij}| = 3{,}846$; both numbers are particular to this basis. Multiplying out gives $Nd = g$ and $N^{\mathsf{T}} d = \tilde{g}$ with g, $\tilde{g}$ as in Eq. (S6.10), and the charge $\frac{1}{2}\operatorname{tr}(N\tilde{d}N^{\mathsf{T}}d) = \frac{1}{2}\operatorname{tr} g\tilde{g} = 22$, which does not depend on the basis.

The operator $S = g\tilde{g}$ equals $S|_U \oplus M_{(1+\zeta^{-1})(1+\zeta)}$ with $(1+\zeta^{-1})(1+\zeta) = 2+\zeta+\zeta^{-1}$, and $\tilde{S} = \tilde{g}g$ equals $\tilde{S}|_U \oplus M_{2+\zeta+\zeta^{-1}}$, which coincides with S on L. Both are self-adjoint with respect to d. Since $\operatorname{Tr}(\zeta) = \operatorname{Tr}(\zeta^{-1}) = \mu(66) = -1$, with μ the Möbius function, the trace is

$$\operatorname{tr} S \;=\; \operatorname{tr} S|_U + \operatorname{Tr}\bigl(2+\zeta+\zeta^{-1}\bigr) \;=\; 6 + \bigl(40 + 2\mu(66)\bigr) \;=\; 6 + 38 \;=\; 44\,, \tag{S6.12}$$

and $\det S = \det \tilde{S} = 1$. In particular neither operator has a kernel, and the kernel clause of the existence criterion (23) holds trivially.

On L the operator S is multiplication by the totally real element $2+\zeta+\zeta^{-1}$, which generates the subfield $\mathbb{Q}(\zeta+\zeta^{-1})$ of degree ten. The minimal polynomial of this element over $\mathbb{Q}$ is $m(x) = x^{10} - 19x^9 + 152x^8 - 666x^7 + 1742x^6 - 2782x^5 + 2665x^4 - 1443x^3 + 390x^2 - 40x + 1$, irreducible of degree ten, with discriminant $\operatorname{disc}(m) = 3^5\, 11^9 = 572{,}981{,}288{,}913$. The characteristic polynomial of $S|_L$ is therefore $m(x)^2$, and that of S and of $\tilde{S}$ is $(x^2-6x+1)\, m(x)^2$, which is Eq. (43). Expanded,

$$\begin{aligned}
\det(x - S) = x^{22} &- 44x^{21} + 894x^{20} - 11136x^{19} + 95209x^{18} - 592708x^{17} + 2781410x^{16} \\
&- 10049472x^{15} + 28311500x^{14} - 62601840x^{13} + 108847544x^{12} - 148482242x^{11} \\
&+ 157932863x^{10} - 129649518x^{9} + 80941466x^{8} - 37668104x^{7} + 12725687x^{6} \\
&- 3015610x^{5} + 479766x^{4} - 48446x^{3} + 2861x^{2} - 86x + 1\,.
\end{aligned} \tag{S6.13}$$

The minimal polynomial of S, and of $\tilde{S}$, is $(x^2-6x+1)\, m(x)$. It is squarefree, and hence both operators are diagonalizable over $\mathbb{R}$. The factors x^2-6x+1 and $m(x)$ are coprime, $\gcd(x^2-6x+1,\, m(x)) = 1$, because the roots $3 \pm 2\sqrt{2}$ of the first do not lie in K: the polynomial x^2-2 is irreducible over $K = \mathbb{Q}(\zeta_{66})$, the conductor of $\mathbb{Q}(\sqrt{2})$ being $8 \nmid 66$. All twelve distinct roots are real and positive, and the ten roots of $m(x)$ lie in the interval $(0,4)$.

We list the eigenvalues of S and the sign of d on each eigenspace in Table S11. On each plane P_k of Eq. (S6.2) the operator S acts as the scalar $\lambda_k = \sigma_k(2+\zeta+\zeta^{-1}) = 2 + 2\cos(2\pi k/66)$, $k \in \mathcal{P}$. These are the ten roots of $m(x)$, each of multiplicity two, and b is definite on the eigenspace P_k with the sign of $\sigma_k(\varepsilon_S \delta^{-1})$, positive for $k=1$ alone. The plane P_1 and the positive eigenline $\ell_{3+2\sqrt{2}}$ in U together span exactly three positive directions, the number of positive directions of $\Lambda \otimes \mathbb{R}$. Every

k	eigenvalue	decimal value	eigenspace of S	sign of d
	$3+2\sqrt{2}$	$5.8284271247\ldots$	line $\ell_{3+2\sqrt{2}} = \mathbb{R}(\sqrt{2}f_1 + f_2)$ in $U \otimes \mathbb{R}$	$+$
	$3-2\sqrt{2}$	$0.1715728752\ldots$	line $\ell_{3-2\sqrt{2}} = \mathbb{R}(\sqrt{2}f_1 - f_2)$ in $U \otimes \mathbb{R}$	$-$
1	λ_1	$3.9909438451\ldots$	plane P_1	$+$
5	λ_5	$3.7776708973\ldots$	plane P_5	$-$
7	λ_7	$3.5721061894\ldots$	plane P_7	$-$
13	λ_{13}	$2.6541359266\ldots$	plane P_{13}	$-$
17	λ_{17}	$1.9048361683\ldots$	plane P_{17}	$-$
19	λ_{19}	$1.5284821289\ldots$	plane P_{19}	$-$
23	λ_{23}	$0.8398861808\ldots$	plane P_{23}	$-$
25	λ_{25}	$0.5525319237\ldots$	plane P_{25}	$-$
29	λ_{29}	$0.1432641339\ldots$	plane P_{29}	$-$
31	λ_{31}	$0.0361426054\ldots$	plane P_{31}	$-$

Table S11: The eigenspaces of $S = g\tilde{g}$ for the charge-22 flux contain exactly three positive directions of d: the eigenline $\ell_{3+2\sqrt{2}}$ in U and the plane P_1 in L. The first two rows are the eigenvalues of the block $S|_U$ of Eq. (S6.9), each of multiplicity one, with f_1, f_2 the hyperbolic basis of U. The remaining rows are the eigenvalues $\lambda_k = 2 + 2\cos(2\pi k/66) = \sigma_k(2 + \zeta + \zeta^{-1})$, $k \in \mathcal{P}$, of S on L, the ten roots of the factor $m(x)$ of the characteristic polynomial (43), each of multiplicity two and carried by the plane $P_k \subset L \otimes \mathbb{R}$ of the embedding pair $\sigma_k, \bar{\sigma}_k$ (Eq. (S6.2)). The last column is the sign of the intersection form d of Eq. (S6.4) on the eigenspace, which on P_k is the sign of $\sigma_k(\varepsilon_S \delta^{-1})$. Every eigenspace is definite. Decimal values are truncated after ten places. The twelve distinct eigenvalues, each counted with its multiplicity, sum to $\operatorname{tr} S = 44$. For $\tilde{S} = \tilde{g}g$ the table is identical except that the two eigenlines in U are $\tilde{\ell}_{3\pm 2\sqrt{2}} = \mathbb{R}(f_1 \pm \sqrt{2}f_2)$.

eigenspace of S and of $\tilde{S}$ is definite, and the vacuum is therefore isolated by the criterion stated below Eq. (23). The positive 3-planes fixed by the flux are the sums of the positive eigenspaces,

$$\begin{aligned} \Sigma \;&=\; \ell_{3+2\sqrt{2}} \oplus P_1 \;=\; \mathbb{R}\,(\sqrt{2}f_1 + f_2) \,\oplus\, P_1\,, \\ \tilde{\Sigma} \;&=\; \tilde{\ell}_{3+2\sqrt{2}} \oplus P_1 \;=\; \mathbb{R}\,(f_1 + \sqrt{2}f_2) \,\oplus\, P_1\,. \end{aligned} \tag{S6.14}$$

The smallest gap between an eigenvalue with positive-definite eigenspace and one with negative-definite eigenspace is $\lambda_1 - \lambda_5 = 0.21327294783632\ldots$. The smallest eigenvalue is λ_{31} and the largest is $3 + 2\sqrt{2}$.

