Factorization without clustering and the six-point Grassmannian string integral

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Abstract

Grassmannian string integrals extend the Veneziano amplitude from points along a line to other configurations. The first case that is not known to be a string theory is $K(3,6)$, an integral over six points in the projective plane. $K(3,6)$ is an amplitude, but it is not an S-matrix element since it is not an invariant function of momenta. Nevertheless, it has some S-matrix-like properties, for example it is known to have codimension-one singularities, like the Veneziano amplitude, and to be otherwise analytic. We show that in addition its residue at every pole and every level is a finite sum of polynomials times integer-shifted string integrals at degenerate configurations, so that it factorizes onto its boundaries as the string amplitude does, each residue being an amplitude of the degenerate configuration. What it lacks, at generic kinematics, is the further split of that residue into two smaller amplitudes summed over states. In field theory that split comes from cluster decomposition, which needs a bipartition with one invariant and a two-body subsystem, and three-label kinematics has neither. In string theory the KLT relations build closed-string amplitudes from pairs of open-string amplitudes, a fact usually traced to the left- and right-moving modes of the worldsheet; for $K(3,6)$ the double copy holds even though no worldsheet is known. Here the part of the color-ordered open-string amplitudes is played by ordered versions of $K(3,6)$ and the part of the closed-string amplitude by the corresponding integral over complex configurations. Finally, we prove that the low-energy coefficients are multiple zeta values to all orders, and find that through weight eight they involve only ordinary zeta values $\zeta(n)$ even though $\zeta(3,5)$ might have appeared.

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1 Introduction

The Veneziano amplitude is an integral over the position of one point on a line,

$$A(s,t) \;=\; \int_0^1 dx\, x^{-\alpha(s)-1}\,(1-x)^{-\alpha(t)-1} \;=\; \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \tag{1}$$

and it was later understood to be the four-point tree amplitude of the relativistic open string [1–6]. Its n-point generalization is the Koba–Nielsen integral over n ordered points $z_1 < z_2 < \cdots < z_n$ on the real line modulo $\mathrm{SL}(2,\mathbb{R})$ [7],

$$K(2,n)(s) \;=\; \int_{X^+(2,n)} \omega_{2,n} \prod_{1\le a<b\le n} (ab)^{\alpha' s_{ab}}, \qquad (ab) = z_b - z_a. \tag{2}$$

Here $X(2,n) = \mathrm{Gr}(2,n)/(\mathbb{C}^*)^n$ is the moduli space $\mathcal{M}_{0,n}$ of n distinct points on $\mathbb{P}^1$, $X^+(2,n)$ is its positive part, and $\omega_{2,n}$ is the Parke–Taylor form. In string theory the main properties of these integrals have a physical or mathematical explanation. The poles of Eq. (1) at $\alpha(s) = n$ are the masses of the string's excited states. The residue at each pole is a sum over the states at that mass of a product of two lower-point amplitudes, which is the form that unitarity and cluster decomposition require of any S-matrix [8]. The $(n-3)!$ independent color orderings and the relations of Kawai, Lewellen and Tye, which express closed-string amplitudes as bilinears in open-string amplitudes, follow from the worldsheet description, the two open-string factors coming from the left- and right-moving modes [9]. The multiple zeta values in the low-energy expansion arise because the amplitudes are periods of the moduli space $\mathcal{M}_{0,n}$, and all such periods are multiple zeta values [10–13].

Cachazo, Early, Guevara and Mizera generalized the scattering equations from points on $\mathbb{P}^1$ to points on $\mathbb{P}^{k-1}$ [14]. Arkani-Hamed, He and Lam wrote down the corresponding generalization of Eq. (2), the Grassmannian string integral [15], over the configuration space $X(k,n) = \mathrm{Gr}(k,n)/(\mathbb{C}^*)^n$ of n points in general position in $\mathbb{P}^{k-1}$ modulo projective transformations, of dimension $(k-1)(n-k-1)$. Every $k\times k$ minor of a matrix representative is raised to a kinematic power:

$$K(k,n)(s) \;=\; \int_{X^+(k,n)} \omega_{k,n} \prod_{|I|=k} (I)^{\alpha' s_I}, \qquad \sum_{I\ni a} s_I = 0 \quad (a = 1,\dots,n). \tag{3}$$

Here $X^+(k,n)$ is the positive part, on which all ordered minors are positive, and $\omega_{k,n}$ is its canonical form. The s_I are the generalized Mandelstam invariants of [14], totally symmetric in the k labels of I. The duality $X(k,n) \cong X(n-k,n)$ [14, 15] sends $(3,5)$ to $(2,5)$ and $(4,6)$ to $(2,6)$, so those members are Koba–Nielsen integrals in other coordinates, while $(3,6)$ is sent to itself. The first member of the family that is not an integral over points on a line, and the subject of this paper, is therefore

$$K(3,6)(s) \;=\; \int_{X^+(3,6)} \omega_{3,6} \prod_{1\le a<b<c\le 6} (abc)^{\alpha' s_{abc}}, \tag{4}$$

a four-dimensional integral over six points in the projective plane. The twenty 3×3 minors (abc) in its integrand vanish when a,b,c are collinear.

The exponents in Eq. (4) are three-label invariants s_{abc}, fourteen of them independent, and they are not constructed from two-label invariants s_{ab}. If one does build them from six massless momenta p_a, with the normalization $s_{abc} = \frac{1}{2}(p_a+p_b+p_c)^2$, then

$$s_{abc} \;=\; \tfrac{1}{2}\big(s_{ab} + s_{bc} + s_{ca}\big), \qquad s_{ab} = (p_a+p_b)^2, \qquad \sum_{b\ne a} s_{ab} = 0 \quad (a = 1,\dots,6), \tag{5}$$

and such sums fill only a nine-dimensional linear subspace $\mathcal{P}$ of the fourteen-dimensional kinematic space. Thus $K(3,6)$ is not the S-matrix element of six particles, and the question of its unitarity does not arise. It nevertheless has the analytic form of an amplitude: it is meromorphic in the fourteen invariants, with poles only on sixteen families of evenly spaced parallel hyperplanes $\alpha'\kappa_a(s) = -n$, $n = 0,1,2,\dots$ [15, 53]. Here the κ_a are sixteen linear forms in the invariants, and we call n the level of the pole. At high energy $K(3,6)$ has Regge behavior with linear trajectories, as shown for stringy canonical forms in [15], and at fixed angle we find it exponentially soft at every kinematic point examined. Its $\alpha' \to 0$ limit is the generalized biadjoint amplitude $m(3,6)$ of [14]. No worldsheet theory and no spectrum of states that would produce $K(3,6)$ are known. In this paper we determine which properties of an S-matrix, transcribed to three-label kinematics, $K(3,6)$ has. In particular, on the slice (5), where the invariants are those of six momenta, the three-particle poles of $K(3,6)$ are double, which is impossible for a tree-level S-matrix element (Eq. (8) below).

For $k \geq 3$ almost nothing is known about the α'-expansion of Eq. (3) beyond its leading order. That order is the canonical function of the Newton polytope of the integrand, which for $K(3,6)$ is $P(3,6)$ [15–18]. At finite α' the integrals (3) bear on whether amplitudes of Veneziano type exist outside string theory, a question studied from the dual-model era [21–25] to the current bootstrap program, which deforms the four-point function directly [26–28]. The Grassmannian integrals instead have a Koba–Nielsen integrand on a different integration domain. We find that three properties of string amplitudes hold for $K(3,6)$ although the structure that explains each of them in string theory is absent. The residues factorize onto amplitudes of the degenerate configuration, although no residue splits over a bipartition of the labels and three-label kinematics has no invariant on which cluster decomposition could be stated. The KLT double copy holds although no worldsheet is known, and the low-energy coefficients are multiple zeta values although the domain is not a moduli space of points on a line.

Factorization without clustering. In quantum field theory the residue of a tree amplitude at a pole is a product summed over the states at that mass:

$$\mathrm{Res}_{s_I = m^2} A_n \;=\; \sum_{\text{states}} A_L(I,P)\, A_R(-P,\bar{I}), \tag{6}$$

with $I \sqcup \bar{I}$ a bipartition of the external labels [8]. Arkani-Hamed, He and Lam showed that integrals of the class (3) have simple poles on the facets of their Newton polytope, with residue the integral of the same class attached to the facet [15]. They remarked that for $k \geq 3$ this facet integral need not split into two lower-point integrals, and left open the residues at the higher levels $n \geq 1$. Cachazo and Early showed that the residue of $m(3,6)$ at a pole where three points become collinear is $m(2,6)$ [78], and Early studied residues of $m(3,n)$ that are products of three lower-point amplitudes [79].

The sixteen facets of $P(3,6)$ are of three kinds. Where three points become collinear (six facets s_{abc}) the residue at $\alpha'\kappa_a = 0$ is the six-point Koba–Nielsen integral on their line. Where two points coincide (six facets t_{abcd}) it is a six-point disk integral with one planar channel absent, summed over the two orderings of a pair of punctures (Eq. (42)). Where a pair of points meets the line through another pair (four facets R) it is a product of three Beta functions. The last identification holds in closed form at every order, as do two of the collinear ones; the rest are exact at leading order and checked numerically beyond it. At every pole and every level we prove in Section 3 (Proposition 1) that

$$\mathrm{Res}_{\alpha'\kappa_a = -n}\, K(3,6) \;=\; \sum_{\lambda} V_\lambda^{(n)}(s)\, F_\lambda(s_\parallel), \tag{7}$$

a finite sum. The V_λ are polynomials of degree at most n in the invariants. The F_λ are the string integrals attached to the facet, with exponents shifted by integers determined by λ. They depend on the kinematics only through the facet exponents $s_\parallel$.

On the slice $\mathcal{P}$ of Eq. (5) the poles $s_{156} = 0$ and $s_{234} = 0$ of complementary triples coincide by momentum conservation, and the three-particle poles become double:

$$K(3,6)\big|_{\mathcal{P}} \;=\; \frac{A_4(P,2,3,4)\,A_4(5,6,1,P)}{(\alpha' s_{156})^2} \;+\; \frac{C_1}{\alpha' s_{156}} \;+\; \cdots\,. \tag{8}$$

The leading coefficient is exactly a product of two massless four-point disk amplitudes, with P the on-shell momentum exchanged between the two triples.

For the open string, (7) holds with the facet a product of two polytopes and $F_\lambda = A_L A_R$. No facet of $P(3,6)$ is such a product over complementary subsets of the labels, the four cube facets being products of three factors [29]. Hence at generic kinematics on each pole no residue of $K(3,6)$, at any level, has the form (6). Channel factorization (6) requires a bipartition with one invariant $s_I = s_{\bar{I}}$, and three-label kinematics has none: $s_{156} \neq s_{234}$. We conclude that factorization holds in the boundary form (7) and not in the channel form (6). Positivity in Weinberg's sense, a statement about elastic residues with $A_L = \overline{A_R}$, is accordingly undefined for $K(3,6)$ rather than violated. These three statements apply to $K(3,6)$ regarded as a function of the three-label invariants.

Double copy without a worldsheet. Mizera showed [36] that the KLT relations [9] are, mathematically, the twisted period relations of Cho and Matsumoto for $\mathcal{M}_{0,n}$ [33–35]. The KLT kernel is the inverse of the intersection matrix of the twisted cycles over which the disk integrals are taken [36, 37]. Arkani-Hamed, He and Lam defined the closed stringy integral of any polytope and raised the question of its single-valued projection [15]. Of the 372 real chambers of $X(3,6)$, the generalized color orderings of [38], sixty are convex, with the generalized Parke–Taylor forms as canonical forms. In Section 4 we write the intersection form for these sixty as a face rule on the D_4 cluster fan, Eq. (64). We obtain it by applying the Kita–Yoshida face sum to the cluster compactification of $X^+(3,6)$, for which that sum has not been derived. At one rational kinematic point we find, numerically, that the period relations hold as functions of α' out to about eighty percent of the radius of convergence, and that the form has rank exactly twenty-six at finite α'. Its inverse on any basis of twenty-six of the sixty is an α'-exact KLT kernel. We find at that kinematic point, through weight six, that the KLT bilinear built from this kernel is the single-valued image of the open-string expansion, $\zeta_{2k} \to 0$, $\zeta_{2k+1} \to 2\zeta_{2k+1}$. We conjecture that the face rule holds for all kinematics in the region of convergence.

By the twisted Riemann relations the KLT bilinear formed with the inverse of the true intersection matrix is an integral over the complex points of $X(3,6)$, the closed stringy integral of [15]. Like the Virasoro–Shapiro amplitude in any spacetime dimension, it is positive as a measure, and this is not a no-ghost condition. Numerically we find the bilinear positive at two kinematic points and at the values of α' examined.

Zeta values without a line. The coefficients of the low-energy expansion of string tree amplitudes are multiple zeta values,

$$\zeta(n_1,\ldots,n_r) \;=\; \sum_{m_1>m_2>\cdots>m_r\geq 1} \frac{1}{m_1^{n_1}\cdots m_r^{n_r}}, \qquad n_1 \geq 2, \tag{9}$$

of weight $n_1 + \cdots + n_r$ and depth r, with the weight set by the order in α' [10, 12, 13, 39]. By contrast, the Aomoto–Gel'fand hypergeometric function of type $(3,6)$, whose parameter space is $X(3,6)$, is at exponents $-\frac{1}{2}$ the period map of a family of K3 surfaces [42–44]. Thus there is no general reason to expect the coefficients of $K(3,6)$ to be multiple zeta values. For arbitrary stringy integrals, Arkani-Hamed, He and Lam asked "for what polynomials do we have multiple zeta values only?" [15]. In Section 5 we compute the α'-expansion of $K(3,6)$, to our knowledge the first beyond leading order for $k \geq 3$. Writing $K(3,6) = \sum_{w\geq 0} \alpha'^{\,w-4} c_w(s)$, we find as functions on the kinematic space

$$c_2 = q_2\,\zeta_2, \quad c_3 = q_3\,\zeta_3, \quad c_4 = q_4\,\zeta_4, \quad c_5 = Q_5\,\zeta_5 + R_5\,\zeta_2\zeta_3, \quad c_6 = A\,\pi^6 + B\,\zeta_3^2, \tag{10}$$

with $\zeta_n = \zeta(n)$, $c_0 = m(3,6)$ and $c_1 = 0$. Here q_w, Q_5, R_5, A, B are rational functions of the fourteen invariants with simple poles on the sixteen κ_a and nowhere else. We determine A and B from values of c_6 at rational kinematic points.

When we expand $K(3,6)$ cone by cone over the tropical fan, only the sum over cones is a combination of multiple zeta values. The individual cone integrals involve polylogarithms at rational arguments and, from weight six, periods with quadratic irrationalities. A gluing of adjacent cones across the internal walls of the fan accounts for the cancellation of the other constants. We establish this cancellation identically in the kinematics through weight five, assuming finitely many numerically identified relations among the cone constants, and at weight six at every kinematic point computed. Independently, the positive parametrization of $X(3,6)$ is linearly reducible in Brown's sense, which with the results of Brown and Panzer [45, 69] implies that c_w lies in $\mathbb{Q}(s) \otimes \mathrm{MZV}_{\leq w}$ at every order w (Theorem 3). That the weight is exactly w at every order remains a conjecture.