For the smoothness clause of (FS), Eq. (22), we now show that no nonzero vector of Λ is orthogonal to Σ. Write $v = u + x$ with $u = u_1 f_1 + u_2 f_2 \in U$ and $x \in L$. The summand u is d-orthogonal to $L \otimes \mathbb{R}$. For $y \in P_1$, Eq. (S6.2) gives $b(x, y) = 2\,\sigma_1(\varepsilon_S \delta^{-1})\operatorname{Re}\bigl(\sigma_1(x)\overline{\sigma_1(y)}\bigr)$ with $\sigma_1(y)$ ranging over all of $\mathbb{C}$, and hence $v \perp P_1$ forces $\sigma_1(x) = 0$. Since Φ_{66} is irreducible, σ_1 is injective on K and $x = 0$. If in addition $v \perp \sqrt{2}f_1 + f_2$, then $u_1 + \sqrt{2}\,u_2 = 0$ with $u_1, u_2 \in \mathbb{Z}$, which gives $u = 0$. The same argument with $f_1 + \sqrt{2}f_2$ applies to $\tilde{\Sigma}$. We conclude that $\Lambda \cap \Sigma^{\perp} = \{0\}$ on both factors. In particular no root of either lattice is orthogonal to the positive 3-plane, and the smoothness clause holds vacuously on Y and on $\tilde{Y}$. In geometric terms, the generic complex structure in the twistor family of the vacuum metric has Picard number zero on each factor.

Finally, the flux has no component along the point classes ($m = \tilde{m} = 0$ in Eq. (19)). Its charge is $N_{\text{flux}} = \frac{1}{2}\operatorname{tr} S = 22 \le 24$, and the tadpole condition (20) is met with $n_{M2} = 24 - 22 = 2$. Since g and $\tilde{g}$ are invertible, the reduced supports on both factors are all of Λ: the flux has support rank 22 and signature $(3, 19)$. It lies neither on the rank-≤ 3 strata of Theorem 3.4 nor on the definite stratum $(0, 19)$ of Theorem 3.5. We do not give the flux matrix in the basis $3U \oplus 2E_8(-1)$ of [1], and whether $W = 0$ at this vacuum is not decided here.

S6.3 The order-66 isometry and Kondō's K3 surface

Besides the flux, the lattice $\Lambda = U \oplus L$ carries the isometry

$$h \;=\; \mathbf{1}_U \oplus M_\zeta \tag{S6.15}$$

of order 66, with M_ζ the matrix (S6.5) and $\mathbf{1}_U$ the identity on U. Its characteristic polynomial is $(x-1)^2\,\Phi_{66}(x)$, its invariant sublattice (the vectors fixed by h) is U, and its coinvariant sublattice (the orthogonal complement of the invariant one) is L, even and unimodular of signature $(2,18)$. The maps g and $\tilde{g}$ commute with h, because on L they are polynomials in M_ζ by Eq. (S6.6) and on U the isometry h is the identity. Moreover h preserves each plane P_k of Eq. (S6.2), on which multiplication by ζ acts through $\sigma_k(\zeta)$, and hence maps each of the two 3-planes Σ and $\tilde{\Sigma}$ of Eq. (S6.14) to itself.

The pair (L, M_ζ) and the pair (Λ, h) both occur in the literature, the first in the arithmetic theory of lattices and the second in the geometry of K3 surfaces. The block (L, M_ζ) is an ideal lattice in the sense of [32] (see also [33]). In that construction, trace forms $\mathrm{Tr}(a\,x\bar{y})$ on ideals of $\mathbb{Z}[\zeta_n]$ such as Eq. (S6.2) realize the isometries of even unimodular lattices with characteristic polynomial Φ_n; here $n = 66$ and the lattice is $L \cong 2U \oplus 2E_8(-1)$. On the geometric side, a K3 surface with a purely non-symplectic automorphism θ of order 66 is unique up to isomorphism and has Picard lattice U [37, 38]. The possible orders of such automorphisms acting trivially on a unimodular Picard lattice were classified in [36, 37]; see also [39, Sec. 3.2]. The database of [34, 35] accordingly has a single entry of order 66, labeled 0.66.0.1 ($k(66) = 1$ in Table 1 of [34]), which gives $(H^2(X,\mathbb{Z}), \langle\theta^*\rangle)$ for this surface X as an integral Gram matrix G_{BH} together with a distinguished generator of the group. That Gram matrix is even, with $\det G_{\mathrm{BH}} = -1$ and signature $(3,19)$. The matrices of the database act on row vectors, and we denote by A the transpose of the stored generator, so that A acts on column vectors as the other matrices of this section do. The matrix A has order 66, determinant 1, trace 1 and characteristic polynomial $(x-1)^2\,\Phi_{66}(x)$, the same as h; its invariant sublattice is again a hyperbolic plane and its coinvariant sublattice is even and unimodular of signature $(2,18)$.

The pair (Λ, h) is isometric to the database entry for Kondō's surface X, by an explicit integer matrix. Let P be the matrix of Eq. (S6.17), whose j-th column holds the coordinates, in the basis of the database, of the image of the j-th vector of the basis $(f_1, f_2, 1, \zeta, \ldots, \zeta^{19})$ of Λ. Then

$$\det P \;=\; 1\,, \qquad P^{\mathsf{T}} G_{\mathrm{BH}}\, P \;=\; d\,, \qquad A^5 P \;=\; P\,h\,, \tag{S6.16}$$

three identities between integer matrices, with d the Gram matrix (S6.4). Thus P is an isometry of Λ onto the lattice of entry 0.66.0.1 that carries h to A^5, the fifth power of the transposed database generator, or equivalently $AP = Ph^{53}$. Explicitly,

$$
P = \left(\begin{array}{cc|cccccccccccccccccccc}
0 & 0 & 9 & 6 & 9 & 10 & 5 & 0 & 2 & 3 & -1 & -2 & 3 & 6 & 3 & 3 & 7 & 7 & 1 & -1 & 1 & -1 \\
0 & 0 & 12 & 13 & 16 & 11 & 4 & 1 & 1 & -4 & -8 & -5 & 0 & 0 & 0 & 5 & 8 & 4 & -1 & -1 & -4 & -11 \\
0 & 0 & 5 & 8 & 4 & -1 & -1 & -4 & -11 & -16 & -13 & -12 & -13 & -11 & -3 & 0 & -2 & -2 & 0 & -3 & -11 & -13 \\
0 & 0 & -9 & -8 & -11 & -10 & -8 & -11 & -15 & -15 & -12 & -14 & -14 & -10 & -4 & -4 & -3 & 1 & 4 & 1 & -1 & 1 \\
0 & 0 & -13 & -9 & -8 & -6 & -4 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 4 & 6 & 8 & 9 & 13 & 14 & 15 & 14 \\
0 & 0 & 6 & 8 & 9 & 13 & 14 & 15 & 14 & 16 & 15 & 14 & 13 & 14 & 14 & 13 & 14 & 15 & 16 & 14 & 15 & 14 \\
0 & 0 & 22 & 20 & 24 & 26 & 19 & 15 & 16 & 16 & 8 & 6 & 9 & 10 & 4 & 4 & 8 & 7 & 0 & -2 & 0 & -5 \\
0 & 0 & 13 & 14 & 16 & 10 & 3 & 0 & -3 & -10 & -16 & -14 & -13 & -14 & -15 & -9 & -8 & -11 & -15 & -14 & -18 & -25 \\
0 & 0 & -9 & -8 & -11 & -15 & -14 & -18 & -25 & -29 & -27 & -27 & -29 & -25 & -18 & -14 & -15 & -11 & -8 & -9 & -15 & -14 \\
0 & 0 & -23 & -21 & -20 & -18 & -14 & -15 & -16 & -16 & -13 & -14 & -13 & -9 & -3 & 0 & 3 & 9 & 13 & 14 & 13 & 16 \\
0 & 0 & -9 & -3 & 0 & 3 & 9 & 13 & 14 & 13 & 16 & 16 & 15 & 14 & 18 & 20 & 21 & 23 & 28 & 30 & 29 & 29 \\
0 & 0 & 11 & 15 & 14 & 18 & 25 & 29 & 27 & 27 & 29 & 25 & 18 & 14 & 15 & 11 & 8 & 9 & 15 & 14 & 13 & 14 \\
0 & 0 & 11 & 8 & 9 & 15 & 14 & 13 & 14 & 16 & 10 & 3 & 0 & -3 & -10 & -16 & -14 & -13 & -14 & -15 & -9 & -8 \\
0 & 0 & -7 & -8 & -4 & -4 & -10 & -9 & -6 & -8 & -16 & -16 & -15 & -19 & -26 & -24 & -20 & -22 & -24 & -19 & -13 & -16 \\
0 & 0 & -15 & -14 & -13 & -14 & -14 & -13 & -14 & -15 & -16 & -14 & -15 & -14 & -13 & -9 & -8 & -6 & -4 & -1 & -1 & -1 \\
0 & 0 & -9 & -8 & -6 & -4 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 4 & 6 & 8 & 9 & 13 & 14 & 15 & 14 & 16 \\
0 & 0 & -1 & 3 & 4 & 4 & 10 & 14 & 14 & 12 & 15 & 15 & 11 & 8 & 10 & 11 & 8 & 9 & 13 & 16 & 13 & 14 \\
0 & 0 & 2 & 2 & 0 & 3 & 11 & 13 & 12 & 13 & 16 & 11 & 4 & 1 & 1 & -4 & -8 & -5 & 0 & 0 & 0 & 5 \\
0 & 0 & -4 & -8 & -5 & 0 & 0 & 0 & 5 & 8 & 4 & -1 & -1 & -4 & -11 & -16 & -13 & -12 & -13 & -11 & -3 & 0 \\
0 & 0 & -7 & -7 & -3 & -3 & -6 & -3 & 2 & 1 & -3 & -2 & 0 & -5 & -10 & -9 & -6 & -9 & -10 & -5 & 0 & -2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{S6.17}
$$