We find at two rational kinematic points that c_7 and c_8 are polynomials in single zeta values. In particular the coefficient of $\zeta(3,5)$, the first multiple zeta value that is conjecturally not such a polynomial, is zero. The same Koba–Nielsen factor integrated against a second dlog form over the same region, at the same kinematics, gives $\zeta(3,5)$ at weight eight with coefficient $-483731/29160$. We conjecture that the diagonal pairing, in which a chamber is integrated against its own canonical form as in Eq. (4), never produces $\zeta(3,5)$.

Beyond six points (Section 7) we expect residues of the form Eq. (7), with no bipartition, for every $K(k,n)$ with $k \geq 3$, because the proof of Eq. (7) uses only that each cone polynomial equals one at the origin and that the facet exponents are positive; we have established these two properties only at $(3,6)$. At seven points the number of independent chamber integrals is $1272 = 42 \cdot 26 + 180$ [72, 75], not a product of lower counts. The 360 convex Parke–Taylor forms at seven points cannot span a cohomology of rank 1272, so a seven-point double copy needs another basis. We compute the coefficients of $K(3,7)$ through weight six at eleven kinematic points, and at weight seven on a two-parameter family of kinematics and at one further point: all are again polynomials in single zeta values. The positive parametrization of $X(3,7)$ that we have examined is not linearly reducible, so the all-orders argument of the six-point case does not apply to it. Generic kinematics and weight eight have not been reached at seven points.

Section 2 sets up $K(3,6)$ in a positive parametrization and on the tropical fan. In Section 4 we also prove (Claim 2) that $K(3,6)$ is not a constant-coefficient combination of seven-point disk or sphere integrals under any linear kinematic map. Independently, $X(3,6)$ and $\mathcal{M}_{0,7}$ differ in Euler characteristic, 26 against 24. Section 6 relates the twenty-six saddle points of the integrand to the rank of the intersection form, shows that their Galois group is S_{26} and excludes a closed form of Gamma type. Section 7 also treats the six-point D_4 deformation. Section 8 opens with the status of each statement. Proofs are in Appendix A and computational details in Appendix B.

2 The Grassmannian string integral $K(3,6)$

A point of $X(3,6) = \mathrm{Gr}(3,6)/(\mathbb{C}^*)^6$ is a configuration of six points in $\mathbb{P}^2$, no three on a line, modulo PGL_3. We represent it by a 3×6 matrix with no vanishing minor, modulo row operations and column rescalings, which leaves $X(3,6)$ four-dimensional. Write (abc) for the 3×3 minor on columns a, b, c, which vanishes where the points a, b, c are collinear. The integral we study is the $(k, n) = (3, 6)$ member of Eq. (3),

$$K(3,6)(s) \;=\; \int_{X^+(3,6)} \omega_{3,6} \prod_{1\le a<b<c\le 6} (abc)^{\,\varepsilon\, s_{abc}}. \tag{11}$$

Here $X^+(3,6)$ is the positive part, on which all twenty ordered minors are positive, and $\omega_{3,6}$ is its canonical form [16], equal to $da\, db\, dc\, dd/(a\, b\, c\, d)$ in the coordinates of Eq. (23) below.

The integral (11) differs from the Koba–Nielsen integral (2) in that the points now lie in the projective plane. The factors raised to kinematic powers, which we call the letters of the integrand, are therefore the twenty minors (abc). They vanish when three points become collinear, whereas the letters $(ab) = z_b - z_a$ of (2) vanish when two points collide. As at $k = 2$, the measure is the canonical form of the positive part and every exponent is proportional to α', written ε in Eq. (11) and below. This section reviews known material, apart from the expansion (30) at one rational point and the derivation of the pole structure (26) from the cone expansion.

2.1 Kinematics and convergence

The twenty exponents s_{abc} are the generalized Mandelstam invariants of [14], totally symmetric in their labels and subject to one momentum-conservation condition per particle,

$$s_{abc} = s_{\sigma(a)\sigma(b)\sigma(c)}, \qquad \sum_{\substack{b<c\\ b,c\neq a}} s_{abc} \;=\; 0 \qquad (a = 1, \ldots, 6), \tag{12}$$

which leave fourteen independent. These conditions make the integrand of Eq. (11) invariant under the torus and under PGL_3. In the parametrization (20) below, the six minors (123), (124), (125), (126), (134), (234) are constant. We solve (12) for their six exponents in terms of the other fourteen, which we take as coordinates $s \in \mathbb{R}^{14}$ on kinematic space. The conditions (12) imply an identity that we use repeatedly: for any four labels with complement $\{e, f\}$,

$$t_{abcd} \;\equiv\; s_{abc} + s_{abd} + s_{acd} + s_{bcd} \;=\; s_{efa} + s_{efb} + s_{efc} + s_{efd}. \tag{13}$$

We carry the string scale in a single parameter ε. Every exponent in Eq. (11) is εs_{abc}, so ε stands for α' in the units of the kinematics, and the terms of order ε^w are the $\alpha'^{\,w}$ corrections. Since the integral depends on ε and s only through these products, we write it as $K(3,6)(\varepsilon s)$ whenever we wish to exhibit the dependence on ε. We drop the overall $(\alpha')^4$ included in [15], and $K(3,6)$ is then a Laurent series in ε beginning at order ε^{-4},

$$K(3,6)(\varepsilon s) \;=\; \sum_{w\ge 0} \varepsilon^{\,w-4}\, c_w(s), \qquad c_w(\lambda s) \;=\; \lambda^{\,w-4}\, c_w(s). \tag{14}$$

We call w the weight, since the constants appearing in c_w turn out, at every order computed below, to have transcendental weight w. The homogeneity relation in (14) follows from the dependence on the products εs_{abc} alone.

Sixteen linear forms in the fourteen invariants cut out the region of convergence. Arkani-Hamed, He and Lam showed that an integral of the type (3) converges absolutely when the piecewise linear tropical function of the integrand is positive on every ray of the normal fan of its Newton polytope [15]. Then the coefficient of the leading power of α' is the canonical function of the polytope [15,16]. For $K(3,6)$ the polytope is $P(3,6)$ [15,17,18] and the fan is the positive tropical

Grassmannian $\mathrm{Trop}^+\mathrm{Gr}(3,6)$, with sixteen rays [19,20]. The values of the tropical function on the rays are the sixteen *channel forms*

$$\begin{aligned} &s_{123},\ s_{234},\ s_{345},\ s_{456},\ s_{561},\ s_{612}, \qquad t_{1234},\ t_{2345},\ t_{3456},\ t_{4561},\ t_{5612},\ t_{6123}, \\ &R = t_{1234} + s_{125} + s_{126}, \quad \tilde{R} = t_{1234} + s_{345} + s_{346}, \\ &R' = t_{2345} + s_{236} + s_{123}, \quad \tilde{R}' = t_{2345} + s_{456} + s_{145}, \end{aligned} \tag{15}$$

in the conventions of [14]. We denote them collectively by $\kappa_a(s)$, $a = 1, \ldots, 16$. They span the fourteen-dimensional dual space and satisfy exactly two linear relations,

$$R + \tilde{R} \ = \ t_{1234} + t_{3456} + t_{5612}, \qquad R' + \tilde{R}' \ = \ t_{2345} + t_{4561} + t_{6123}, \tag{16}$$

one at each of the two non-simple vertices of $P(3,6)$, where the five facets named in the relation meet [14,15]. The region of convergence is the open cone

$$\mathcal{C} \ = \ \{\, s \in \mathbb{R}^{14} \ : \ \kappa_a(s) > 0, \quad a = 1, \ldots, 16 \,\}, \tag{17}$$

and for $s \in \mathcal{C}$ the series (14) converges for $|\varepsilon| < 1/\max_a \kappa_a(s)$, the first singularity in ε occurring where some $\varepsilon\kappa_a$ reaches -1.

Continued in s, $K(3,6)$ is meromorphic with poles only on the hyperplanes $\varepsilon\kappa_a(s) = -n$, $n = 0, 1, 2, \ldots$ (Proposition 1). We call the hyperplane $\kappa_a(s) = 0$ and its shifts $\varepsilon\kappa_a(s) = -n$ the *walls* of the facet a. In what follows "facet a" refers to the codimension-one face of $P(3,6)$ normal to the ray on which the tropical function equals κ_a, and "pole" to the singularity of $K(3,6)$ on a wall. Every coefficient $c_w(s)$ has poles only on the sixteen walls $\kappa_a(s) = 0$, and these walls correspond to the sixteen boundary divisors of $X^+(3,6)$ at which the residues of Section 3 are taken. Geometrically [14], the wall $s_{abc} = 0$ corresponds to configurations in which the points a, b, c are collinear, and the wall $t_{abcd} = 0$ to those in which the two complementary points coincide. The wall $R = 0$ corresponds in the same way to two points meeting at a point of the line through two others.

2.2 The positive parametrization

Let us first recall how the Koba–Nielsen integral looks in a positive parametrization. For $n = 4$ we take the 2×4 matrix with columns $(1,0)$, $(1,1)$, $(1,1+y)$, $(0,1)$, that is, punctures at $0, 1, 1+y, \infty$. Its minors are $(12) = (14) = (24) = (34) = 1$, $(23) = y$ and $(13) = 1 + y$, and the positive part is $y > 0$. With $s = s_{12} = s_{34}$, $t = s_{23} = s_{14}$ and $s_{13} = s_{24} = -s - t$ we find

$$\begin{aligned} K(2,4) \ &= \ \int_0^\infty \frac{dy}{y}\, y^{\varepsilon t}\, (1+y)^{-\varepsilon(s+t)} \ = \ \frac{\Gamma(\varepsilon s)\,\Gamma(\varepsilon t)}{\Gamma(\varepsilon s + \varepsilon t)} \\ &= \ \frac{1}{\varepsilon}\Big(\frac{1}{s} + \frac{1}{t}\Big) \exp\Big[\sum_{k\geq 2} \frac{(-\varepsilon)^k \zeta(k)}{k}\big(s^k + t^k - (s+t)^k\big)\Big]. \end{aligned} \tag{18}$$

This is the Veneziano amplitude (1), with the two channel forms s and t positive in the region of convergence and poles at $\varepsilon s, \varepsilon t = 0, -1, -2, \ldots$. For $n = 5$ we take columns $(1,0)$, $(1,1)$, $(1,1+a)$, $(1,1+a+ab)$, $(0,1)$. The ten minors are $(12) = (15) = (25) = (35) = (45) = 1$, the two monomials $(23) = a$ and $(34) = ab$, and $(24) = a(1+b)$, $(13) = 1+a$, $(14) = 1+a+ab$. Up to the factor a in (24) the last three are the polynomial letters, and

$$K(2,5) \ = \ \int_{\mathbb{R}^2_{>0}} \frac{da\,db}{a\,b}\, a^{\varepsilon s_{51}}\, b^{\varepsilon s_{34}}\, (1+a)^{\varepsilon s_{13}}\, (1+b)^{\varepsilon s_{24}}\, (1+a+ab)^{\varepsilon s_{14}}, \tag{19}$$

where $s_{23}+s_{24}+s_{34}=s_{51}$ has been used. The five channel forms are the planar invariants $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$ and the Newton polytope is a pentagon. The leading order is the five-term biadjoint amplitude, and the coefficients of the higher orders in α' are multiple zeta values, which, unlike those of Eq. (18), are not generated by expanding a ratio of Gamma functions [13].

For $K(3,6)$ we use the positive parametrization of Giménez Umbert and Sturmfels [46]. We take the six points to be the columns of

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1+a & 1+a+ab \\ 1 & 0 & 0 & 1 & 1+a+c & 1+a+c+ab+bc+cd \end{pmatrix}, \qquad a,b,c,d>0. \tag{20}$$

All twenty minors are then positive. Six are constant, $(123)=(124)=(125)=(126)=(134)=(234)=1$; four are monomials,

$$(145)=a, \qquad (156)=ab, \qquad (345)=c, \qquad (456)=acd. \tag{21}$$

The remaining ten are, up to monomial factors, ten polynomials with unit coefficients,

$$\begin{aligned}
&(135)=P_1=1{+}a, && (235)=P_4=1{+}a{+}c, && (256)=P_8=ab{+}bc{+}cd,\\
&(136)=P_2=1{+}a{+}ab, && (245)=P_5=a{+}c, && (346)=c\,P_9, \quad P_9=1{+}b{+}d,\\
&(146)=a\,P_3, \quad P_3=1{+}b, && (236)=P_6=1{+}a{+}c{+}ab{+}bc{+}cd, && (356)=c\,P_{10}, \quad P_{10}=b{+}d{+}ad,\\
& && (246)=P_7=a{+}c{+}ab{+}bc{+}cd. &&
\end{aligned} \tag{22}$$

Collecting the monomials, Eq. (11) becomes

$$K(3,6)(s) \;=\; \int_{\mathbb{R}^4_{>0}} \frac{da\,db\,dc\,dd}{a\,b\,c\,d}\; a^{\varepsilon n_a}\, b^{\varepsilon n_b}\, c^{\varepsilon n_c}\, d^{\varepsilon n_d} \prod_{t=1}^{10} P_t(a,b,c,d)^{\,\varepsilon\, s_t}, \tag{23}$$

with s_t the invariant attached to P_t in Eq. (22) and

$$n_a=s_{145}{+}s_{146}{+}s_{156}{+}s_{456}, \qquad n_b=s_{156}, \qquad n_c=s_{345}{+}s_{346}{+}s_{356}{+}s_{456}, \qquad n_d=s_{456}. \tag{24}$$

The first two rows of Eq. (20), with the first column deleted, are the five-point matrix of Eq. (19) with the last puncture moved to the front, up to the sign of that column. Hence the polynomial letters of Eq. (19) are P_1, P_2, P_3. Equivalently, projecting the six points from the first of them, (0:0:1), onto a line gives the five punctures $\infty, 0, 1, 1+a, 1+a+ab$.

2.3 The cone expansion and the leading order

We compute both $K(3,6)$ at finite α' and its α'-expansion from a second form of the integral, the cone form (25) below. The normal fan of $P(3,6)$ has sixteen rays and forty-eight maximal cones, two of which are bipyramids and the rest simplices [19]. Splitting each bipyramid into three simplicial cones about its axis gives fifty-two cones, and each of them is unimodular: its four primitive ray vectors have determinant ± 1, as we check in Appendix B.2. We decompose $\mathbb{R}^4_{>0}$ accordingly and map each cone to a unit cube, which gives the cone form of the integral,

$$K(3,6)(\varepsilon\, s) \;=\; \sum_{C=1}^{52} \int_{(0,1)^4} \prod_{i=1}^{4} y_i^{\,\varepsilon\,\kappa_{C,i}(s)-1} \prod_t Q_{C,t}(y)^{\,\varepsilon\, s_t}\, d^4 y. \tag{25}$$

Here the four exponents $\kappa_{C,i}$ of the cone C are the channel forms (15) of its four rays, and each $Q_{C,t}$ is a polynomial with unit coefficients and $Q_{C,t}(0)=1$.