with the vertical rule after the images of f_1 and f_2, which are the two basis vectors of the invariant hyperbolic plane of the database lattice, listed last in its basis; the entries satisfy $\max_{i,j} |P_{ij}| = 30$.

An isometry carrying h to A^a exists exactly for $a \in \{5, 61\}$: the matrix P above realizes $a = 5$, and a second integer matrix of the same kind realizes $a = 61$, that is, carries h to A^{-5}. Only these two exponents occur, because such an isometry carries the positive-definite eigenplane of h onto that of A^a. This eigenplane of h in $L \otimes \mathbb{R}$ is P_1, on which h acts through $\zeta^{\pm 1}$, whereas for the generator as stored in the database the positive-definite eigenplane is that of $\zeta^{\pm 13}$, and $5 \cdot 13 \equiv -1$, $61 \cdot 13 \equiv 1 \pmod{66}$. As lattices with a cyclic group of isometries of order 66, therefore, $(\Lambda, \langle h \rangle) \cong (H^2(X, \mathbb{Z}), \langle \theta^* \rangle)$ for Kondō's surface X. Under this identification X is the member of the twistor family of the vacuum metric whose period line lies in $P_1 \otimes \mathbb{C}$, and h acts on this period line through $\sigma_1(\zeta)$ or its conjugate. By the argument given for the smoothness clause in Section S6.2, the vectors of Λ orthogonal to P_1 form the sublattice U, so that X is algebraic, with Picard lattice U and transcendental lattice L, whereas the generic member of the family has Picard number zero. Transported by P, the flux becomes a pair of integral matrices PgP^{-1}, $P\tilde{g}P^{-1}$ on the database lattice that commute with its generator, and their product has the characteristic polynomial (S6.13).

S7 Data for the Hulek–Verrill sector

The data in this section support Proposition 4.1 of the main text, the three enclosures of Section 4.2 and the count of 28 vacua at twelve positions in Section 4.3. Besides the items listed in the next paragraph, Section S7.2 contains the table of the twelve vacuum positions cited in Section 4.3 (Table S15) and the Krawczyk operator of Eq. (16) written out in the two real coordinates $(\operatorname{Re}\varphi, \operatorname{Im}\varphi)$ that every enclosure below uses.

In this section we collect the data used in Section 4. Section S7.1 gives the integral frame and the monodromy by which we count the S_6-invariant fluxes, the list of on-cone fluxes of charge at most 3, the Picard–Fuchs operator in the form used for the period evaluations, the origin of the cut $n_2 \leq 751$, and the bounds on the period fields over the box R used in the exclusion of Section 4.2.

Section S7.2 lists the 28 vacua in R, gives the full enclosures of the three vacua used in Section 4.2 and the first-order bound that excludes a vacuum in R for the fluxes on the ray $r = 2$, states what is known at charge 1, and describes the catalog of special points. Section S7.3 gives the map from our frame to the period basis of [15] and the vanishing classes at the three conifold points, among them the charge-1 flux at $\varphi = 1/36$. Throughout, a flux is a vector $g \in \mathbb{Z}^5$ in the orbit basis of Section 4.1, a position is a value of the slice modulus φ, the cut and the box R are those of Eq. (47), and every period evaluation assumes hypothesis (O) of Proposition 4.1.

S7.1 The invariant lattice, the operator and the counts

On the six-parameter family of Eq. (44) the period vector $\hat{\Pi}$ in the integral frame of [8, 13] has $29 = 1 + 6 + 15 + 6 + 1$ components, grouped as $(\hat{\Pi}^0, \hat{\Pi}^I, \hat{\Pi}^{IJ}, \hat{\Pi}_I, \hat{\Pi}_0)$ with $I < J$ running over $1, \ldots, 6$. The intersection pairing Σ on this frame is symmetric, integral and even, with $\det \Sigma = 2^{16}3^6 = 47{,}775{,}744$ and signature $(17, 12)$, and a flux $G \in \mathbb{Z}^{29}$ has $N_{\text{flux}} = \frac{1}{2}\, G^{\mathsf{T}} \Sigma\, G \in \mathbb{Z}$. We take the integrality of this frame from [8, 13]. Three questions about it are left open here: whether the integral span of the frame is the whole flux lattice or a sublattice of finite index in it, the half-integral shift of the flux by $c_2/2$, and the closed form of the monodromy matrices about the conifold points in this frame. On the invariant sublattice, however, the conifold monodromies follow from those of [15] by the frame map of Section S7.3. Under the same map the flux lattice of [15] for this sector contains ours with index 6. That finer lattice is described there as conjecturally integral, so that the first of these three questions is open in the literature as well [8, 13, 15]. The counts below refer to integer vectors g in this frame.

The group S_6 permutes the six indices, and its orbits on the 29 components are the five blocks $\hat{\Pi}^0$, $\hat{\Pi}^I$, $\hat{\Pi}^{IJ}$, $\hat{\Pi}_I$, $\hat{\Pi}_0$, of sizes 1, 6, 15, 6 and 1. Let B be the 29×5 matrix whose column o is the indicator vector of orbit o,

$$B \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_6 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{15} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B^{\mathsf{T}} B \;=\; \operatorname{diag}(1, 6, 15, 6, 1), \tag{S7.1}$$

where $\mathbf{1}_k$ denotes a column of k ones. The S_6-invariant fluxes are exactly $G = Bg$ with $g \in \mathbb{Z}^5$, since a vector fixed by every permutation is constant on orbits. Hence the invariant sublattice is primitive of rank 5.

In terms of g the Euclidean norm of $G = Bg$ in the frame is $n_2(g) = G^{\mathsf{T}} G = g^{\mathsf{T}} (B^{\mathsf{T}} B)\, g$, the positive form of Eq. (47). Hence the caps $|g| \le (27, 11, 7, 11, 27)$ in the same equation are the integer parts of $\sqrt{751/w_i}$ for the diagonal entries w_i of $B^{\mathsf{T}} B$. Of the tuples with $n_2(g) \le 751$, those that also satisfy $1 \le N_{\text{flux}} \le 30$, that is, the tuples in the cut, have $|g_2| \le 8$ and $|g_3| \le 5$. The intersection form of the sublattice is $\mathrm{Gram}_5 = B^{\mathsf{T}} \Sigma\, B$, whose entry for the orbits o and p is the sum of Σ_{ij} over $i \in o$ and $j \in p$. This sum gives the matrix of Eq. (45), and $N_{\text{flux}} = \frac{1}{2}\, g^{\mathsf{T}} \mathrm{Gram}_5\, g$ is the form (46). The matrix Gram_5 has determinant $6480 = 2^4 3^4 5$, and its characteristic polynomial $x^5 - 122x^4 - 11497x^3 - 25728x^2 + 30276x - 6480$ has three positive and two negative real roots, so that its signature is $(3, 2)$.