The pole structure of the coefficients c_w and the class of constants appearing in them both follow from (25). Expanding the Q factors in ε produces powers of logarithms integrated against $\prod y_i^{\varepsilon\kappa_{C,i}-1}$, and the only source of singularities in s is the endpoint $y_i \to 0$, which gives $1/(\varepsilon\kappa_{C,i})$. Hence, as for the Koba–Nielsen integral, each $c_w(s)$ is a sum of constants multiplied by rational functions of the kinematics with rational coefficients, with at most simple poles, located on the sixteen hyperplanes $\kappa_a(s) = 0$ and nowhere else, at every order:

$$c_w(s) \;=\; \frac{N_w(s)}{\prod_{a=1}^{16} \kappa_a(s)}, \qquad N_w \text{ a polynomial, homogeneous of degree } w + 12. \tag{26}$$

Here the degree of N_w is fixed by the homogeneity relation in (14), and the constants, which we identify in Section 5, are independent of s.

At leading order only the endpoint singularities of (25) contribute, and

$$c_0(s) \;=\; \sum_{C=1}^{52} \prod_{i=1}^{4} \frac{1}{\kappa_{C,i}(s)} \;=\; m_6^{(3)}(\mathbb{I}|\mathbb{I}) \;\equiv\; m(3,6), \tag{27}$$

the canonical function of $P(3,6)$ [15], which is the generalized biadjoint scalar amplitude of Cachazo, Early, Guevara and Mizera in the canonical ordering [14, 46]. Grouped by maximal cone, the sum (27) has forty-eight terms, and forty-six of them are products of four compatible poles, the $k = 3$ Feynman diagrams. The remaining two terms come from the two bipyramids. Each is the sum, over the three simplicial cones of the bipyramid used in (25), of the products of four inverse channel forms out of the five related by Eq. (16). A bipyramid can also be cut into two simplicial cones through its equatorial triangle, and the two triangulations give the same sum. This equality is the "$3 = 2$" identity of [14]. The next coefficient vanishes identically, $c_1(s) = 0$. Any correction to Eq. (27) carries at least one factor $\varepsilon s_t \log Q_{C,t}$, which contributes one power of ε and, since $Q_{C,t}(0) = 1$, also removes one endpoint pole, so the first correction is at order ε^{-2}; it is proportional to ζ_2 (Section 5).

2.4 The expansion at a rational point

Most results below hold for general kinematics s, but we refer throughout to two rational points of $\mathcal{C}$. The first is a generic point:

$$\begin{aligned} s^*: \quad & s_{135} = -\tfrac{2}{5},\ s_{136} = -\tfrac{3}{7},\ s_{146} = -\tfrac{4}{9},\ s_{235} = -\tfrac{5}{11},\ s_{236} = -\tfrac{6}{13},\ s_{245} = -\tfrac{1}{3},\ s_{246} = -\tfrac{2}{7}, \\ & s_{256} = -\tfrac{3}{8},\ s_{346} = -\tfrac{4}{11},\ s_{356} = -\tfrac{5}{13},\ s_{145} = \tfrac{1}{4},\ s_{156} = \tfrac{2}{5},\ s_{345} = 1,\ s_{456} = \tfrac{1}{2}. \end{aligned} \tag{28}$$

All sixteen channel forms are positive there, and at $\varepsilon = 1$ the integral (23) has the value $K(3,6)(s^*) = 127.3994148\ldots$, evaluated numerically to about sixty significant figures. The second, which we call the base point, is s^* rounded to thirds, with all ten polynomial exponents in Eq. (23) equal to $-\frac{1}{3}$,

$$s_0: \quad s_{135} = s_{136} = \cdots = s_{356} = -\tfrac{1}{3}, \qquad s_{145} = s_{156} = \tfrac{1}{3}, \qquad s_{345} = 1, \qquad s_{456} = \tfrac{2}{3}, \tag{29}$$

which gives $(n_a, n_b, n_c, n_d) = (1, \frac{1}{3}, 1, \frac{2}{3})$. At s_0 the sixteen channel forms take the six values $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$ and the series (14) converges for $|\varepsilon| < \frac{1}{2}$. Through weight eight we find

$$\begin{aligned} & c_0 = \tfrac{130817}{800}, \qquad c_1 = 0, \qquad c_2 = -\tfrac{241301}{1200}\,\zeta_2, \qquad c_3 = \tfrac{228617}{1200}\,\zeta_3, \qquad c_4 = \tfrac{734017}{3600}\,\zeta_4, \\ & c_5 = -\tfrac{312803}{32400}\,\zeta_5 - \tfrac{3999979}{16200}\,\zeta_2\zeta_3, \qquad c_6 = \tfrac{142168667}{734832000}\,\pi^6 + \tfrac{12936979}{97200}\,\zeta_3^2, \\ & c_7 = -\tfrac{133437637}{583200}\,\zeta_7 - \tfrac{189359}{12150}\,\zeta_2\zeta_5 + \tfrac{18539231}{97200}\,\zeta_3\zeta_4, \\ & c_8 = -\tfrac{7173761239}{83980800}\,\zeta_8 + \tfrac{11628931}{145800}\,\zeta_3\zeta_5 - \tfrac{74762681}{437400}\,\zeta_2\zeta_3^2, \end{aligned} \tag{30}$$

with $\zeta_n = \zeta(n)$. The first line follows from Eq. (27) and from the closed forms (89) and (90), valid on all of $\mathcal{C}$, and the second from the weight-five and weight-six coefficient functions (93) and (96). The last two lines are obtained numerically at this point and identified as the rational combinations shown (Appendix B).

We use three further rational points of $\mathcal{C}$ below: $\tilde{s}$ of (99) for a second expansion at weight eight, and $\hat{s}$ of (62) and $\hat{s}'$ of (144) for the statements at finite α'. All five points lie in the subcone of $\mathcal{C}$ on which the ten polynomial exponents s_t of (23) are negative, and Table S1 of the Supplementary Material lists their coordinates.

3 Factorization at the boundary

The poles of $K(3,6)$ lie on sixteen families of parallel hyperplanes, one family for each facet of the four-dimensional polytope $P(3,6)$. We determine the residue at every one of them at finite α' and compare it with (6) at generic kinematics on each wall.

Let us first describe the polytope $P(3,6)$. It has sixteen facets, sixty-six two-faces, ninety-eight edges and forty-eight vertices. Forty-six of the vertices are simple, and at each of the other two, five facets meet. Indeed $P(3,6)$ is the simple D_4 cluster polytope with two edges contracted to points [15, 19]. Each facet corresponds to a ray of the fan and to a pole $\kappa_a(s) = 0$ of $K(3,6)$. The facets come in three kinds [14, 17]:

$$\begin{array}{lll} s_{i\,i+1\,i+2} = 0, & \text{the points } i,\ i{+}1,\ i{+}2 \text{ become collinear,} & \text{six facets;} \\ t_{i\,i+1\,i+2\,i+3} = 0, & \text{the two complementary points coincide,} & \text{six facets;} \\ R = 0, & \text{two points coalesce at a point of the line through two others,} & \text{four facets.} \end{array} \tag{31}$$

We label an R facet by its incidence condition, $R_{ab,cd,ef}$ being the wall on which the pair ab meets the line $\overline{ef}$. Equivalently cd meets $\overline{ab}$, or ef meets $\overline{cd}$: the three conditions define the same stratum [14]. In the notation of Eq. (15) the four R facets are $\tilde{R} = R_{12,34,56}$ and $R = R_{12,56,34}$ at one non-simple vertex, and $\tilde{R}' = R_{16,23,45}$ and $R' = R_{16,45,23}$ at the other.

Each facet is itself a three-dimensional polytope (Figure 1). An s facet is the associahedron K_5, with fourteen vertices and nine faces. A t facet is K_5 with one edge contracted, with thirteen vertices and nine faces. An R facet is a cube, the boundary divisor of root type $A_2 \times A_2 \times A_2$ in the language of cubic surfaces [51, 52]. In the cluster language of [29] the facets of the D_4 polytope correspond to the configuration spaces of the Dynkin diagram with one node deleted. Deleting an outer node leaves A_3, whose polytope is K_5, and deleting the central node leaves $A_1 \times A_1 \times A_1$, the cube. The six t facets are the A_3 facets that contain a contracted edge.

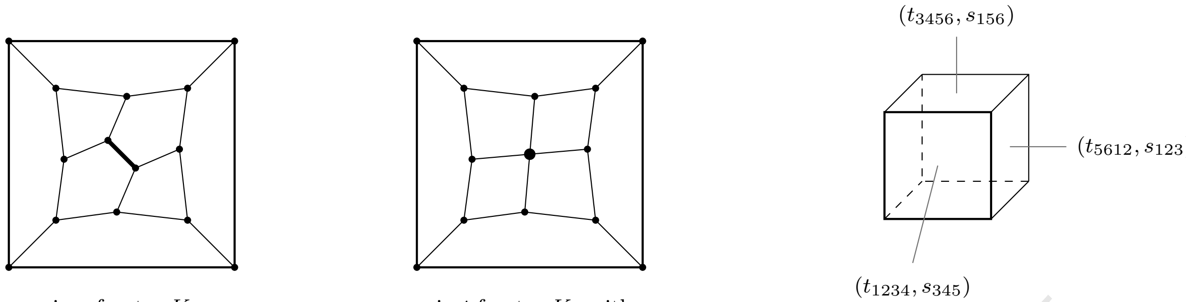
3.1 Residues at every level

In the cone form Eq. (25) the ray of a facet a is dual to one integration variable y_a in every cone that contains it. Setting $y_a = 0$ there, which keeps in each letter the monomials free of y_a, leaves a three-dimensional string integral of the same type over the fan of the facet,

$$K_{\partial a}(\varepsilon s) \;=\; \sum_{w \geq 0} \varepsilon^{\,w-3}\, c_w^{\partial a}(s), \tag{32}$$

a function of the kinematics on the wall $\kappa_a(s) = 0$. We compare the residue of $K(3,6)$ with this facet integral through the ratio

$$N_a \;=\; \frac{\mathrm{Res}_{\kappa_a = 0}\; c_w}{c_w^{\partial a}} \tag{33}$$



six s facets: K_5 residue $K(2,6)$

six t facets: K_5 with one edge contracted residue Eq. (42)

four R facets: cube residue Eq. (40)

Figure 1: The three kinds of facet of $P(3,6)$, none of which is a product of two polytopes attached to complementary label sets; the cube is a product of three. Left: when three of the six points become collinear ($s_{ijk} = 0$, six facets) the facet is the associahedron K_5 and the residue of $K(3,6)$ is the six-point disk integral on the line, Eq. (41). Middle: when two complementary points coincide ($t_{ijkl} = 0$, six facets) the facet is K_5 with one edge contracted, with thirteen vertices, and the residue is Eq. (42). Right: when a pair of points meets the line through another pair ($R = 0$, four facets) the facet is a cube and the residue is the product of three Veneziano factors (40). The two associahedral facets are drawn as Schlegel diagrams through a square face; the heavy edge of K_5 on the left is the one contracted to the heavy vertex in the middle. On the cube, drawn for $R_{12,34,56}$, each pair of opposite faces is labeled by the two channels of $P(3,6)$ that cut it out, and each pair gives one Veneziano factor in Eq. (40).

at each order w. The wall is the first of the family of parallel hyperplanes $\varepsilon\kappa_a(s) = -n$ of Section 2.1. Near a generic point of one of them we write $K(3,6)(\varepsilon s) = R_a^{(n)}(s)/(\varepsilon\kappa_a(s) + n) + O(1)$ and call $R_a^{(n)}$, restricted to the hyperplane, the level-n residue at the wall a.

Proposition 1. *$K(3,6)(\varepsilon s)$ is a meromorphic function of the fourteen exponents with at most simple poles, lying only on the hyperplanes $\varepsilon\kappa_a(s) = -n$, $n = 0, 1, 2, \ldots$, for the sixteen channel forms κ_a of Eq. (15), simultaneous only where compatible facets of $P(3,6)$ meet. The level-n residue is*

$$R_a^{(n)}(s) \;=\; \sum_{C \ni a} \int_{(0,1)^3} \frac{1}{n!}\,\frac{\partial^n}{\partial y_a^n}\Big[\prod_t Q_{C,t}(y)^{\,\varepsilon s_t}\Big]_{y_a=0}\; \prod_{i\neq a} y_i^{\,\varepsilon\kappa_{C,i}(s)-1}\; d^3y', \tag{34}$$

the sum running over the cones of Eq. (25) that contain the ray a and y' denoting the other three coordinates. The representation holds where the link exponents $\kappa_{C,i}$, $i \neq a$, are positive and defines $R_a^{(n)}$ elsewhere by continuation. At $n = 0$ the residue is the string integral (32) of the facet with unit normalization: $\mathrm{Res}_{\kappa_a=0}\, c_w = c_w^{\partial a}$ *for every a and every w, that is $N_a = 1$, or*

$$\lim_{\kappa_a \to 0}\; \varepsilon\kappa_a\, K(3,6)(\varepsilon s) \;=\; K_{\partial a}(\varepsilon s). \tag{35}$$

The proof is elementary. The form κ_a appears only in the integrals over the cones containing the ray a, and in each exactly once, as the exponent of the single coordinate y_a. Every letter has unit coefficients and $Q_{C,t}(0) = 1$, so $Q_{C,t} \geq 1$ on the closed cube and $\prod_t Q_{C,t}^{\varepsilon s_t}$ is smooth in y_a on $[0,1]$. We expand it about $y_a = 0$ to any finite order N. Against $y_a^{\varepsilon\kappa_a - 1}$ the remainder integrates to a function analytic for $\varepsilon\kappa_a > -N-1$. The term of degree m integrates to $1/(\varepsilon\kappa_a + m)$, a simple pole at $\varepsilon\kappa_a = -m$ with the coefficient displayed in (34) and no other singularity, the y' integral converging while the other three exponents are positive. Away from the sixteen families $K(3,6)$ is analytic by the general theory of Euler–Mellin integrals [15, 53].

At $n=0$ we must also check that setting $y_a=0$ in each letter gives the corresponding letter of the facet integrand, and that the cones through a, projected along a, form the normal fan of the facet. In fact the projected cones form a unimodular refinement of that fan, under which the cone form is invariant (Appendix A.1). The statement $N_a=1$ at $n=0$ is the self-reductive property of stringy canonical forms [15] and the boundary factorization of cluster string integrals [29]. We prove it here cone by cone for a polytope that is not simple. The argument holds at finite α', assumes no spectrum of intermediate states, and uses only the behavior of the cone integrals near $y_a=0$.