On the diagonal slice the period vector is invariant under the action of S_6 on the frame [8], hence constant on orbits: $\hat{\Pi} = B\, \Pi(\varphi)$, with Π the five-component period vector of Section 4.1. We write $\langle x, y \rangle = x^{\mathsf{T}} \mathrm{Gram}_5\, y$ for the pairing of the invariant vectors Bx and By. For an invariant flux the superpotential $W = \int G \wedge \Omega$ and its covariant derivative $D_\varphi W = \partial_\varphi W + (\partial_\varphi K) W$, with

$e^{-K} = \langle \Pi, \bar{\Pi} \rangle$ on the slice, then read

$$\begin{aligned} W &= G^{\mathsf{T}} \Sigma \hat{\Pi} = g^{\mathsf{T}} \mathrm{Gram}_5 \, \Pi = u \cdot \Pi, && u = \mathrm{Gram}_5 \, g, \\ e^{-K} D_\varphi W &= \langle \Pi, \bar{\Pi} \rangle \, (u \cdot \Pi') - \langle \Pi', \bar{\Pi} \rangle \, (u \cdot \Pi) = G_g(\varphi), && A_a = \langle \Pi, \bar{\Pi} \rangle \, \Pi'_a - \langle \Pi', \bar{\Pi} \rangle \, \Pi_a. \end{aligned} \tag{S7.2}$$

In the first line u is the flux written in the period frame, as in Eq. (50). By the second line the five fields A_a of Eq. (49) are these bilinears in the periods, with $G_g = \sum_a u_a A_a$, and $G_g = 0$ is $D_\varphi W = 0$. The F-terms of such a flux transverse to the slice vanish on the slice by the S_6 symmetry, and F-flatness in this sector therefore reduces to the single equation (50).

The monodromy matrices $T_1, \ldots, T_6$ about the divisors $\varphi_i = 0$ are integral, preserve Σ and have determinant 1, and their product $T = T_1 \cdots T_6$ is maximally unipotent [8]. Restricted to the invariant sublattice, T acts on g as the matrix

$$M_d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ -20 & -60 & -60 & 1 & 0 \\ 30 & 120 & 180 & -6 & 1 \end{pmatrix}, \qquad \begin{aligned} M_d^{\mathsf{T}} \, \mathrm{Gram}_5 \, M_d &= \mathrm{Gram}_5, \\ (M_d - 1)^5 = 0 &\neq (M_d - 1)^4. \end{aligned} \tag{S7.3}$$

We call the group $\langle M_d \rangle \cong \mathbb{Z}$, generated by T on the invariant fluxes, the *residual monodromy* of the sector. The *classes* of Section 4 are the orbits of $\langle M_d \rangle$ on $\mathbb{Z}^5$ that meet the cut, and we identify fluxes under no other group. In particular the monodromies about the conifold points are not in $\langle M_d \rangle$, the quotient by S_6 is already built into the sublattice, and g and $-g$ are distinct classes. Since M_d preserves Gram_5, the charge N_{flux} is constant on a class, whereas the norm n_2 is not, and a class may therefore have several members that satisfy the cut. We count each class once, however many such members it has.

In Section 4 we therefore count fluxes in two ways (Table S12): as *signed flux vectors*, the 5-tuples $g \in \mathbb{Z}^5$ in the cut with g and $-g$ both counted, and as classes under the residual monodromy. In all, 3,509,207 integer vectors satisfy $n_2(g) \leq 751$, of which 59,360 have $1 \leq N_{\text{flux}} \leq 30$, and these fall into 58,388 classes. At charge at most 3 the 4,102 signed flux vectors form 4,048 classes. Each of the 230 on-cone vectors listed below is the only member of its class in the cut, and the 3,872 off-cone vectors form 3,818 classes (3,764 with one member in the cut and 54 with two), so that $4{,}048 = 3{,}818 + 230$. In the off-cone exclusion of Section 4.2 we test all 55,516 off-cone signed flux vectors in the cut, and hence every class they form.

The 230 on-cone vectors of charge at most 3 are obtained as follows. On the cone $g_2 = g_3 = 0$ the charge is $N_{\text{flux}} = g_1(g_1 + g_5)$, Eq. (55), and g_4 does not enter it. The condition $1 \leq N_{\text{flux}} \leq 3$ then leaves three families: $(\pm 1, 0, 0, g_4, g_5)$ with $g_5 \in \{0, \pm 1, \pm 2\}$ and $g_1 g_5 \geq 0$, at charges 1, 2 and 3; $(\pm 2, 0, 0, g_4, \mp 1)$, at charge 2; and $(\pm 3, 0, 0, g_4, \mp 2)$, at charge 3. In each family the cut $n_2 = g_1^2 + 6g_4^2 + g_5^2 \leq 751$ allows every $|g_4| \leq 11$, the cap of Eq. (47), which gives $2 \cdot 23 \cdot 3 + 2 \cdot 23 + 2 \cdot 23 = 230$ vectors. All of them have $r = g_5/g_1 \leq 2$. The interval test (52) excludes 228 of them, and the remaining two, $\pm(1, 0, 0, 0, 2)$ on the ray $r = 2$, are treated in Section S7.2.

The operator of hypothesis (O) is L_5 of Eq. (51). Ordered by powers of φ it reads $L_5 = \sum_{j=0}^{3} \varphi^j Q_j(\theta)$ with $\theta = \varphi \, d/d\varphi$ and

$$\begin{aligned} Q_0 &= \theta^5, \\ Q_1 &= -\big(56\,\theta^5 + 140\,\theta^4 + 168\,\theta^3 + 112\,\theta^2 + 40\,\theta + 6\big), \\ Q_2 &= 784\,\theta^5 + 3920\,\theta^4 + 8076\,\theta^3 + 8548\,\theta^2 + 4628\,\theta + 1020, \\ Q_3 &= -\big(2304\,\theta^5 + 17280\,\theta^4 + 50688\,\theta^3 + 72576\,\theta^2 + 50688\,\theta + 13824\big). \end{aligned} \tag{S7.4}$$

	signed flux vectors g	classes under $\langle M_d \rangle$
$n_2(g) \le 751$	3,509,207	
$n_2(g) \le 751$ and $1 \le N_{\text{flux}} \le 30$ (the cut)	59,360	58,388
$N_{\text{flux}} = 1$	626	626
$N_{\text{flux}} = 2$	1,996	1,994
$N_{\text{flux}} = 3$	1,480	1,428
$N_{\text{flux}} \le 3$, all	4,102	4,048
off the cone, $(g_2, g_3) \ne (0, 0)$	3,872	3,818
on the cone, $g_2 = g_3 = 0$	230	230
$N_{\text{flux}} \le 30$, off the cone	55,516	
classes with a vacuum in R		28
classes with no vacuum in R		58,360

Table S12: The S_6-invariant fluxes counted in the two ways used in Section 4, as signed flux vectors and as classes; the two counts differ only where a monodromy orbit has more than one member inside the cut. A signed flux vector is a 5-tuple $g \in \mathbb{Z}^5$ in the orbit basis of Eq. (S7.1), with g and $-g$ counted separately. A class is an orbit of the residual monodromy $\langle M_d \rangle$ of Eq. (S7.3), an infinite cyclic unipotent group, on $\mathbb{Z}^5$ that meets the cut $n_2 \le 751$, $1 \le N_{\text{flux}} \le 30$ of Eq. (47), counted once; the S_6 quotient is already taken in the sublattice, g and $-g$ are distinct classes, N_{flux} is constant on a class and n_2 is not. At $N_{\text{flux}} \le 3$ each of the 230 on-cone vectors is a class by itself and 54 off-cone classes have two members in the cut, so that $4{,}102 = 3{,}872 + 230$ and $4{,}048 = 3{,}818 + 230$. The last two rows give the result of the row-by-row enumeration of Section 4.3, in which, under hypothesis (O) of Proposition 4.1, we either prove a vacuum in R or exclude one for each of the classes. Empty cells are counts the text does not use.

The coefficient of $d^5/d\varphi^5$ in L_5 is φ^5 times $1 - 56\varphi + 784\varphi^2 - 2304\varphi^3 = (1 - 4\varphi)(1 - 16\varphi)(1 - 36\varphi)$. Hence the singular points are 0, 1/36, 1/16, 1/4 and ∞. There are no apparent singularities. We collect the local exponents in the scheme

$$
\begin{array}{c|ccccc}
\varphi & 0 & 1/36 & 1/16 & 1/4 & \infty \\
\hline
 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 1 & 1 & 1 & 1 \\
 & 0 & 3/2 & 3/2 & 3/2 & 3/2 \\
 & 0 & 2 & 2 & 2 & 2 \\
 & 0 & 3 & 3 & 3 & 2
\end{array}
\tag{S7.5}
$$

Thus $\varphi = 0$ is a point of maximal unipotent monodromy, in accord with Eq. (S7.3), and 1/36, 1/16, 1/4 are conifold points.

The holomorphic solution of L_5 at $\varphi = 0$ is the fundamental period restricted to the diagonal, $\varpi_0 = \sum_{n \ge 0} c_n \varphi^n$ with

$$
\begin{aligned}
c_n &= \sum_{n_1 + \cdots + n_6 = n} \left(\frac{n!}{n_1! \cdots n_6!} \right)^2 \\
&= 1,\ 6,\ 66,\ 996,\ 18306,\ 384156,\ 8848236,\ 218040696,\ 5651108226,\ \ldots \quad (n = 0, 1, 2, \ldots).
\end{aligned}
\tag{S7.6}
$$

The series determines the operator: among operators L of order 5 and degree 3 in φ, with coefficients of degree at most 5 in θ, the conditions that the coefficients of $\varphi^0, \ldots, \varphi^{60}$ in $L\varpi_0$ vanish have a one-dimensional space of solutions, spanned by (S7.4). No nonzero operator of order at most 4 and degree at most 10, or of order 5 and degree at most 2, satisfies these conditions. The solution (S7.4) agrees coefficient for coefficient with the operator given for this slice in [13], in [14, Eq. (8.24)] and in [15, Eq. (6.4.21)]. That L_5 annihilates ϖ_0 to all orders is a theorem [41, Thm. 1]; the series (S7.6) and the operator also appear under entry 130 of the tables of [42], and independent derivations are given in [43, 44]. That L_5 also annihilates the other four periods of the sector, the property we use in every period evaluation, is the part of hypothesis (O) that we take from [13], as discussed after Proposition 4.1. The box R, the three vacua of Section S7.2 and the real interval sampled there all lie at distance less than 1/36 from the origin, so that every period evaluation in this sector takes place inside the disk about $\varphi = 0$ on which the local solutions of L_5 converge.

The cut $n_2 \le 751$ comes from the bound $\mu_{\rm lo}$ of Eq. (48), which we now state in full. At a point φ of the slice let $H_5(\varphi)$ be the Gram matrix of the Hodge norm on the invariant sublattice, $\|Bg\|^2_{\rm Hodge} = g^{\mathsf T} H_5(\varphi)\, g$, and let $\mu(\varphi)$ be the smallest root of $\det\big(H_5(\varphi) - \mu\, B^{\mathsf T} B\big) = 0$. Then $\|Bg\|^2_{\rm Hodge} \ge \mu(\varphi)\, n_2(g)$ for every g. At an F-flat point the flux is self-dual, $2N_{\rm flux} = G^{\mathsf T}\Sigma\, G = \|G\|^2_{\rm Hodge}$, and a vacuum at φ with $N_{\rm flux} \le 30 = \chi/24$ therefore has $n_2(g) \le 60/\mu(\varphi)$. We define $\mu_{\rm lo}$ as the minimum of μ over the 21 points $\varphi = (a+ib)/1024$ with $a = 13, \ldots, 19$ and $b \in \{-4, 0, 4\}$, which span R. The minimum is attained at the corners $13/1024 \pm i/256$, and in ball arithmetic we obtain

$$\begin{aligned} \mu_{\rm lo} \in \Big[\frac{5756447315870245}{2^{56}},\ \frac{5756447891589797}{2^{56}}\Big] &= [\,0.0798867543765,\ 0.0798867623663\,], \\ \frac{60}{\mu_{\rm lo}} &\in (751.0631076,\ 751.0631828). \end{aligned} \tag{S7.7}$$

Both ends of the last interval have integer part 751. We have not proven a lower bound for μ over all of R, only at these 21 points. For this reason the cut is hypothesis (i) of Proposition 4.1 and is not derived from the tadpole bound.

For the interval test (52) we tabulate outward-rounded enclosures of the ten real components $\operatorname{Re} A_a$, $\operatorname{Im} A_a$ on each of the 3,072 sub-boxes of R. We find that five of the ten keep one sign on all of R, with the bounds of Table S13. For each of the off-cone flux vectors g in the cut, the test (52) fails on every sub-box. We define the *margin* of g in the test (the distance referred to in Section 4.2) as the distance from 0 to an interval of (52) that does not contain 0, minimized over the sub-boxes. The smallest margin among the 3,872 off-cone vectors with $N_{\rm flux} \le 3$ is 22794.4498, for $g = (-3, -1, 0, -4, 0)$, and the smallest among all 55,516 is 5063.7822, for $g = (-2, 1, 0, 9, -1)$, in the normalization of Table S13. On every sub-box the distance from 0 to the failing interval is roughly half the magnitude of the field $\sum_a u_a A_a$ there or more.

S7.2 The vacua in R and the special points

In Table S14 we list the 28 vacua in R found by the row-by-row enumeration of Section 4.3, one row per class. For each of the remaining 58,360 classes we exclude a vacuum in R. Every flux in the table lies on the cone, where $u = (2g_1 + g_5,\ 6g_4,\ 30g_1,\ 0,\ g_1)$ and $N_{\rm flux} = g_1^2(1 + r)$ with $r = g_5/g_1$. Fluxes with proportional u, in particular g and $-g$, have the same zero. The 28 vacua therefore lie at twelve positions. The four positions on the real axis belong to $g_4 = 0$ and $r = 3,\ 7/2,\ 4,\ 9/2$, with $\operatorname{Re}\varphi^*$ decreasing as r grows. For $g_4 \ne 0$ the zero lies off the axis, at $|\operatorname{Im}\varphi^*|$ between 0.0023 and 0.0029 and thus inside the bound 1/256 of R, and the zeros for opposite signs of g_4 are complex conjugates of each other. The charges that occur are 4 and 5, with two vacua each, and 16, 18, 20 and 22, with six each. The vacuum nearest to $\varphi = 1/64$ is that of $\pm(2, 0, 0, 0, 7)$, at distance $2.848\ldots \times 10^{-4}$.

Table S15 groups the 28 vacua of Table S14 by position; the special points with which we compare the twelve positions are described at the end of this subsection.

component	$\mathrm{Re}\,A_1$	$\mathrm{Im}\,A_2$	$\mathrm{Re}\,A_3$	$\mathrm{Im}\,A_4$	$\mathrm{Re}\,A_5$
bound on all of R	≥ 3231	≥ 820.7	≤ -125.5	≤ -2022.6	≤ -8619.2

off-cone signed flux vectors in the cut	how many	smallest margin	attained at g
$N_{\text{flux}} \leq 3$	3,872	22794.4498	$(-3, -1, 0, -4, 0)$
$N_{\text{flux}} \leq 30$	55,516	5063.7822	$(-2, 1, 0, 9, -1)$

Table S13: Five components of the period fields keep a fixed sign on the whole box R, and the interval test fails for every off-cone flux vector in the cut, with the smallest margins listed in the lower table. Upper table: proven one-sided bounds, valid at every point of R of Eq. (47), on five of the ten real components of the fields A_a of Eq. (S7.2), in the normalization of the period vector in which the fields are tabulated; only the signs and the relative sizes enter the argument. Lower table: the margin of a flux vector g is the distance from 0 to an interval of Eq. (52) that fails to contain 0, minimized over the 3,072 sub-boxes; each entry is the smallest margin over the off-cone signed flux vectors ($(g_2, g_3) \neq (0,0)$, inside the cut) obeying the charge bound of its row, together with the vector attaining it. All entries are ball-arithmetic bounds rounded outward, under hypothesis (O).

The Krawczyk radii that appear in Section 4 come from different computations. The radii of Table S14 are those of the enumeration. The largest of them, $2.149\ldots \times 10^{-22}$ for the classes $\pm(2, 0, 0, \pm1, 9)$, is the radius given as at most 2.2×10^{-22} in Section 2.3. In the enumeration the vacuum of the class of $(-1, 0, 0, 0, -3)$, the flux named in Proposition 4.1, is enclosed in a ball of radius 8.14×10^{-23}. For the same vacuum computed on its own, and for the two vacua above R, the enclosures have the radii between 5.634×10^{-23} and 7.853×10^{-23} of Table S16. These are the radii below 10^{-22} of Proposition 4.1 and Section 4.2.