Iterating (35) down the faces of $P(3,6)$, we obtain the multiple residues. On every one of the ninety-eight edges of $P(3,6)$ the restricted integrand has the single letter $1+y$, and at every vertex it has none. Hence the triple residue along the three facets through an edge e is one Veneziano factor, and the quadruple residue at a vertex is a pure number,

$$\lim \prod_{i=1}^{3} (\varepsilon\kappa_{a_i})\, K(3,6) \;=\; B(\varepsilon\kappa_e, \varepsilon\kappa_{e'}), \qquad \lim \prod_{i=1}^{4} (\varepsilon\kappa_{a_i})\, K(3,6) \;=\; 1, \tag{36}$$

with $\varepsilon\kappa_e$, $\varepsilon\kappa_{e'}$ the two exponents of the one-dimensional integral along the edge. The limits in (36) are taken along facets whose successive intersections are faces of $P(3,6)$. At the two non-simple vertices the iterated residue vanishes instead when it is taken through the pair R, $\tilde{R}$, which share no two-face, or through the three t facets, which share no edge. In the cone form (25) an edge is dual to an internal wall of the fan, and the Beta function in (36) arises from gluing the two maximal cones that share it, by the identity Eq. (91).

The D_4 cluster string integral of [15] is obtained from $K(3,6)$ by raising the two cluster variables of $\mathrm{Gr}(3,6)$ that are not Plücker coordinates to powers as well, with exponents $(\varepsilon c, \varepsilon c')$. Along each of the two extra edges of its polytope the factor analogous to (36) is

$$\frac{\Gamma(\varepsilon\kappa_R)\,\Gamma(\varepsilon\kappa_{\tilde{R}})}{\Gamma(-\varepsilon c)}, \tag{37}$$

with $\kappa_R, \kappa_{\tilde{R}}$ the two R-type forms that meet along it. At $c=0$ the factor $1/\Gamma(-\varepsilon c)$ vanishes, the poles associated with this edge are absent, and the edge contracts to a non-simple vertex of $P(3,6)$. This contraction is the "$3=2$" identity of [14] at finite α'.

For the higher levels we make the bracket in (34) explicit by expanding each letter in the coordinate dual to the wall, $Q_{C,t} = q_t^{(0)} + y_a\, q_t^{(1)} + y_a^2\, q_t^{(2)} + \cdots$, with $q_t^{(0)}(y')$ the facet's letter and the $q_t^{(j)}(y')$ polynomials with integer coefficients. The level-one bracket is $\prod_t (q_t^{(0)})^{\varepsilon s_t} \sum_t \varepsilon s_t\, q_t^{(1)}/q_t^{(0)}$. At level n it is $\prod_t (q_t^{(0)})^{\varepsilon s_t}$ times a polynomial of degree n in the εs_t whose coefficients are polynomials in the ratios $q^{(j)}/q^{(0)}$. Dividing by $q_t^{(0)}$ lowers the exponent of a facet letter by one, and each monomial of $q_t^{(j)}$ raises link exponents by integers. Hence

$$R_a^{(n)}(s) \;=\; \sum_{\lambda} V_\lambda^{(n)}(s)\, F_\lambda(s_\parallel), \tag{38}$$

a finite sum. The vertex polynomials $V_\lambda^{(n)}$ have degree at most n in the exponents and rational coefficients, the F_λ are string integrals of the facet with exponents shifted by integers, and λ runs over the ways of distributing n powers of y_a among the letters. The F_λ depend on the kinematics only through the facet data $s_\parallel$: the link exponents and the exponents of the surviving letters.

Facets	Degeneration	Facet polytope	Residue $\lim \varepsilon\kappa_a K(3,6)$
s_{ijk}, six	i,j,k collinear	associahedron K_5	six-point disk integral on the line ijk, Eq. (41)
t_{ijkl}, six	complementary pair coincides	K_5, one edge contracted	six-point disk integral, channel $s_{p\delta}$ absent, both orderings of (p,δ), Eq. (42)
R, four	a pair meets a line	cube	three Veneziano factors, Eq. (40)

Table 1: The residues of $K(3,6)$ at its sixteen poles, at the leading level $n=0$, where the residue is the string integral of the facet with unit normalization, $N_a = 1$ in (33); the residues at the higher levels $\varepsilon\kappa_a = -n$ follow from these by Proposition 1. Every residue is a string integral of the degenerate configuration of six points, and none is a product over a bipartition of the six labels. Here K_5 is the associahedron, and at a t facet p denotes the point to which the complementary pair coalesces and δ the direction along which the two points approach. The cube entry holds in closed form at every order in α', as do the s-facet entries at s_{156} and s_{345}; the other four s-facet entries and the six t-facet entries are exact at leading order and are checked numerically at finite α' (Appendix B.4), the finite-α' object at a t facet being the facet integral $K_{\partial a}$ of (32) for the contracted associahedron.

Now a six-point disk integral has nine independent invariants, the t-facet function of Table 1 has eight, and three Beta functions have six arguments, while the wall has dimension thirteen. Only eight of the nine channel forms adjacent to a t facet are independent on its wall, by (16). There remain $13-9=4$, $13-8=5$ and $13-6=7$ directions in an s, t or R wall along which every F_λ is constant. Along these transverse directions the level-n residue is a polynomial of degree at most n; along every other it is transcendental.

In the chart of Eq. (23) the transverse directions at the wall s_{123} are the exponents s_{135}, s_{136}, s_{235}, s_{236}, each with two labels on the degenerating line and one off it. At the wall s_{156} no single exponent is transverse, and the four directions are

$$\partial_{136}-\partial_{135}, \qquad \partial_{146}-\partial_{145}, \qquad \partial_{356}-\partial_{135}-\partial_{256}, \qquad \partial_{456}-\partial_{145}-\partial_{256}, \tag{39}$$

with $\partial_{abc} = \partial/\partial s_{abc}$. Numerically, at rational points on the wall s_{156} at levels one and two, and on t_{4561} and one cube at level one, we find the degree to be exactly n in every case computed (Appendix B).

The derivation of (38) applies word for word to the Koba–Nielsen integral. There the facet of the associahedron at the channel s_I is the product of two smaller associahedra, one for I and the intermediate state and one for $\bar{I}$ and the same state. Then the surviving letters separate, and every F_λ in (38) is a product $A_L A_R$ of two shifted disk integrals in complementary variables. In the integral representation, cluster decomposition is expressed by this product structure of the facet.

3.2 The sixteen residues

Arkani-Hamed, He, Lam and Thomas identified the facet integrals of the D_4 cluster string integral at generic exponents [29]. The string integral of an A_3 boundary is a six-point disk integral and that of an A_1^3 boundary a product of three Beta functions. To this we add the explicit map of their arguments to the invariants s_{abc} of $K(3,6)$; at the s facets it already appears at leading order in [78]. We also add the higher levels, and the residues at the contracted t facets, which occur only at the Grassmannian point $c=c'=0$ of the D_4 family. Table 1 lists the leading residues.

The cube facets are the simplest: the letters of the facet integrand separate into three sets, each depending on one coordinate of the cube. For $R_{12,34,56}$ the three pairs of opposite faces are cut out by the channels (t_{1234}, s_{345}), (t_{3456}, s_{156}) and (t_{5612}, s_{123}), and

$$\lim_{R_{12,34,56}\to 0} \varepsilon R_{12,34,56}\, K(3,6) \;=\; B(\varepsilon t_{1234}, \varepsilon s_{345})\, B(\varepsilon t_{3456}, \varepsilon s_{156})\, B(\varepsilon t_{5612}, \varepsilon s_{123}), \tag{40}$$

three four-point open-string amplitudes with no variable in common. On the wall the three pairs of arguments sum to $-\varepsilon s_{346}$, $-\varepsilon s_{256}$ and $-\varepsilon s_{124}$, and each factor has the form (18).

At an s facet three points become collinear, and the residue is the six-point Koba–Nielsen integral on the line L they span. Let us take $s_{456} = 0$. The six punctures on L are 4, 5, 6 and the three points where the lines $\overline{12}$, $\overline{13}$, $\overline{23}$ cross L. The kinematics on the wall determines one dihedral order of these six punctures, and we find

$$\begin{gathered} \lim_{s_{456}\to 0} \varepsilon s_{456}\, K(3,6) \;=\; K(2,6)(4,5,6,(12),(13),(23)), \\ s_{a\,(ij)} = s_{aij}, \qquad s_{ab} = s_{1ab} + s_{2ab} + s_{3ab}, \end{gathered} \tag{41}$$

for $a, b \in \{4,5,6\}$ and $\{i,j\} \subset \{1,2,3\}$. In the residue the three-index invariant s_{aij} is the two-body invariant between the puncture a and the crossing point (ij). The nine planar variables of this hexagon are the restrictions to the wall of the nine channel forms of $P(3,6)$ adjacent to s_{456}. Modulo (12) and $s_{456} = 0$, the nine s_{aij} that enter are independent. The six-point integral on the wall then factorizes on its own nine channels in the ordinary way: the $k = 2$ bipartition factorization occurs inside the facet amplitude and not in $K(3,6)$ itself. At leading order in α', Eq. (41) is the residue $\mathrm{Res}_{s_{456}=0}\, m(3,6) = m(2,6)$ of [78], $m(2,6)$ being the biadjoint amplitude, with the same map of invariants and the same four transverse directions; it continues that identification to finite α', in closed form at two of the six s walls and numerically at the other four, and Proposition 1 extends it to every level.

At a t facet two points collide, say 1 and 2 at $t_{3456} = 0$. We write p for the point they coalesce to and δ for the direction along which they approach. The facet is the contracted associahedron of Figure 1, with thirteen vertices where K_5 has fourteen, so its canonical function cannot be a single six-point planar amplitude. We find instead, at leading order,

$$\mathrm{Res}_{t_{3456}=0}\, c_0 \;=\; m'(p\,\delta\,3\,4\,5\,6) \;+\; m'(\delta\,p\,3\,4\,5\,6), \tag{42}$$

where $m'(\alpha)$ is the planar biadjoint amplitude in the ordering α with the five trees through the channel $s_{p\delta}$ removed, the two spurious poles canceling in the sum. In the deformed family of (37), where $s_{p\delta}$ is kept away from zero, the facet is a genuine K_5 and the residue a single six-point disk integral.

Apart from the cube entry (40) and the s-facet entries at s_{156} and s_{345}, which hold at every order in α', the entries of Table 1 are exact at leading order. Beyond it we have compared $K_{\partial a}$ with the disk integrals numerically at rational points on each wall. At a t wall the disk integral in this comparison is a single $K(2,6)$ in one dihedral ordering whose planar variables obey the one linear relation that Eq. (16) induces there. Its leading order is (42) (Appendix B.4).

3.3 Facets and cluster decomposition

In a theory with a unitary S-matrix obeying the cluster decomposition principle, an amplitude has a pole when the momentum P_I carried by a subset I of the particles approaches the mass shell of a one-particle state [8]. The residue is

$$\mathrm{Res}_{s_I=m^2}\, A_n \;=\; \sum_{\text{states}} A_L(I,P)\, A_R(-P,\bar{I}), \tag{43}$$

summed over the spin states of that particle. Here the sum over intermediate states follows from unitarity. The bipartition $I \sqcup \bar{I}$ follows from cluster decomposition, since the pole arises from the region in which the particles of I are produced far from those of $\bar{I}$.

Arkani-Hamed, He and Lam noted that residues of stringy canonical forms need not split into two lower-point integrals [15]. Early studied residues of the generalized biadjoint amplitude $m(3,n)$ that are products of three lower-point amplitudes [79], among them the $\alpha' \to 0$ limit of (40). We find that no residue of $K(3,6)$ has the form (43). This holds at all sixteen poles and all levels, for generic kinematics on the wall. The residue is instead Eq. (38), which shares with (43) the structure of a finite sum of products over intermediate labels. The second factor in each term, however, is the string integral of the facet. No facet of $P(3,6)$ is a product of two polytopes attached to complementary subsets of the labels. At the twelve s and t walls the facet integral, (41) or the sum over two orderings (42), is a single function of all six labels. At the four R walls the residue is a product of three factors, not two, and the arguments of each Beta function in (40) involve labels from more than one of the three pairs.

We must therefore distinguish two factorization statements. Boundary factorization, Proposition 1, holds at every pole and every level. Channel factorization in Weinberg's sense, (43), requires in addition a bipartition $I \sqcup \bar{I}$ with an invariant $s_I = s_{\bar{I}}$ attached to it, a two-body subsystem whose asymptotic states are summed, and a positive inner product on those states. Three-label kinematics has no bipartition invariant: a pair of labels has no invariant, and two complementary triples carry two different invariants, $s_{156} \neq s_{234}$. The poles of $K(3,6)$, none of which is a channel, are incidences of six points and the fifteen lines they span [14]. Even the collinearity walls $s_{156} = 0$ and $s_{234} = 0$, each labeled by a single triple, are different facets with different residues, and they coincide only on the slice of Section 3.5. Nor does Proposition 1 involve a state space or an inner product: V_λ is a polynomial and F_λ a six-label facet integral, never amplitudes A_L, A_R of complementary label sets. We conclude that $K(3,6)$ factorizes at every pole, in the sense of Proposition 1, and that cluster decomposition, which presupposes such a bipartition and subsystem, is neither obeyed nor violated. The same holds for positivity, which in Weinberg's argument is a property of elastic residues, those of (43) with $A_L = \overline{A_R}$: $K(3,6)$ has no elastic residue on which the condition could be imposed. Both conclusions apply to $K(3,6)$ regarded as a function of the three-label invariants s_{abc}, with no six momenta underneath. If one insists on reading s_{156} as the three-particle invariant of six particles, the same facts imply instead that $K(3,6)$ is not their S-matrix element in any unitary theory with cluster decomposition, as shown in Section 3.5.

Let us return to the sum (38). At level n the sum can be regrouped so that λ runs over the excitations transverse to the facet, the monomials of degree at most n in the four, five or seven transverse variables. Each such monomial is paired with a combination of twisted periods of the facet at integer-shifted exponents. At $n = 0$ the second factor is the facet's own twisted period, whose natural pairing is the facet's intersection form. Its rank is six on the s facets, the 3! of $\mathcal{M}_{0,6}$, four on the t facets, and one on the cubes. These ranks are consistent with, though not derived from, the wall structure of the rank-twenty-six intersection form of Section 4.

3.4 The split locus and the four-Beta locus

On special subspaces of kinematic space, $K(3,6)$ itself is a product of lower-dimensional integrals at finite α'. For the generalized biadjoint amplitudes such splittings are known [46, 54]. The first subspace is the six-dimensional split locus of Giménez Umbert and Sturmfels [46], cut out in the chart of Eq. (23) by

$$s_{135} = s_{136} = s_{146} = s_{236} = s_{245} = s_{246} = s_{256} = s_{356} = 0, \tag{44}$$

and on this locus we find

$$K(3,6)\big|_{\text{split}} = \frac{\Gamma(\varepsilon n_a)\,\Gamma(\varepsilon n_c)\,\Gamma\big(-\varepsilon(n_a{+}n_c{+}s_{235})\big)}{\Gamma(-\varepsilon s_{235})}\;\frac{\Gamma(\varepsilon s_{156})\,\Gamma(\varepsilon s_{456})\,\Gamma\big(-\varepsilon(s_{156}{+}s_{456}{+}s_{346})\big)}{\Gamma(-\varepsilon s_{346})}, \tag{45}$$

with $n_a = s_{145} + s_{156} + s_{456}$ and $n_c = s_{345} + s_{346} + s_{456}$ the exponents (24) restricted to the locus.

Let us derive (45) by hand. Of the ten letters $P_t(a, b, c, d)$ of Eq. (23), five mix the pair of variables (a, c) with the pair (b, d). Their exponents are s_{136}, s_{236}, s_{246}, s_{256} and s_{356}. Setting these to zero decouples the integrand into a block in (a, c) and a block in (b, d). If we set $s_{135} = s_{146} = s_{245} = 0$ as well, the letters $1 + a$, $1 + b$ and $a + c$ disappear and each block collapses to a Dirichlet integral,

$$\int_{\mathbb{R}^2_{>0}} \frac{da\,dc}{a\,c}\; a^x\, c^y\, (1 + a + c)^{-x-y-z} \;=\; \frac{\Gamma(x)\,\Gamma(y)\,\Gamma(z)}{\Gamma(x + y + z)}, \tag{46}$$

the first with letter $1 + a + c$ and exponent εs_{235}, the second with letter $1 + b + d$ and exponent εs_{346}. The S_6 images of (44) give 360 such loci in all. As $\alpha' \to 0$ the two blocks become the two triangle amplitudes into which $m(3, 6)$ splits in [46], and (45) is the lift of that splitting to finite α', of the kind known for the ordinary string [55, 56]. The splitting (45) separates the integration variables and not the external labels, since the first block depends on invariants involving all six points.

Part of the reduction to (45) already occurs on the codimension-five subspace where only the five mixing exponents vanish: there $K(3, 6)$ is a five-point disk integral times a ratio of Gamma functions. Imposing $s_{245} = 0$ as well defines a codimension-six locus $\mathcal{L}_4$. There the substitutions $c = (1 + a)u$, $d = (1 + b)v$ turn each block into two Beta functions,

$$K(3,6)\big|_{\mathcal{L}_4} \;=\; B(\varepsilon t_{4561}, \varepsilon t_{5612})\, B(\varepsilon t_{3456}, \varepsilon s_{234})\, B(\varepsilon s_{156}, \varepsilon s_{126})\, B(\varepsilon s_{456}, \varepsilon t_{2345}), \tag{47}$$

each channel form standing for its restriction to $\mathcal{L}_4$ through Eq. (12), for instance $t_{4561} = s_{145} + s_{146} + s_{156} + s_{456}$ and $s_{126} = -(s_{156} + s_{146} + s_{346} + s_{456})$. The eight arguments are the channel forms of eight of the sixteen walls of $P(3, 6)$, which remain poles on $\mathcal{L}_4$. The other eight walls, among them all four cubes, are not poles of the restricted function. Setting $s_{135} = s_{146} = 0$ in (47), we recover (45).

On $\mathcal{L}_4$ the wall s_{156} is thus a pole of one Veneziano factor, the third in (47): the amplitude (18) with εs_{156} in the role of $-\alpha(s)$ and $q \equiv \varepsilon s_{126}$ in that of $-\alpha(t)$. Of the four directions (39) transverse to this wall, $\partial_{146} - \partial_{145}$ stays inside $\mathcal{L}_4$, and along it only q varies. The residue at $\varepsilon s_{156} = -n$ is

$$R^{(n)}_{s_{156}}\Big|_{\mathcal{L}_4} \;=\; \frac{(-1)^n}{n!}\,(q-1)(q-2)\cdots(q-n)\; B(\varepsilon t_{4561}, \varepsilon t_{5612})\, B(\varepsilon t_{3456}, \varepsilon s_{234})\, B(\varepsilon s_{456}, \varepsilon t_{2345}), \tag{48}$$

a polynomial in the one transverse variable of degree exactly n, the most that Proposition 1 allows, times three spectator amplitudes. The cone sums (34), evaluated numerically at levels one and two, reproduce (48) (Appendix B.4).

To expand $R^{(1)} \propto 1 - q$ and $R^{(2)} \propto (q - 1)(q - 2)$ in partial waves we borrow the kinematics of the unit-intercept Veneziano model, that is, four scalars of mass squared $-1/\alpha'$, for which $\cos\theta = 1 + 2t/(s + 4)$ in units $\alpha' = 1$. At the level-n pole this relation becomes the embedding

$$q \;=\; -1 + \frac{(n + 3)(1 - \cos\theta)}{2}. \tag{49}$$

We expand in Gegenbauer polynomials in d dimensions, $R^{(n)} \propto \sum_J c_J\, C_J^{((d-3)/2)}(\cos\theta)$, where c_J here denotes a partial-wave coefficient, not the c_w of (14). At level one, $1 - q = 2\cos\theta$ is pure

$J=1$, and at level two $(q-1)(q-2)=\frac{1}{4}(25\cos^2\theta-1)$ gives

$$\frac{c_0}{c_2} \;=\; \frac{(26-d)(d-3)}{50}, \tag{50}$$

which vanishes at the level-two no-ghost boundary $d=26$ of the unit-intercept Veneziano model [30–32] and is positive below it. It is not an independent test, since on $\mathcal{L}_4$ the block is a Veneziano amplitude, whose partial waves are non-negative at every level for $d \le 26$ by the no-ghost theorem [31,32], and we have merely expanded levels one and two. The same holds for the Beta-function towers on the four cube walls (40) and on the ninety-eight edges (36).

Let us move off $\mathcal{L}_4$ in the one exponent s_{136}, keeping the other five conditions. The letter $P_2 = 1+a+ab$ then depends on variables of both blocks, and the wall s_{156} is no longer a pole of a Beta function. Its level-one residue is still linear in q along $\partial_{146}-\partial_{145}$, but its zero moves away from $q=1$ at first order in s_{136}, for either sign of s_{136}. In the embedding (49) the level-one residue then has a $J=0$ component, of either sign. Displacing one rational point of $\mathcal{L}_4$, given in Appendix B.1, to $s_{136}=+\frac{1}{6}$ and to $s_{136}=-\frac{1}{6}$, we find in four dimensions

$$\frac{c_0}{c_1} \;=\; -\frac{11}{168} \qquad \text{and} \qquad \frac{c_0}{c_1} \;=\; +\frac{13}{192} \tag{51}$$

respectively. Both values are exact, and both grow in magnitude with d through the Gegenbauer index. The sign of the $J=0$ component is embedding-dependent: it depends on how the Veneziano variable is continued off the locus. The choice $q=\varepsilon s_{126} \leftrightarrow -\alpha(t)$ is one; the other natural one, $q' = q+\varepsilon s_{136}$, gives $+\frac{1}{56}$ and $-\frac{1}{64}$ at the same two points.

The values (51) are formal partial-wave coefficients, obtained by treating the residue at s_{156} as if it were a two-body residue, which off $\mathcal{L}_4$ it is not. The residue is therefore inelastic, so that its formal partial waves may have either sign and the signs in (51) do not bear on unitarity or on positivity. For the R walls no such analysis is possible, since we know no closed form for $K(3,6)$ on a subspace with an R pole. By (38) and (40) the residue at a cube wall is a sum of vertex polynomials times products of three shifted Beta functions, and such a sum is not the two-body residue (43) of any channel.

3.5 The momentum slice

In general the s_{abc} obey only the conservation conditions (12). Let us nevertheless suppose that they come from six massless momenta p_a with $\sum_a p_a = 0$, through

$$s_{abc} \;=\; p_a\cdot p_b + p_b\cdot p_c + p_c\cdot p_a \;=\; \tfrac{1}{2}\,(p_a+p_b+p_c)^2, \qquad s_{ab} \;\equiv\; (p_a+p_b)^2, \tag{52}$$

the one half fixing the units of α'. Momentum conservation implies Eq. (12), and these kinematics form a linear subspace $\mathcal{P}$ of kinematic space, of dimension nine: fifteen s_{ab} less six conditions. In particular, generic kinematics of $K(3,6)$ are not of this form for momenta in any dimension.

On $\mathcal{P}$ the sixteen walls reduce to eleven distinct ones. The walls of complementary triples coincide, their common channel form being half a three-particle invariant, and the six t forms become two-particle invariants of adjacent labels:

$$\begin{gathered} s_{156}=s_{234}, \qquad s_{123}=s_{456}, \qquad s_{345}=s_{126}, \\ t_{3456}=s_{12}, \quad t_{4561}=s_{23}, \quad t_{5612}=s_{34}, \quad t_{6123}=s_{45}, \quad t_{1234}=s_{56}, \quad t_{2345}=s_{61}. \end{gathered} \tag{53}$$

The two cubes at each non-simple vertex coincide as well, $R_{12,34,56}=R_{12,56,34}=\frac{1}{2}(s_{12}+s_{34}+s_{56})$. These two facets, however, are not compatible: they meet only at the non-simple vertex of the relation (16), and the residue (40) at one is regular at the other. Hence $K(3,6)|_{\mathcal{P}}$ has simple poles at $\varepsilon(s_{12}+s_{34}+s_{56}) = -2n$, whose residue is the sum of the two cube residues.

The facets s_{156} and s_{234} are compatible and meet in a two-face of $P(3,6)$ that is a square, with one pair of opposite sides on t_{4561}, t_{5612} and the other on t_{1234}, t_{2345}. Iterating (35) as in (36), we find that the double residue of $K(3,6)$ on that face is the string integral of the square, a product of two Beta functions in the channel forms of opposite sides. On $\mathcal{P}$ the two simple poles coincide. Applying Proposition 1 twice, we obtain at $s_{156} = s_{234} \to 0$

$$\begin{aligned} K(3,6)\big|_{\mathcal{P}} \;&=\; \frac{C_2}{(\varepsilon s_{156})^2} + \frac{C_1}{\varepsilon s_{156}} + O(1), \\ C_2 \;=\; B(\varepsilon s_{23}, \varepsilon s_{34})\, B(\varepsilon s_{56}, \varepsilon s_{61}) \;&=\; A_4(P,2,3,4)\, A_4(5,6,1,P). \end{aligned} \tag{54}$$

The leading coefficient C_2 is exact on the wall, and each factor is the massless four-point disk integral (18) of one triple and the on-shell momentum $P = -(p_2+p_3+p_4)$, so the coefficient of the double pole is precisely the bipartition product (43) and does not depend on how the two triples are oriented relative to each other.

The simple-pole coefficient C_1 contains the finite parts of the two six-point facet integrals at the face, and it does depend on the relative orientation. Numerically, at a rational point of the slice and at the pole, each finite part changes by about eleven percent from one relative orientation of the two triples to the next, that is, as the invariants s_{ab} with $a \in \{1,5,6\}$, $b \in \{2,3,4\}$ are varied at fixed invariants within each triple, while C_2 does not change (Appendix B). Thus C_1 is not a product of a function of the invariants of $(1,5,6,P)$ and one of those of $(P,2,3,4)$, and depends jointly on the invariants of all six labels.

At the two-particle poles of (53) the leading-level residue has the factorized form expected of a massless tree amplitude. When $t_{4561} = s_{23} = 0$ the facet data reduce exactly to the five-point kinematics of $(P,4,5,6,1)$ with $P = p_2 + p_3$, insensitive to how p_2 and p_3 share P. The residue is then a three-point coupling A_3 times a function of five momenta. That function is not a multiple of the five-point disk integral of $(P,4,5,6,1)$, because the facet integral has an extra boundary exponent $\frac{1}{2}(s_{61}+s_{45})$, and we have not identified it. We have examined $K(3,6)|_{\mathcal{P}}$ only near its poles. Suppose the s_{abc} are interpreted through (52) as invariants of six massless particles. Then the pole structure alone shows that $K(3,6)$ is not a tree-level S-matrix element of those particles, independently of any positivity question. The propagator of each three-particle channel appears squared, with the bipartition product A_4A_4 of (54) as numerator.

3.6 High energy

At large values of the invariants $K(3,6)$ has the two classic high-energy properties of string amplitudes discussed for stringy canonical forms in [15], Regge behavior and exponential softness at fixed angle, the second at every kinematic point we have examined. Let us scale all fourteen invariants together, $s \to \lambda s$ with s in the convergence region $\mathcal{C}$. The integrand $u(x;s)^{\varepsilon\lambda}$, with $u = \prod (abc)^{s_{abc}}$, vanishes on the boundary of the positive part, and u attains its maximum at an interior point x_+. Every critical point of u in the positive part is one of the twenty-six solutions of the scattering equations of Section 6. The point x_+ is the $k=3$ Gross–Mende saddle [57]:

$$\log K(3,6)(\varepsilon\lambda s) \;=\; \varepsilon\lambda\, S_+(s) + O(\log\lambda), \qquad S_+(s) \;=\; \sum_{a<b<c} s_{abc} \log{(abc)}\Big|_{x_+}, \tag{55}$$

and the amplitude is exponentially soft at fixed angles wherever $S_+ < 0$.

In the chart (23) we have, with $y = (y_1, y_2, y_3, y_4) = (\log a, \log b, \log c, \log d)$ and the exponents (24),

$$\log u \;=\; n_a\, y_1 + n_b\, y_2 + n_c\, y_3 + n_d\, y_4 \;+\; \sum_{t=1}^{10} s_t \,\log P_t(e^{y_1}, e^{y_2}, e^{y_3}, e^{y_4}), \tag{56}$$

where each P_t has positive coefficients. Thus $\log P_t(e^y)$, the logarithm of a sum of exponentials of linear functions of y, is convex. Its Hessian is positive semidefinite, with kernel the directions along which all monomials of P_t scale with the same weight. On the subcone of $\mathcal{C}$ where the ten polynomial exponents s_t of Eq. (23) are all negative, (56) is therefore a linear function plus a concave one. In fact $\log u$ is strictly concave there, because the kernels for the four letters $P_1 = 1 + a$, $P_3 = 1 + b$, $P_4 = 1 + a + c$ and $P_9 = 1 + b + d$ of Eq. (22) already intersect only in zero. A strictly concave function has at most one critical point. Hence on this subcone x_+ is the only critical point of u in the positive part. The subcone contains s^*, s_0 and the points $\tilde{s}$, $\hat{s}$, $\hat{s}'$ of (99), (62) and (144); it is the case treated in [15].

Where some polynomial exponent is positive, there can be several critical points, and (55) then holds with x_+ the global maximum. At one integer point of $\mathcal{C}$ with three positive polynomial exponents, given in Appendix B.1, u has two local maxima in the positive part. Between them lies a third critical point, at which the Hessian of $\log u$ has one positive eigenvalue. We have not shown that $S_+ < 0$ throughout $\mathcal{C}$. It holds at s^*, where $S_+(s^*) = -2.809033467$, at the integer point just mentioned, and at each of 300 integer points of $\mathcal{C}$ drawn at random, at 293 of which at least one polynomial exponent is positive.