The enclosures of this section come from the Krawczyk operator (16) of Section 2.3 of the main text. The simplest instance below is a single equation $\Psi(\varphi) = 0$ in one complex modulus, such as Eq. (50). As Ψ involves the conjugate periods it is not holomorphic, and we take $n = 2$, $x = (\mathrm{Re}\,\varphi, \mathrm{Im}\,\varphi)$ and $F = (\mathrm{Re}\,\Psi, \mathrm{Im}\,\Psi)$. With X the square of half-side ϱ about $\tilde{x} = (\mathrm{Re}\,\tilde{\varphi}, \mathrm{Im}\,\tilde{\varphi})$, (16) reads

$$\mathcal{K}(\tilde{x}, X) \;=\; \tilde{x} - Y \begin{pmatrix} \mathrm{Re}\,\Psi(\tilde{\varphi}) \\ \mathrm{Im}\,\Psi(\tilde{\varphi}) \end{pmatrix} + \left(I_2 - Y \begin{pmatrix} [\partial_1 \mathrm{Re}\,\Psi](X) & [\partial_2 \mathrm{Re}\,\Psi](X) \\ [\partial_1 \mathrm{Im}\,\Psi](X) & [\partial_2 \mathrm{Im}\,\Psi](X) \end{pmatrix} \right) \begin{pmatrix} [-\varrho, \varrho] \\ [-\varrho, \varrho] \end{pmatrix}, \qquad \text{(S7.8)}$$

where $[\,\cdot\,](X)$ denotes a ball enclosure over X computed from the period balls and their tail bounds.

In Table S16 we give the three vacua used in Section 4.2, each computed independently of the enumeration: the charge-4 vacuum in R and the charge-3 and charge-2 vacua above R. For each we apply the Krawczyk test (16) in the two real coordinates $(\mathrm{Re}\,\varphi, \mathrm{Im}\,\varphi)$. It first succeeds on a box of radius 10^{-13} about an approximate zero, which proves that the box contains exactly one zero of G_g, and we then refine the enclosure to the radii shown. The test also succeeds on the box of radius $1/15625 = 6.4 \times 10^{-5}$ about φ^*, the *uniqueness box* of the table, so that each zero is the only one within that distance. In every case the enclosure of $\mathrm{Im}\,\varphi^*$ contains 0, the lower bound on $|W|$ is positive, and $e^{-K} = \langle \Pi, \bar{\Pi} \rangle > 0$, so that the Kähler potential $K = -\log\langle \Pi, \bar{\Pi} \rangle$ is defined at φ^* and the point lies in the physical domain of the slice. From the signs in the row $\varphi^* - 19/1024$ we see that the charge-4 vacuum is inside R and the other two are above its upper edge. All three lie below the conifold point $1/36$, since the entries of the row $1/36 - \varphi^*$ are positive.

On the cone with $g_4 = 0$ the real zero of G_g increases as r decreases (Table S14), and for the values $r < 3$ treated below, it lies above the upper edge of R. For $g = (2, 0, 0, 0, 5)$, with $r = 5/2$,

g	N_{flux}	n_2	r	$\operatorname{Re}\varphi^*$	$\operatorname{Im}\varphi^*$	radius	$\lvert W\rvert >$	$\lvert\varphi^* - \frac{1}{64}\rvert$
$(1,0,0,0,3)$	4	10	3	$0.0172802004854\ldots$	0^{a}	8.143×10^{-23}	4.07199	0.0016552
$(-1,0,0,0,-3)$	4	10	3	$0.0172802004854\ldots$	0^{a}	8.143×10^{-23}	4.07199	0.0016552
$(1,0,0,0,4)$	5	17	4	$0.0146978699953\ldots$	0^{a}	1.013×10^{-22}	5.53163	0.0009271
$(-1,0,0,0,-4)$	5	17	4	$0.0146978699953\ldots$	0^{a}	1.013×10^{-22}	5.53163	0.0009271
$(2,0,0,0,6)$	16	40	3	$0.0172802004854\ldots$	0^{a}	8.143×10^{-23}	8.14398	0.0016552
$(-2,0,0,0,-6)$	16	40	3	$0.0172802004854\ldots$	0^{a}	8.143×10^{-23}	8.14398	0.0016552
$(2,0,0,1,6)$	16	46	3	$0.0174022589821\ldots$	$-0.0028531301774\ldots$	1.473×10^{-22}	9.13667	0.0033614
$(-2,0,0,-1,-6)$	16	46	3	$0.0174022589821\ldots$	$-0.0028531301774\ldots$	1.473×10^{-22}	9.13667	0.0033614
$(2,0,0,-1,6)$	16	46	3	$0.0174022589821\ldots$	$0.0028531301774\ldots$	1.473×10^{-22}	9.13667	0.0033614
$(-2,0,0,1,-6)$	16	46	3	$0.0174022589821\ldots$	$0.0028531301774\ldots$	1.473×10^{-22}	9.13667	0.0033614
$(2,0,0,0,7)$	18	53	7/2	$0.0159098050613\ldots$	0^{a}	9.169×10^{-23}	9.59372	0.0002848
$(-2,0,0,0,-7)$	18	53	7/2	$0.0159098050613\ldots$	0^{a}	9.169×10^{-23}	9.59372	0.0002848
$(2,0,0,1,7)$	18	59	7/2	$0.0159855530896\ldots$	$-0.0026736706207\ldots$	1.674×10^{-22}	10.47273	0.0026979
$(-2,0,0,-1,-7)$	18	59	7/2	$0.0159855530896\ldots$	$-0.0026736706207\ldots$	1.674×10^{-22}	10.47273	0.0026979
$(2,0,0,-1,7)$	18	59	7/2	$0.0159855530896\ldots$	$0.0026736706207\ldots$	1.674×10^{-22}	10.47273	0.0026979
$(-2,0,0,1,-7)$	18	59	7/2	$0.0159855530896\ldots$	$0.0026736706207\ldots$	1.674×10^{-22}	10.47273	0.0026979
$(2,0,0,0,8)$	20	68	4	$0.0146978699953\ldots$	0^{a}	1.013×10^{-22}	11.06326	0.0009271
$(-2,0,0,0,-8)$	20	68	4	$0.0146978699953\ldots$	0^{a}	1.013×10^{-22}	11.06326	0.0009271
$(2,0,0,1,8)$	20	74	4	$0.0147410635170\ldots$	$-0.0024940074138\ldots$	1.876×10^{-22}	11.85305	0.0026460
$(-2,0,0,-1,-8)$	20	74	4	$0.0147410635170\ldots$	$-0.0024940074138\ldots$	1.876×10^{-22}	11.85305	0.0026460
$(2,0,0,-1,8)$	20	74	4	$0.0147410635170\ldots$	$0.0024940074138\ldots$	1.876×10^{-22}	11.85305	0.0026460
$(-2,0,0,1,-8)$	20	74	4	$0.0147410635170\ldots$	$0.0024940074138\ldots$	1.876×10^{-22}	11.85305	0.0026460
$(2,0,0,0,9)$	22	85	9/2	$0.0136228530299\ldots$	0^{a}	1.164×10^{-22}	12.55070	0.0020021
$(-2,0,0,0,-9)$	22	85	9/2	$0.0136228530299\ldots$	0^{a}	1.164×10^{-22}	12.55070	0.0020021
$(2,0,0,1,9)$	22	91	9/2	$0.0136430107257\ldots$	$-0.0023215845083\ldots$	2.149×10^{-22}	13.26867	0.0030525
$(-2,0,0,-1,-9)$	22	91	9/2	$0.0136430107257\ldots$	$-0.0023215845083\ldots$	2.149×10^{-22}	13.26867	0.0030525
$(2,0,0,-1,9)$	22	91	9/2	$0.0136430107257\ldots$	$0.0023215845083\ldots$	2.149×10^{-22}	13.26867	0.0030525
$(-2,0,0,1,-9)$	22	91	9/2	$0.0136430107257\ldots$	$0.0023215845083\ldots$	2.149×10^{-22}	13.26867	0.0030525

Table S14: All 28 vacua in the box R lie on the cone $g_2 = g_3 = 0$, at twelve positions, each enclosed in a Krawczyk ball of radius below 2.2×10^{-22} that contains exactly one zero of G_g, and each with $W \neq 0$. One row per flux class of the cut of Eq. (47) with an F-flat point in R, under hypothesis (O); g and $-g$ are distinct classes with the same zero, and fluxes with proportional $u = \operatorname{Gram}_5 g$ share a zero. Columns: the representative g; its charge $N_{\text{flux}} = g_1^2(1+r)$; the norm $n_2(g)$; $r = g_5/g_1$; the position φ^* of the zero, truncated, every digit shown being a digit of the zero (0^{a}: the enclosure of $\operatorname{Im}\varphi^*$ is centered at a value smaller in modulus than the radius and contains 0; these are the on-axis vacua); the radius of the ball, to four figures; a number below the proven lower bound on $|W|$ at the vacuum, in the normalization of the frame of Section S7.1; and the distance from φ^* to the special point $1/64$, rounded to the digits shown. The largest radius is $2.149\ldots \times 10^{-22}$ and the smallest distance to $1/64$ is $2.848\ldots \times 10^{-4}$.

fluxes g	N_{flux}	r	$\operatorname{Re}\varphi^*$	$\operatorname{Im}\varphi^*$	radius	$\lvert W\rvert \geq$	$\lvert\varphi^* - 1/64\rvert$
$\pm(1,0,0,0,3)$, $\pm(2,0,0,0,6)$	4, 16	3	0.0172802004...	0	8.143×10^{-23}	4.071993, 8.143986	0.0016552
$\pm(2,0,0,1,6)$	16	3	0.0174022589...	−0.0028531301...	1.473×10^{-22}	9.136674	0.0033614
$\pm(2,0,0,-1,6)$	16	3	0.0174022589...	+0.0028531301...	1.473×10^{-22}	9.136674	0.0033614
$\pm(2,0,0,0,7)$	18	7/2	0.0159098050...	0	9.169×10^{-23}	9.593722	0.0002848
$\pm(2,0,0,1,7)$	18	7/2	0.0159855530...	−0.0026736706...	1.674×10^{-22}	10.472738	0.0026979
$\pm(2,0,0,-1,7)$	18	7/2	0.0159855530...	+0.0026736706...	1.674×10^{-22}	10.472738	0.0026979
$\pm(1,0,0,0,4)$, $\pm(2,0,0,0,8)$	5, 20	4	0.0146978699...	0	1.013×10^{-22}	5.531634, 11.063269	0.0009271
$\pm(2,0,0,1,8)$	20	4	0.0147410635...	−0.0024940074...	1.876×10^{-22}	11.853052	0.0026460
$\pm(2,0,0,-1,8)$	20	4	0.0147410635...	+0.0024940074...	1.876×10^{-22}	11.853052	0.0026460
$\pm(2,0,0,0,9)$	22	9/2	0.0136228530...	0	1.164×10^{-22}	12.550706	0.0020021
$\pm(2,0,0,1,9)$	22	9/2	0.0136430107...	−0.0023215845...	2.149×10^{-22}	13.268674	0.0030525
$\pm(2,0,0,-1,9)$	22	9/2	0.0136430107...	+0.0023215845...	2.149×10^{-22}	13.268674	0.0030525

Table S15: All 28 vacua in R lie on the cone $g_2 = g_3 = 0$, at twelve positions none of which is a special point of the slice, and all have $W \neq 0$; the nearest approach to the one special point in R, $\varphi = 1/64$, is at least 2.8×10^{-4}. One row per position φ^*; the first column lists the fluxes $g = (g_1, 0, 0, g_4, g_5)$ of the invariant sector with a vacuum at that position (g and $-g$ share their zeros, as do proportional fluxes), followed by their charges N_{flux} of Eq. (46) and the ratio $r = g_5/g_1$ of Eq. (55). Each φ^* is the midpoint of a Krawczyk ball, of the radius listed beside it, that is proven to contain exactly one zero of G_g of Eq. (50); the entry 0 for $\operatorname{Im}\varphi^*$ means a midpoint smaller than the radius in absolute value. The column $|W| \geq$ is the proven lower bound on $|u \cdot \Pi|$ at the vacuum, one value per listed flux up to sign. The last column is the distance to 1/64, the only one of the twelve special points of the slice (the conifold points, the diagonal positions of the vacua of Grimm and van de Heisteeg, and the point $\varphi = 1$ of Jockers, Kotlewski and Kuusela) that lies inside R. By charge the count is two vacua each at 4 and 5 and six each at 16, 18, 20 and 22; fluxes with $g_4 \neq 0$ have their vacua off the real axis in conjugate pairs. All rows assume the cut and the box of Eq. (47) and hypothesis (O) of Proposition 4.1; the fluxes are counted as classes under the monodromy T of Section 4.1, with g and $-g$ distinct. Digit strings ending in dots are truncated; Table S14 lists all 28 rows.

$u = (9, 0, 60, 0, 2)$, $N_{\text{flux}} = 14$ and $n_2 = 29$, we enclose the zero in

$$\begin{aligned} \operatorname{Re}\varphi^* &\in [\,0.018831602234793067654712,\ 0.018831602234793067654849\,], \\ \varphi^* - \tfrac{19}{1024} &\in [\,2.7691473479306765,\ 2.7691473479306766\,] \times 10^{-4}, \end{aligned} \tag{S7.9}$$

with $|\operatorname{Im}\varphi^*| \leq 7.77 \times 10^{-23}$. Its uniqueness box $[307, 309]/16384$ in $\operatorname{Re}\varphi$ lies entirely above the edge $19/1024 = 304/16384$. This is the $r = 5/2$ zero of Section 4.2, about 2.8×10^{-4} above R.

The ray $r = 2$ of the cone contains $\pm(1, 0, 0, 0, 2)$ and their multiples, with u proportional to $(4, 0, 30, 0, 1)$. For these fluxes G_g vanishes above R at the charge-3 vacuum of Table S16, and it has no zero inside R. To see this, on each of 48 blocks $\mathcal{B}$ tiling R, with center c and radius ρ (the largest distance from c to a point of $\mathcal{B}$), we verify in ball arithmetic that

$$|G_g(c)| \;>\; \Big(\sup_{\mathcal{B}} |\partial_\varphi G_g| + \sup_{\mathcal{B}} |\partial_{\bar\varphi} G_g|\Big)\, \rho. \tag{S7.10}$$

Since $|G_g(\varphi) - G_g(c)|$ is at most the right side for $\varphi \in \mathcal{B}$, G_g has no zero on $\mathcal{B}$. The difference of the two sides of (S7.10), which is the margin of the first-order bound referred to in Section 4.2, is at least 675.9 and at most 7604.3 over the 48 blocks. By contrast, the off-cone family of Eq. (54), with $u/N \to (5, 0, 30, 0, 1)$, approaches the ray $r = 3$ of the charge-4 vacuum, and its members of large N do have a zero in R, at charges beyond the cut.

The simplest flux of charge 1 is $g = (1, 0, 0, 0, 0)$, on the ray $r = 0$ of the cone, with $n_2 = 1$ and $u = (2, 0, 30, 0, 1)$. Inside R it is one of the 228 on-cone vectors excluded by the interval test, and

	charge 4, in R	charge 3, above R	charge 2, above R
flux g	$(1,0,0,0,3)$ [b]	$(1,0,0,0,2)$	$(1,0,0,0,1)$
$u = \mathrm{Gram}_5\, g$	$(5,0,30,0,1)$	$(4,0,30,0,1)$	$(3,0,30,0,1)$
N_{flux}, n_2, r	4, 10, 3	3, 5, 2	2, 2, 1
$\mathrm{Re}\,\varphi^*$ (midpoint)	0.01728020048542370039 409738415466206210358	0.02058287610565320248 969098182774725634291	0.02463659201867469650 402740477419252692710
radius in $\mathrm{Re}\,\varphi$	6.706×10^{-23}	5.634×10^{-23}	5.857×10^{-23}
radius in $\mathrm{Im}\,\varphi$	7.853×10^{-23}	6.273×10^{-23}	6.138×10^{-23}
first Krawczyk box, radius	10^{-13}	10^{-13}	10^{-13}
uniqueness box, radius	1/15625	1/15625	1/15625
$\lvert W\rvert \geq$	4.071993111705892	2.6566902537836627	1.2969535535636363
e^{-K}	95.5736478647722	81.44738082664485	68.61402034602041
$\varphi^* - 19/1024$	-0.0012744870145763	$+0.0020281886056532$	$+0.0060819045186747$
$\lvert\varphi^* - 1/64\rvert$	0.0016552004854237	0.004957876105653	0.009011592018675
$1/36 - \varphi^*$	0.010497577	0.007194902	0.003141186
radius in Table S14	8.14×10^{-23}		