In a Regge limit, where some invariants grow at fixed values of the others, $K(3,6)$ behaves instead as a power, $\Gamma(\beta)\,\lambda^{-\beta}$ times a function of the fixed invariants. The exponent β is always one of the sixteen channel forms κ_a evaluated on the fixed invariants, the analog of the linear trajectory $\alpha(t) = t$. Which of the sixteen appears is determined by the direction of the limit through the tropical function of the integrand, and changes across walls in the space of directions, as for the Koba–Nielsen integral [15, 58].

4 Double copy and the intersection kernel

Kawai, Lewellen and Tye derived their relations, which express a closed-string tree amplitude as a bilinear in open-string amplitudes [9], by a worldsheet argument. Mizera showed that the same relations follow from twisted de Rham theory, with no reference to a worldsheet [36, 37]. In that formulation the bilinear identities among disk integrals are the twisted period relations of Cho and Matsumoto [33] for $\mathcal{M}_{0,n}$, and the KLT kernel is the inverse of the Kita–Yoshida intersection matrix [34, 35] of the corresponding twisted cycles. Relations of this kind require four ingredients: a space, a multivalued function on it, twisted homology and cohomology of finite rank, and bases of cycles and cocycles. Arkani-Hamed, He and Lam defined the closed stringy integral of any polytope and showed that its leading order agrees with that of the open integral [15]. They also asked whether the single-valued projection between closed- and open-string amplitudes extends to stringy integrals.

For the configuration space $X(3,6)$ with the integrand of Eq. (11) all four ingredients are available, and no worldsheet is known. The real points of $X(3,6)$ with the twenty minors removed have 372 connected components, the generalized color orderings of Cachazo, Early and Zhang [38], and each component defines a twisted cycle. Sixty are convex, labeled by the dihedral orderings of six points, with the generalized Parke–Taylor forms as canonical forms. The remaining 312 do not enter below. We give the intersection matrix of the sixty convex chambers in closed form and find, numerically, that its rank at finite α' is exactly twenty-six, so that its inverse on any basis of twenty-six of them is a KLT kernel. We then show that no combination of seven-point disk or sphere integrals with constant coefficients and a linear kinematic map equals $K(3,6)$, and that $X(3,6)$ is not $\mathcal{M}_{0,7}$ in other coordinates. The combinatorial statements below hold for all kinematics. Every statement about the values of integrals at finite α' is numerical and is made at the kinematic point $\hat{s}$ of Eq. (62). At a second point $\hat{s}'$ we check the sign of the KLT bilinear of Section 4.3.

4.1 Chamber integrals and the period relation

Write $u(x;s) = \prod_J (J)^{s_J}$ for the product over the twenty minors in Eq. (11), a multivalued function on $X(3,6)$. Let Ω_a be the canonical form of a convex chamber a, the dlog four-form with unit leading residues on its boundary. We define the chamber integrals, the analogs here of the color-ordered open-string amplitudes, by

$$Z(\gamma|a)(\varepsilon) \;=\; \int_\gamma u(x;s)^\varepsilon\,\Omega_a, \qquad Z(I|I)(\varepsilon) = K(3,6)(\varepsilon s), \tag{57}$$

where $I = X^+(3,6)$ is the positive chamber and Ω_I is the form $\omega_{3,6}$ of Eq. (11). On a real chamber γ we take for $u^{\pm\varepsilon}$ the positive branch, each minor replaced by its absolute value. The sixty convex chambers form one orbit of the relabeling group S_6, and one integral with relabeled arguments serves for all of them.

The forms Ω_a define classes in the twisted cohomology of $\nabla_\varepsilon = d + \varepsilon\, d\log u\wedge$ and the loaded chambers $\gamma \otimes u^\varepsilon$ classes in the twisted homology. Both spaces have dimension twenty-six, the number of critical points of $\log u$ [15, 59]. These two spaces and their counterparts at $-\varepsilon$ admit three pairings. The first is the period pairing of a loaded chamber with a form, the integral (57). The second, the intersection pairing of two cocycles Ω_a, Ω_b, localizes on the critical points and is proportional to the generalized biadjoint amplitude $m(a|b)$ [14, 37, 60], the leading order of (57). The third, the intersection pairing of two loaded cycles,

$$H(\gamma,\delta)(\varepsilon) \;=\; \langle\, \gamma\otimes u^\varepsilon\,,\, \delta\otimes u^{-\varepsilon}\,\rangle, \tag{58}$$

is the topological intersection number of γ, regularized in the sense of Kita and Yoshida [34, 35], against δ carrying the inverse local system. It is a rational function of the phases $e^{2\pi i\varepsilon\kappa}$ that u^ε acquires around the boundary divisors, and requires no integration. Cho and Matsumoto showed that the three pairings are not independent [33]: for any basis B of twenty-six convex chambers,

$$H(\gamma,\delta)(\varepsilon) \;=\; \Big(\frac{\varepsilon}{2\pi i}\Big)^4 \sum_{a,b\in B} Z(\gamma|a)(\varepsilon)\,\big(m_B^{-1}\big)_{ab}\, Z(\delta|b)(-\varepsilon), \tag{59}$$

with m_B the 26×26 block of biadjoint amplitudes. The left side is a finite trigonometric sum over the cones of the D_4 cluster fan described below, while the right side involves the chamber integrals, which we evaluate numerically.

The simplest instance of (59) is $\gamma = \delta = I$, the analog of the quadratic relation between Beta functions at opposite arguments [36]. For the self-intersection number $H(I,I)$ the face rule (64) below gives a sum over the D_4 cluster fan of the positive chamber. With $\kappa_r(s)$ the sixteen channel forms of Eq. (15), one on each ray,

$$H(I,I)(\varepsilon) \;=\; \sum_C \prod_{r\in C} \frac{1}{e^{2\pi i\varepsilon\kappa_r} - 1} \;=\; \sum_{v\in\mathbb{Z}^4} e^{-2\pi i\varepsilon\,\ell(v)}, \tag{60}$$

where the first sum runs over all cones C of the fan including the empty one, and ℓ is the piecewise-linear tropical function of the integrand. The lattice-point form shows that any unimodular refinement of the fan gives the same answer. The cluster fan is complete and simplicial, the cone over a triangulated three-sphere, with f-vector $(1, 16, 66, 100, 50)$ and 233 cones in all. The triangulation of Eq. (25), with 241 cones in all, gives the same function. At leading order (60) reduces to $(2\pi i\varepsilon)^{-4}c_0$, the canonical function of $P(3,6)$, and the right side of (59) to $(2\pi i\varepsilon)^{-4}m(I|I)$. Thus at order ε^{-4} the period relation is the statement that the 60×60 matrix $m(a|b)$ has rank twenty-six with $m(I|I) = c_0 = m(3,6)$, which holds exactly [14, 46].

At $\gamma = \delta$ both sides of (59) are even functions of ε, for any kinematics. On the left this follows from $(e^{-2\pi ix} - 1)^{-1} = -1 - (e^{2\pi ix} - 1)^{-1}$ together with

$$\sum_{C \supseteq S} (-1)^{|C|-|S|} \;=\; (-1)^{|S|} \tag{61}$$

for every cone S of a complete simplicial fan in four dimensions, whose face poset is that of a three-sphere (Appendix A.4). On the right it follows from the symmetry of m_B^{-1}. Hence the odd orders of (59) are satisfied identically. The order ε^{-2} involves only the weight-two coefficient of $Z(I|I)$. All twenty-six chamber integrals first enter at order ε^0, that is, at weight four.

We evaluate (59) at the rational point $\hat{s}$ of the convergence region $\mathcal{C}$,

$$87\,\hat{s}: \quad \begin{array}{l} (s_{135}, s_{136}, s_{145}, s_{146}, s_{156}) = (-35, -26, 38, -20, 38), \\ (s_{235}, s_{236}, s_{245}, s_{246}, s_{256}) = (-38, -32, -38, -29, -20), \\ (s_{345}, s_{346}, s_{356}, s_{456}) = (87, -29, -23, 58), \\ (s_{123}, s_{124}, s_{125}, s_{126}, s_{134}, s_{234}) = (120, -43, -67, 83, -88, 64), \end{array} \tag{62}$$

the last six being fixed by the first fourteen through momentum conservation. There the sixteen channel forms, in units of $1/87$, take the values 34, 38, 53, 58, 64, 69, 75, 83, 87, 93, 111, 114, 120, 145, 163, 171, from $t_{5612} = 34/87$ to $R_{16,23,45} = 57/29$. Both sides of (59) therefore converge for $|\varepsilon| < 29/57$. The left side is an exact Laurent series $H(I,I) = \sum_e h_e\,(2\pi i\varepsilon)^e$ beginning

$$h_{-4} \;=\; \frac{9725966538575915127}{80757169778288000} \;=\; c_0(\hat{s}), \qquad h_{-2} \;=\; \frac{12218413985132565727}{87217743360551040000}. \tag{63}$$

Numerically, at $\hat{s}$, we find that (59) holds at $\gamma = \delta = I$ order by order through ε^2, where all twenty-six forms and the weight-six coefficients enter. We also find that it holds as an identity between functions of ε at $\varepsilon = 1/20$, $1/7$, $1/4$ and $2/5$. At the last value, $114/145$ of the radius of convergence, the terms beyond order ε^2 make up more than one percent of $H(I,I)$.

4.2 The intersection form on the cluster fan

To write the left side of (59) in closed form for every pair of chambers, we fix $\gamma = I$ and let c run over the other fifty-nine convex chambers. For fifteen orderings the closures of c and I share no cone of the fan, and these are precisely the fifteen with $m(I|c) = 0$. For the rest the shared cones form the closed star of a single face F_c. Its dimension is one, two, three or four for six, twelve, fourteen and twelve orderings respectively. On that star we fix the branch of u^ε on c as follows. In the cone charts of Eq. (25) each minor is a monomial in the cone coordinates times a polynomial equal to one at the origin. Near the origin we reach c from I by reversing the signs of the coordinates along the rays of F_c, that is, by continuing across the divisors of F_c and no others. We then write the intersection number as the face sum of Kita and Yoshida [34, 35, 59], with one crossing factor for each ray of F_c and one same-side factor for each remaining ray of a shared cone:

$$H(I,c)(\varepsilon) \;=\; \prod_{r \in F_c} \frac{1}{2i\sin(\pi\varepsilon\kappa_r)} \sum_{C \supseteq F_c} \prod_{r \in C \setminus F_c} \frac{1}{e^{2\pi i\varepsilon\kappa_r} - 1}, \tag{64}$$

and a general entry $H(\gamma,\delta)$ follows by relabeling. For $c = I$ itself the shared face is $F_c = \varnothing$ and (64) reduces to (60). The face-and-side data in (64) are properties of the D_4 cluster fan and do not depend on s, which enters only through the sixteen linear forms $\kappa_r(s)$.

The rule (64), which we call the face rule, is the Kita–Yoshida formula for hyperplane arrangements applied to the cluster compactification of $X^+(3,6)$, where it has not been derived, so we test it. Numerically, at $\hat{s}$ and at several values of ε inside the radius of convergence, the 60×60 matrix $H(\varepsilon)$ built from (64) is real and symmetric, and

$$\operatorname{rank} H(\varepsilon) \;=\; 26 \tag{65}$$

exactly, at finite ε: the twenty-seventh singular value vanishes to the fifty significant figures carried. We also evaluate the right side of (59) at $\varepsilon = 1/7$ for all sixty choices of γ and of δ, which requires the integrals of all twenty-six basis forms over all sixty chambers at both signs of ε. The result reproduces (64) entry by entry, including its vanishing entries (Appendix B.3).

Rank twenty-six at finite α' means that the sixty loaded convex chambers span the whole twenty-six-dimensional space of twisted cycles. Hence a double copy built on convex chambers involves twenty-six independent chamber integrals. On any basis B of twenty-six convex chambers the block $H_B(\varepsilon)$ of the intersection matrix is invertible, and

$$S_B(\varepsilon) \;=\; H_B(\varepsilon)^{-1} \tag{66}$$

is the KLT kernel of $K(3,6)$, exact in α'. With H_B given by the face rule (64) it is a 26×26 matrix of trigonometric functions of the sixteen $\varepsilon\kappa_r$. We conjecture that (64) is the intersection matrix of the loaded convex chambers of $X(3,6)$ for all kinematics in the convergence region, with F_c the face of the D_4 cluster fan along which the closures of c and I meet. The evidence is at the one point $\hat{s}$: the rank (65), and the entrywise agreement of (64) with the period relations. Neither the rule (64) nor these checks cover the 312 non-convex chambers.

Other assignments of phases to the face-and-side data of (64) are possible, such as a half-monodromy $\mp e^{\pm i\pi\varepsilon\kappa_r}/(e^{2\pi i\varepsilon\kappa_r} - 1)$ in place of the crossing factor. At $\hat{s}$ each gives a matrix that is complex or of full rank, that changes between the 233-cone and 241-cone fans, where (64) does not, and that differs from the right side of (59) by amounts of order one. Of these assignments only (64) satisfies the period relations, and this uniqueness is the $k = 3$ counterpart of Mizera's result [36] for the string KLT kernel.

4.3 The KLT bilinear and the single-valued map

Given the intersection form, we define the closed-string analog of $K(3,6)$, the closed function $\mathcal{M}(\varepsilon)$, on a basis B of twenty-six convex chambers by

$$\mathcal{M}(\varepsilon) \;=\; \sum_{\gamma,\delta\in B} Z(\gamma|I)(\varepsilon)\, S_B(\varepsilon)_{\gamma\delta}\, Z(\delta|I)(\varepsilon) \;=\; \sum_{w\geq 0} \varepsilon^{\,w-4}\, \mathcal{M}_{w-4}, \tag{67}$$

the KLT bilinear of the twenty-six chamber integrals of the single form Ω_I. Here both factors are taken at $+\varepsilon$, and no complex conjugate arises because s, the chambers and Ω_I are real. For the disk integrals over $\mathcal{M}_{0,n}$ this combination is the closed-string sphere integral [9, 36]. Its low-energy expansion is the image of the open-string one under the single-valued map [39–41], the ring homomorphism on motivic multiple zeta values with

$$\operatorname{sv}\zeta_{2k} = 0, \qquad \operatorname{sv}\zeta_{2k+1} = 2\,\zeta_{2k+1}. \tag{68}$$

Numerically, at $\hat{s}$, we find

$$\frac{\mathcal{M}_{w-4}}{(2\pi)^4} \;=\; \mathrm{sv}\, c_w(\hat{s}), \qquad w \le 6. \tag{69}$$

Written out with the coefficient functions of Section 5, $c_2 = q_2\zeta_2$, $c_3 = q_3\zeta_3$, $c_4 = q_4\zeta_4$, $c_5 = Q_5\zeta_5 + R_5\zeta_2\zeta_3$ and $c_6 = A\pi^6 + B\zeta_3^2$, the content of (69) is

$$\begin{aligned}
&\frac{\mathcal{M}_{-4}}{(2\pi)^4} = c_0, \quad \frac{\mathcal{M}_{-2}}{(2\pi)^4} = 0, \qquad \frac{\mathcal{M}_{-1}}{(2\pi)^4} = 2\, q_3\, \zeta_3, \\
&\frac{\mathcal{M}_{0}}{(2\pi)^4} = 0, \quad \frac{\mathcal{M}_{1}}{(2\pi)^4} = 2\, Q_5\, \zeta_5, \quad \frac{\mathcal{M}_{2}}{(2\pi)^4} = 4\, B\, \zeta_3^2,
\end{aligned} \tag{70}$$

together with $\mathcal{M}_{-3} = 0$, which is empty since $c_1 = 0$. From $\mathcal{M}_0$ on, all twenty-six chamber integrals and the subleading terms of the kernel (66) enter, and the second line of (70) is the nontrivial test. The weight-four zero holds to 1.2×10^{-31} of $\mathcal{M}_{-4}$, and Appendix B.3 gives the accuracy at weights five and six.

In forming the second line of (70) the terms that cancel are comparable in magnitude to the terms that remain, or larger. At $\hat{s}$ the rational coefficients are

$$\begin{aligned}
q_3 &= \frac{217934183405996377667}{1264657278727990080}, & Q_5 &= \frac{344051981026090274 6657}{2798886240553262256 00}, \\
R_5 &= -\frac{12070767941147022436823413}{4786095471346078457760 0}, & 945\, A &= \frac{669955990777728785850979 3897}{3506444682165126954316 8000}, \\
B &= \frac{303102876085025257208994119}{20819515300355441291256 00},
\end{aligned} \tag{71}$$

so that at weight five the remaining term $2Q_5\zeta_5 \approx 25.5$ is twenty times smaller in magnitude than the canceling term $R_5\zeta_2\zeta_3 \approx -498.7$, and at weight six the remaining term $4B\zeta_3^2 \approx 841.5$ is accompanied by a canceling term $A\pi^6 \approx 194.4$. We have not assembled the orders ε^3 and ε^4.

Substituting (59) into (67), we express $\mathcal{M}$ through the period matrix at $+\varepsilon$ times the inverse of the period matrix at $-\varepsilon$. This product has the form of the single-valued period construction of Brown and Dupont [12, 41], with $u \to u^{-1}$ in the role of complex conjugation. For $\mathcal{M}_{0,n}$ they prove that the result is the image of the open-string expansion under (68). Equation (69) is evidence that the same identification holds on $X(3,6)$, and we conjecture that it does, to all orders and for all kinematics in the convergence region.

For real exponents and real chambers the twisted Riemann relations [33, 36, 41, 61] identify the bilinear (67), formed with the inverse of the intersection matrix, with the integral of the single-valued integrand over the complex points of $X(3,6)$,

$$\mathcal{M}(\varepsilon) \;=\; \int_{X(3,6)(\mathbb{C})} |u|^{2\varepsilon}\, \Omega_I \wedge \overline{\Omega}_I \;=\; 16 \int_{X(3,6)(\mathbb{C})} |u|^{2\varepsilon}\, |\omega_I|^2\, d^8x, \qquad \Omega_I = \omega_I\, d^4x, \tag{72}$$

in the normalization of (64). In complex dimension n the constant is $(2i)^n$, which is real and positive for $n = 4$. Equation (72) is the closed stringy integral that Arkani-Hamed, He and Lam associate with the polytope $P(3,6)$ [15]. The right-hand side of (72) is the integral of a positive density, and hence $\mathcal{M} > 0$ wherever it converges absolutely. Near a boundary divisor D with local coordinate y the integrand behaves as $|y|^{2(\varepsilon\kappa_D + K_D - 1)}\, d^2y$, where $K_D - 1$ is the order of vanishing of Ω_I on D, equal to -1 on the sixteen walls. The integral converges absolutely when $\varepsilon\kappa_D > -K_D$ for every D. The convex chambers meet forty-nine further divisors, relabelings of the sixteen walls. On thirty-six of them Ω_I is regular, and on the thirteen whose exponents are most negative it vanishes to first or second order. Along the ray through $\hat{s}$ the domain of absolute convergence of (72) is therefore $0 < \varepsilon < 87/88$, beyond the radius of convergence $29/57$ of the expansion.

The residues of $\mathcal{M}$ along each wall are, at every level, integrals of positive densities over the complex facet, positive where they converge. This is positivity of a measure, which the Virasoro–Shapiro amplitude has in any spacetime dimension, and not a no-ghost condition. The no-ghost condition of the closed string is positivity of $\sum \mathcal{M}_L \mathcal{M}_R$ on the elastic slice, level by level, and $K(3,6)$ has no elastic slice on which to state it (Section 3.3). A bilinear period identity such as (59) involves neither a state space nor a positive inner product.

Since $H_B(\varepsilon)$ is indefinite, the sign of the bilinear (67) built from the face rule is a further test of the rule: if (64) is the intersection matrix, then (72) holds and $\mathcal{M} > 0$ wherever (72) converges absolutely. Numerically, at $\hat{s}$, we find $\mathcal{M}(\varepsilon)$ positive and decreasing at fourteen values $\frac{1}{10} \leq \varepsilon \leq \frac{1}{2}$. It is regular through $\varepsilon = 87/218$, where $\varepsilon\kappa_D = -1$ on a relabeled divisor and the entries of H_B have a pole. There the inertia of H_B changes from $(11,15)$ to $(12,14)$ positive and negative eigenvalues. At the second point $\hat{s}'$ of Eq. (144) we find $\mathcal{M}(\varepsilon) > 0$ at six values in the same range, with inertia $(16,10)$, changing to $(17,9)$ at $\varepsilon = \frac{1}{2}$. The positivity $\mathcal{M}(\varepsilon) > 0$ is thus the $k = 3$ counterpart of $\sum_{\alpha,\beta} A(\alpha)\, S[\alpha|\beta]\, \tilde{A}(\beta) > 0$ for the string: the period vector of the positive chamber has positive norm with respect to an indefinite form whose inertia depends on the kinematics.

The period relation (59), the kernel (66) and the expansion (69) involve only the twisted cohomology of the integration domain, as the intersection-theoretic formulation [36,37,41] leads one to expect. Here they hold on $X(3,6)$, which is not $\mathcal{M}_{0,7}$ in other coordinates (Section 4.5). The basis of convex Parke–Taylor forms, however, is special to six points: sixty convex orderings are more than enough to span a twenty-six-dimensional cohomology, but at seven points there are 360 convex orderings and $|\chi(X(3,7))| = 1272$ (Section 7), so the convex Parke–Taylor forms cannot span the cohomology and a double copy of $K(3,7)$ would require a different basis of forms.

4.4 Seven-point disk and sphere integrals

The natural candidate for a presentation of $K(3,6)$ as an ordinary string integral in other coordinates is a seven-point integral. Like $X(3,6)$, the moduli space $\mathcal{M}_{0,7}$ is four-dimensional with fourteen independent invariants, momentum conservation at seven points having rank seven on the twenty-one s_{ij}. Let $Z_7(\alpha|\beta)$ denote the seven-point disk integral of the Koba–Nielsen factor against the Parke–Taylor form of the ordering β over the domain of the ordering α [13,62,63], and $J_7(\alpha|\beta)$ its sphere analog. We ask whether

$$K(3,6)(\varepsilon s) \;\overset{?}{=}\; \sum_{\alpha,\beta} c_{\alpha\beta}\, Z_7(\alpha|\beta)(\varepsilon L(s)), \tag{73}$$

with $c_{\alpha\beta}$ constants and L a linear map from the fourteen s_{abc} to the seven-point Mandelstam invariants. The target of L is the massless on-shell space, $\sum_{j\neq i} s_{ij} = 0$ and $s_I = s_{\bar{I}}$. This is not an extra assumption: the Koba–Nielsen factor of Z_7 is SL(2)-invariant only there.

Claim 2. *There are no constants $c_{\alpha\beta}$ and no linear map L from the fourteen generalized Mandelstam invariants of $X(3,6)$ to the massless seven-point Mandelstam invariants such that*

$$m(3,6) \;=\; \sum_{\alpha,\beta} c_{\alpha\beta}\, m_7(\alpha|\beta) \circ L. \tag{74}$$

Consequently $K(3,6)$ is not a constant-coefficient combination $\sum c_{\alpha\beta} Z_7(\alpha|\beta) \circ L$ of seven-point disk integrals, nor of seven-point sphere integrals $J_7(\alpha|\beta) \circ L$, whenever the leading order of the combination does not vanish.

We pass from (73) to (74) by taking $\varepsilon \to 0$, since L commutes with the scaling and Z_7 and J_7 have the same leading order $m_7(\alpha|\beta)$. Only seven points are compatible with homogeneity in s

when the coefficients are constant. A biadjoint amplitude $m_N(\alpha|\beta)$ is a sum of products of $N-3$ propagators, while $c_0 = m(3,6)$ is a sum of products of four channel forms,

$$m_N(\alpha|\beta)(\lambda s) = \lambda^{-(N-3)}\, m_N(\alpha|\beta)(s), \qquad m(3,6)(\lambda s) = \lambda^{-4}\, m(3,6)(s), \tag{75}$$

and constant coefficients with a linear L require $N-3=4$. The proviso on the leading order is empty at $N=7$: $K(3,6)$ begins at ε^{-4} with $c_0 \neq 0$, and a seven-point combination whose leading order cancels begins one order later.

Three classes of presentation are not covered by the Claim and remain open. The first is kinematic polynomial prefactors of degree $N-7$ at $N \geq 8$, which the reduction of a correlator of local vertex operators to the $(N-3)!$ Parke–Taylor basis does produce [62]. The second is constant-coefficient combinations of Z-integrals at $N \geq 9$ whose orders ε^0 through ε^{N-8}, counting the leading order of each integral as order zero, vanish identically [63], divided by ε^{N-7} to restore degree -4. The four-point instance of such a cancellation is $Z(1234|1234) - Z(1423|1234) - Z(1342|1234) = 3\zeta_2\, u + O(\varepsilon)$. The third, at $N=7$ itself, is coefficients that are ratios of homogeneous polynomials of equal degree. Each of the three classes requires a different argument: the proof below determines L from the polar divisor of $m(3,6)$, and coefficients with poles and zeros of their own would change that divisor.

We now prove Claim 2. Suppose that (74) holds; we derive a contradiction. The poles of $m(3,6)$ are the sixteen channel hyperplanes $\kappa_a(s) = 0$ of Section 3. These sixteen linear forms span the fourteen-dimensional dual space, no two are proportional and no three are linearly dependent. On the right-hand side of (74) the poles lie on hyperplanes $s_I \circ L = 0$. Hence every pole hyperplane $\kappa_a = 0$ of the left-hand side is among them, and each κ_a is a multiple of some $s_I \circ L$ and vanishes on $\ker L$. Since the sixteen κ_a span, $\ker L = 0$, and with fourteen dimensions on both sides L is a bijection. We may therefore assign to each facet F_a of $P(3,6)$ a seven-point channel I_a with

$$\kappa_a \;\propto\; s_{I_a} \circ L, \qquad a = 1, \ldots, 16, \tag{76}$$

and the assignment $a \mapsto I_a$ is injective because no two κ_a are proportional.

Next we take double residues. There are fifty-six seven-point channels s_I, with $|I| = 2$ or 3 modulo $s_I = s_{\bar{I}}$. On the on-shell space no two are proportional and no three are dependent, and an iterated residue at any pair of them is therefore well defined. Since a planar cubic tree never carries two channels that cross, the double residue of each $m_7(\alpha|\beta) \circ L$ at an incompatible pair vanishes term by term. On the left-hand side, $m(3,6)$ is the canonical function of $P(3,6)$ [15], and its double residue at (κ_a, κ_b) is nonzero exactly when F_a and F_b meet in a codimension-two face [16]. Since $P(3,6)$ has sixty-six two-faces, the double residue is nonzero on sixty-six of the $\binom{16}{2} = 120$ pairs and zero on the other fifty-four. It follows that

$$F_a \cap F_b \text{ a two-face of } P(3,6) \quad \Longrightarrow \quad I_a,\ I_b \text{ compatible.} \tag{77}$$

The contradiction comes from the two non-simple vertices of $P(3,6)$ (Section 3). At each of them five facets meet, three of type t and two of type R. Their channel forms satisfy one of the two relations (16): at one vertex

$$R + \tilde{R} \;=\; t_{1234} + t_{3456} + t_{5612}, \tag{78}$$

with R and $\tilde{R}$ as in Eq. (15), and at the other vertex its image under a cyclic shift. The three t facets in (78) are pairwise adjacent and each R facet is adjacent to all three, while R and $\tilde{R}$ are not adjacent to each other. Pulling (78) back through L^{-1} with (76) and (77), we obtain five distinct seven-point channels obeying a linear relation with five nonzero coefficients: a pairwise compatible triple, and two further channels each compatible with all three. A pairwise compatible triple of seven-point channels is a cubic tree with one propagator removed. Exactly three further channels are compatible with the whole triple, namely the three resolutions of the resulting quartic vertex. With 1260 such triples this gives

$$1260 \times \binom{3}{2} \;=\; 3780 \tag{79}$$

candidate quintuples. We find by enumeration that every one of them is linearly independent on the on-shell space (Appendix B.7), and hence none satisfies a relation of the form (78). This completes the proof of Claim 2.

The adjacency condition (77) is by itself a second obstruction, independent of the five-term relation: we find that no injective map from the sixteen facets to the fifty-six channels sends all sixty-six adjacent pairs to compatible pairs. Already the nine facets $t_{1234}, \ldots, t_{6123}$, s_{234}, s_{456}, s_{612} admit no compatible image. Those nine include no R facet, and their pairwise adjacency is the same in the simple D_4 cluster polytope of [15], which therefore has the same obstruction. We also repeat the enumeration of quintuples at eight, nine and ten points. It gives 236250, 5336100 and 80302530 of them, none dependent, so through ten points nested collisions of ordinary punctures produce no relation of the form (78), and we see no mechanism by which more points would.

All the ingredients of the proof of Claim 2 are in the literature: the positive tropical Grassmannian with its two bipyramids [19, 20], the poles R, $\tilde{R}$ and the bipyramid term of [14], the leading order as a canonical function [15, 16], and the facets of the polytopes $P(k, n)$ [17]. Arkani-Hamed, He and Lam also remarked that residues of Grassmannian string integrals need not factorize into two lower-point integrals [15]. Our proof uses that non-factorization, in the form of the five-term relations at the two non-simple vertices, to exclude a seven-point presentation with constant coefficients and a linear kinematic map.