Table S16: The vacuum of Proposition 4.1 and the two vacua of lower charge just above the box R, each proven to exist and to be locally unique by the Krawczyk test in $(\mathrm{Re}\,\varphi, \mathrm{Im}\,\varphi)$ under hypothesis (O), with $W \neq 0$ and $e^{-K} > 0$. All three fluxes lie on the cone $g_2 = g_3 = 0$ and satisfy the cut of Eq. (47); the two of charge 3 and 2 have their zero outside R, whose upper edge is $\mathrm{Re}\,\varphi = 19/1024$. Rows, in order: the flux and its period-frame image u; charge, norm and $r = g_5/g_1$; the midpoint of the final enclosure of $\mathrm{Re}\,\varphi^*$, printed in full (the zero differs from it by at most the radius in the next row, so the trailing digits specify the midpoint and not the zero); and the radii of the final enclosure in the two coordinates (the enclosure of $\mathrm{Im}\,\varphi^*$ is centered within its radius of 0 and contains 0). The next two rows give the radius of the box on which the test first succeeds and that of the uniqueness box, a larger box about φ^* on which it also succeeds, so that no second zero lies within $1/15625 = 6.4 \times 10^{-5}$. Then follow a lower bound on $|W| = |u \cdot \Pi|$ and the value of $e^{-K} = \langle \Pi, \bar{\Pi} \rangle$ at φ^*, both in the normalization of the frame of Section S7.1 (only $|W| > 0$ and $e^{-K} > 0$ are used). The three distance rows give the signed distance to the upper edge of R, the distance to the special point 1/64 and the distance below the conifold point 1/36, computed from the midpoint and rounded to the digits shown. [b] The enumeration of Section 4.3 represents this class by $-g = (-1,0,0,0,-3)$, the flux named in Proposition 4.1; the zero is the same, and the last row gives the radius of the enumeration's ball for it.

every other flux of charge 1 in the cut is excluded there by Proposition 4.1. Outside R we have only the following floating-point observation, which does not prove absence. Evaluating G_g at 241 points of the real interval [0.0128, 0.0271], which reaches from inside R to just below the conifold point 1/36, we find no sign change for this flux, whereas for the charge-2 and charge-3 fluxes of Table S16 the sampled values change sign across their zeros. We therefore cannot exclude a charge-1 vacuum at smooth points of the slice outside R. The scanned interval does not reach the conifold points, where the periods degenerate and Eq. (50) as used here does not apply. At the conifold point $\varphi = 1/36$, however, the flux $g = (1,0,0,0,0)$ is the vanishing class, with $W = \partial_\varphi W = 0$ (Section S7.3), and gives the solution with $W = 0$ found at that point in [14, 15], so that at charge 1 the existence of a vacuum remains open only at smooth points of the slice outside R.

For the comparison with special points in Section 4.3 we use a fixed catalog of eleven points of the slice with exact algebraic φ: the conifold points 1/36, 1/16 and 1/4 of Eq. (S7.5), and the positions on the diagonal of the ten exact vacua of [8], which have S_6-breaking fluxes outside this sector and two of which lie at conifold points. The slice has a twelfth special point with exact algebraic φ that is not in the catalog, the point $\varphi = 1$, at which fluxes in this sector with $D_\varphi W = 0$

and $W \neq 0$ are found in [13, Secs. 3.4 and 4]; like the three conifold points it lies outside R. Of the eleven only $1/64 = 16/1024$, the position of one of the ten vacua of [8], lies in R. The comparison inside R therefore reduces to this one point, and by the last column of Table S14 none of the 28 vacua lies at $1/64$. The ten vacua of [8], which we recover as a check (Section 2.4), have fluxes that break S_6 and hence lie outside the invariant sublattice, although nine of the ten fluxes satisfy the norm cut, with $n_2 = G^{\mathsf{T}}G \leq 751$; the one labeled $(13, 2, 4)$ in [8] has $n_2 = 1596$. We do not extend the catalog to further complex-multiplication points of the slice, and the comparison with special points is therefore limited to these eleven points (one of the conditions collected at the end of Section 4).

S7.3 The invariant sector at the conifold points

Period bases for the invariant sector of the diagonal slice other than the frame of Section S7.1 are in print: the rational basis of [13] adapted to the $\mathbb{Z}_6$ quotient, the basis of [14, Sec. 8], and the basis Π_{th} of [15, Eq. (6.4.23)]. The basis Π_{th} is given there together with an integral intersection form Σ_{th} of determinant 180, the monodromy matrices about the five singular points, and a flux satisfying the vacuum conditions at each singular point other than $\varphi = 0$ [15, Table 6.3]; flux vacua at these four points are likewise reported in [14, Sec. 8]. Reading Π_{th} with $t = \log\varphi/(2\pi i)$ on $0 < \varphi < 1/36$ and with the normalization of our Π, we find

$$\Pi_{\mathrm{th}} = C\,\Pi, \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \\ 1 & 0 & 30 & 0 & 1 \end{pmatrix}, \qquad \det C = 6, \qquad C^{\mathsf{T}}\,\Sigma_{\mathrm{th}}\,C = \mathrm{Gram}_5, \tag{S7.11}$$

as exact identities, together with $C^{-1}M_0\,C = M_d$ for the monodromy M_0 about $\varphi = 0$ of [15] and M_d of Eq. (S7.3), an identity that fixes the branch of t. In both frames W is the pairing of the flux vector with the period vector through the intersection form, so that a flux g of ours is the flux Cg of [15]. The lattice $(\mathbb{Z}^5, \mathrm{Gram}_5)$ is thereby an even sublattice of index 6 in $(\mathbb{Z}^5, \Sigma_{\mathrm{th}})$, since the Smith form of C is $\mathrm{diag}(1, 1, 1, 1, 6)$ and $6480 = 6^2 \cdot 180$. The finer lattice is described in [15] as conjecturally integral, and which of the two, if either, is the invariant part of $H^4(X, \mathbb{Z})$ is the first of the open questions of Section S7.1; independently of this question, the counts and exclusions of Section 4 refer to integer g. The identities (S7.11) assume the conventions of [15, Eq. (6.4.23)] for the branch of t and for the normalization of the local solutions, as we read them there.

Conjugated by C, the monodromy matrices of [15, Table 6.3] become integral isometries M_c of $(\mathbb{Z}^5, \mathrm{Gram}_5)$ acting on Π, and about the three conifold points c they are reflections,

$$M_c\,x \;=\; x - \frac{2\langle x, g_c\rangle}{Q(g_c)}\,g_c, \tag{S7.12}$$

c	$1/36$	$1/16$	$1/4$
g_c	$(1,0,0,0,0)$	$(6,1,0,0,0)$	$(15,5,1,0,0)$
$N_{\mathrm{flux}} = Q(g_c)/2$	1	6	15

with $\mathrm{rank}(M_c - 1) = 1$ in each case, so that g_c spans the vanishing classes of the sector at c; explicitly,

$$M_{1/36} = \begin{pmatrix} -1 & 0 & -30 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{S7.13}$$

The classes at $1/16$ and $1/4$, which lie off the cone, depend on the path of continuation and are given here for the loops of [15, Table 6.3], which are based in $0 < \varphi < 1/36$ and pass the intermediate conifold points on one side. For loops passing on the other side these classes are replaced by their images under the intermediate reflections, which preserve Gram_5 and therefore the charge.

The class at $1/36$ is the charge-1 flux $g = (1, 0, 0, 0, 0)$ of Section 4.2 and Section S7.2, and $Cg = (1, 0, 0, 0, 1)$ is the flux listed at this point in [15, Table 6.3], of charge 1 in both frames. Since $M_{1/36}$ is a reflection with $M_{1/36}\, g = -g$, continuation around $1/36$ takes $W_g = \langle g, \Pi\rangle$ to $\langle g, M_{1/36}\Pi\rangle = \langle M_{1/36}\, g, \Pi\rangle = -W_g$, and the solutions of L_5 near $1/36$ that change sign in this way are the multiples of the local solution of exponent $3/2$ in the scheme (S7.5). Hence $W_g = u \cdot \Pi$, with $u = (2, 0, 30, 0, 1)$, and its derivative $\partial_\varphi W_g$ both vanish at $\varphi = 1/36$: the charge-1 flux, the vanishing class of this conifold point, gives the solution with $W = \partial_\varphi W = 0$ reported at that point in [14, Sec. 8]; it is the flux listed there in [15, Table 6.3]. In the same way W_g for $g = (6, 1, 0, 0, 0)$ and for $g = (15, 5, 1, 0, 0)$ is a multiple of the local solution of exponent $3/2$ at $1/16$ and at $1/4$, respectively, along the loops just described.

We have also checked the vanishing of W_g at $1/36$ for $g = (1, 0, 0, 0, 0)$ in our own frame, without the monodromy matrices of [15], by transporting Π toward $1/36$ in floating-point arithmetic. In this check, which like the scan of Section S7.2 is numerical and is not a proof, the jet of W_g is proportional to that of $(1/36-\varphi)^{3/2}\big(1+O(1/36-\varphi)\big)$ with coefficient $\lambda = 393.94$ in the normalization of Section S7.1, up to a rank-one defect (the departure of the two jets from exact proportionality) of 7.1×10^{-59}, whereas for the other fluxes tested the defect is 0.045 or more. The solutions at the three conifold points lie where Eq. (50) as used in Section 4 does not apply, and they are not among the vacua counted there.

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