4.5 Euler characteristic and point count

The hypotheses of Claim 2, constant coefficients and a linear kinematic map, are restrictive, but two elementary invariants distinguish $X(3, 6)$ from $\mathcal{M}_{0,7}$ independently of any integral or presentation. The first invariant is the Euler characteristic, 26 for $X(3, 6)$ against $4! = 24$ for $\mathcal{M}_{0,7}$. The second is the number of points over a finite field,

$$\big|\mathcal{M}_{0,7}(\mathbb{F}_q)\big| = (q-2)(q-3)(q-4)(q-5), \qquad \big|X(3,6)(\mathbb{F}_q)\big| = (q-2)(q-3)(q^2-9q+21), \tag{80}$$

the $X(3, 6)$ count being Glynn's [64]. Setting $q = 1$ in the two counts gives the two Euler characteristics. The quadratic factor $q^2 - 9q + 21$ has discriminant -3 and is irreducible over $\mathbb{Z}$. A polynomial point count is the E-polynomial of the variety [65], so the irreducible quadratic factor is a property of the variety $X(3, 6)$ itself and does not depend on the presentation.

We can trace the quadratic factor in (80) by fibering both spaces over $\mathcal{M}_{0,5}$, $X(3, 6)$ by projecting from one of the six points and $\mathcal{M}_{0,7}$ by forgetting two of the seven punctures. In an affine chart the fiber of $\mathcal{M}_{0,7}$ is the complement of nine lines in the pattern $4 + 4 + 1$, with four triple and twelve double points. The fiber of $X(3, 6)$ is also the complement of nine lines, three in each of two pencils together with three more, but they meet in three quadruple and twelve double points. Their characteristic polynomials are

$$\chi_{\mathcal{M}_{0,7}\text{-fiber}}(t) = (t-4)(t-5), \qquad \chi_{X(3,6)\text{-fiber}}(t) = t^2 - 9t + 21, \tag{81}$$

with $|\chi| = 12$ and 13. The arrangement of every $\mathcal{M}_{0,N}$, an iterated fibration by punctured lines, is supersolvable, and its characteristic polynomial factors into linear forms over $\mathbb{Z}$ [66]. The fiber arrangement of $X(3,6)$ is not supersolvable, and its count retains the irreducible quadratic factor of (81).

Projecting from two of the six points, we present $X(3,6)$ inside $\mathcal{M}_{0,5}\times\mathcal{M}_{0,5}$. Write the columns of Eq. (20) as $p_1 = (0{:}0{:}1)$, $p_2 = (0{:}-1{:}0)$ and $p_j = (1{:}\sigma_j^+{:}\sigma_j^-)$ for $j = 3,\dots,6$, with $(\sigma_3^+,\sigma_3^-) = (0,0)$ and $(\sigma_4^+,\sigma_4^-) = (1,1)$. Projection from p_1 sends the other five points to the punctures $\infty,\sigma_3^+,\dots,\sigma_6^+$ of a left line, and projection from p_2 to $\infty,\sigma_3^-,\dots,\sigma_6^-$ of a right line. Together they give the classical birational identification of $\mathbb{P}^2$ with $\mathbb{P}^1\times\mathbb{P}^1$ [47,48]. The twenty minors are then

$$\begin{gathered}(12j) = (134) = (234) = 1, \qquad (1jk) = \sigma_k^+ - \sigma_j^+, \qquad (2jk) = \sigma_k^- - \sigma_j^-, \\ (ijk) = \det\left(1,\sigma^+,\sigma^-\right)_{ijk} \equiv A_{ijk}\end{gathered} \tag{82}$$

for $3\le i<j<k\le 6$, and the integral reads

$$K(3,6) \;=\; \int_{X^+} \Omega \prod_{3\le j<k\le 6} (\sigma_k^+ - \sigma_j^+)^{\varepsilon s_{1jk}} (\sigma_k^- - \sigma_j^-)^{\varepsilon s_{2jk}} \prod_{3\le i<j<k\le 6} A_{ijk}^{\varepsilon s_{ijk}}, \tag{83}$$

with $\Omega = d\sigma_5^+\, d\sigma_6^+\, d\sigma_5^-\, d\sigma_6^- / [(\sigma_6^+ - \sigma_5^+)A_{345}A_{456}]$ and $X^+ = \{1<\sigma_5^+<\sigma_6^+,\ A_{345}>0,\ A_{456}>0\}$. Thus $X(3,6)$ is the open subset of $\mathcal{M}_{0,5}\times\mathcal{M}_{0,5}$ on which no three of $p_3,\dots,p_6$ are collinear. Its letters are the five differences on each line and the four areas $A_{345}, A_{346}, A_{356}, A_{456}$, each a difference of the left and right cross-ratios $u_L = \sigma_{ij}^+/\sigma_{ik}^+$, $u_R = \sigma_{ij}^-/\sigma_{ik}^-$ of $(\infty, i; j, k)$,

$$A_{ijk} \;=\; \sigma_{ik}^+\sigma_{ik}^-\,(u_L - u_R), \qquad \sigma_{ik}^\pm = \sigma_k^\pm - \sigma_i^\pm, \tag{84}$$

and p_i, p_j, p_k are collinear exactly when $u_L = u_R$. In the form (83), $K(3,6)$ is thus an integral over five punctures on each of two lines, with one Koba–Nielsen factor for each line, and the two lines are coupled only through the four area letters A_{ijk}.

The form of the letters in a presentation, however, is not an invariant. If we place the fifth point of $X(3,5)$ at $(1{:}x{:}y)$ in the gauge of Eq. (82), its five letters are x, $x-1$, y, $y-1$ and the area letter $y-x = A_{345}$. Yet $X(3,5)\cong X(2,5)$, and $K(3,5) = K(2,5)$ is the five-point disk integral, in which $y-x$ is an ordinary collision of two punctures. Hence the area letters of $K(3,6)$ do not by themselves distinguish it from a string integral, whereas $\chi = 26 \ne 24$ and the two counts (80) are independent of the presentation. None of these arguments rules out a presentation with $N\ge 8$ punctures and kinematic prefactors (Section 8.4).

5 The low-energy expansion and its arithmetic

The coefficients in the low-energy expansion of the Veneziano amplitude are Riemann zeta values, and those of the n-point disk integrals are the multiple zeta values of (9). Both facts are consequences of a theorem on the periods of the integration domain: every period of the moduli space $\mathcal{M}_{0,n}$ of n points on a line is a multiple zeta value [10–12]. The class of constants in the α'-expansion is therefore determined by the integration domain. One can ask which ring the constants lie in for any integral of this type, whether or not a physical theory produces it. For $k=2$ the whole expansion is understood in these terms, with the weight of the zeta values set by the order in α' [13,39]. For the periods of $X(3,6)$ there is no such theorem. A double integral of dlog forms can already be a period of a cusp form, as Brown and Duhr showed for a configuration in $\mathbb{P}^2$ consisting of a plane elliptic curve and a set of lines [67]. For the Grassmannian integrals with $k\ge 3$ little was known beyond leading order [14,15], and whether the higher orders involve multiple zeta values only was left open in [15].

We compute the expansion of $K(3,6)$ as a function of the kinematics through weight six and at two rational points through weight eight. Every coefficient is a rational combination of multiple zeta values. The individual cone integrals of (25) contain constants that are not zeta values. These cancel in the sum over cones by the gluing of adjacent cones described in Section 5.2, identically in s through weight five granted finitely many numerically identified relations among the cone constants, and at every kinematic point computed at weight six. Independently of the gluing, linear reducibility of the positive parametrization implies that at every order w the coefficient c_w is a combination of multiple zeta values of weight at most w with coefficients rational in s (Theorem 3). Through weight five we sharpen the theorem by giving the coefficient functions explicitly, assuming the same relations among the cone constants in place of linear reducibility.

In the expansion (14),

$$K(3,6)(\varepsilon s) \;=\; \sum_{w\geq 0} \varepsilon^{\,w-4}\, c_w(s), \qquad c_0 = m(3,6), \quad c_1 = 0, \tag{85}$$

each c_w is homogeneous of degree $w-4$ in the fourteen invariants. Expanding the cone form (25) in ε under the integral gives

$$c_w(s) \;=\; \sum_r W_r(s)\, C_r, \tag{86}$$

a finite sum. Here the C_r are numbers independent of s, namely integrals of products of logarithms of the cone polynomials over faces of the unit cube. The $W_r(s)$ are rational functions built from inverse powers of the channel forms $\kappa_a(s)$. As shown in Section 2.3, at every weight c_w has the form (26), a polynomial $N_w(s)$ of degree $w+12$ over the product of the sixteen channel forms, with all sixteen poles at most simple; through weight seven we find in addition that, apart from $c_1 = 0$, all sixteen poles are present.

Through weight eight the multiple zeta values are spanned, weight by weight, by

$$\zeta_2; \quad \zeta_3; \quad \zeta_4; \quad \zeta_5,\ \zeta_2\zeta_3; \quad \zeta_6,\ \zeta_3^2; \quad \zeta_7,\ \zeta_2\zeta_5,\ \zeta_3\zeta_4; \quad \zeta_8,\ \zeta_3\zeta_5,\ \zeta_2\zeta_3^2,\ \zeta(3,5), \tag{87}$$

of dimensions at most $1,1,1,2,2,3,4$, conjecturally exactly [12]. Through weight seven every element is a polynomial in single zeta values. Weight eight is the first weight with a generator that is conjecturally not such a polynomial, which we take to be

$$\zeta(3,5) \;=\; \sum_{n>m\geq 1} \frac{1}{n^3 m^5} \;=\; 0.20466\ldots. \tag{88}$$

Two expansions whose coefficients are multiple zeta values can therefore first differ in depth at weight eight.

5.1 The coefficients

Table 2 collects the coefficients through weight eight, as functions of s where we have them and at the base point s_0 of Eq. (29). Weight two we can do by hand. Each weight-two constant in (86) is the integral of one logarithm of one cone polynomial along one coordinate axis of a cone. On the 208 axes of the fifty-two cones every one of the ten polynomials restricts to either 1 or $1+y$. Hence every nonzero weight-two constant is the same number,

$$\int_0^1 \log(1+y)\,\frac{dy}{y} \;=\; \frac{\zeta_2}{2}, \tag{89}$$

and $c_2(s) = q_2(s)\,\zeta_2$ identically, with q_2 an explicit rational function.

w	$c_w(s)$	$c_w(s_0)$ for $K(3,6)$	for the form ω' of (100)
0	$m(3,6)$	$\frac{130817}{800}$	$\frac{60767}{800}$
1	0	0	0
2	$q_2(s)\,\zeta_2$	$-\frac{241301}{1200}\,\zeta_2$	$-\frac{106901}{1200}\,\zeta_2$
3	$q_3(s)\,\zeta_3$	$\frac{228617}{1200}\,\zeta_3$	$\frac{86567}{1200}\,\zeta_3$
4	$q_4(s)\,\zeta_4$	$\frac{734017}{3600}\,\zeta_4$	$\frac{93551}{800}\,\zeta_4$
5	$Q_5(s)\,\zeta_5 + R_5(s)\,\zeta_2\zeta_3$	$-\frac{312803}{32400}\,\zeta_5 - \frac{3999979}{16200}\,\zeta_2\zeta_3$	$-\frac{1646603}{32400}\,\zeta_5 - \frac{1569379}{16200}\,\zeta_2\zeta_3$
6	$A(s)\,\pi^6 + B(s)\,\zeta_3^2$	$\frac{142168667}{734832000}\,\pi^6 + \frac{12936979}{97200}\,\zeta_3^2$	$\frac{3681569}{40824000}\,\pi^6 + \frac{5486479}{97200}\,\zeta_3^2$
7		$-\frac{133437637}{583200}\,\zeta_7 - \frac{189359}{12150}\,\zeta_2\zeta_5 + \frac{18539231}{97200}\,\zeta_3\zeta_4$	$-\frac{165333299}{1166400}\,\zeta_7 + \frac{1249157}{24300}\,\zeta_2\zeta_5 + \frac{5607931}{97200}\,\zeta_3\zeta_4$
8		$-\frac{7173761239}{83980800}\,\zeta_8 + \frac{11628931}{145800}\,\zeta_3\zeta_5 - \frac{74762681}{437400}\,\zeta_2\zeta_3^2$	Eq. (101), with $\zeta(3,5)$

Table 2: The low-energy expansion of $K(3,6)$ and of one comparison integral at the base point s_0 of Eq. (29), where all ten polynomial exponents of Eq. (23) equal $-\frac{1}{3}$. Through weight seven the two expansions involve the same products of single zeta values $\zeta_n = \zeta(n)$; at weight eight the comparison integral requires $\zeta(3,5)$, in the convention (88), and $K(3,6)$ does not. Second column: the form of the coefficient as a function of the fourteen invariants on the convergence region, with $q_2, \ldots, B$ rational functions of the shape (26); $q_2, \ldots, R_5$ are exact (Appendix A) and A, B are determined from values of c_6 (Section 5.3). Third column: the values at the base point s_0 of Eq. (29), repeating Eq. (30) for comparison; through weight six they are values of the coefficient functions, and at weights seven and eight they are given by integer relations in the basis (87) found from numerical values of $c_7(s_0)$ and $c_8(s_0)$ (Appendix B.6). Fourth column: the integral over the same region, with the same Koba–Nielsen factor and kinematics, of the four-form ω' of Eq. (100) in place of the canonical form, obtained numerically (Appendix B).

From weight three on, the constants C_r include integrals over faces of the unit cube of dimension greater than one. On the two-dimensional face spanned by a pair of axes of a cone, each cone polynomial restricts to a polynomial $Q(u,v)$ of one of five shapes, such as $1+v+uv$. Each such restriction contributes the double integral $E(Q) = \int_0^1\int_0^1 \log \frac{Q(u,v)}{Q(u,0)\,Q(0,v)}\,\frac{du\,dv}{uv}$. At weight three the C_r lie a priori in the span of ζ_3, $\zeta_2 \log 2$ and $\log^3 2$. We express every weight-four C_r as a rational combination of sixteen elements of the ring of polylogarithms at sixth roots of unity and of three integrals over three-dimensional faces, which we do not reduce to polylogarithms. Nevertheless

$$c_3(s) \;=\; q_3(s)\,\zeta_3, \qquad c_4(s) \;=\; q_4(s)\,\zeta_4, \tag{90}$$

identically in s: the coefficient function of every other direction vanishes as a rational function. Indeed, every weight-three C_r is individually a rational multiple of ζ_3, given the evaluations $E(1+v+uv) = \frac{13}{24}\zeta_3$ and $E(1+u+v) = -\frac{5}{12}\zeta_3$ of two of the five pair integrals, which we identify numerically and do not derive. The weight-four constants are not all multiples of ζ_4, and at that weight the proof in Appendix A.2 consists of seventeen polynomial identities among the $W_r(s)$. Its inputs are the expression of each cone constant in terms of the nineteen constants listed there, together with one relation among those. We find that the coefficients $W_r(s)$ of the individual constants C_r that are not zeta values do not vanish. The identities are linear relations among these W_r under which every direction other than ζ_4 cancels in the sum over cones.

5.2 Gluing across walls of the fan

The constants that are not zeta values arise because a cone integral covers only half of each coordinate line: the coordinates y_i of (25) run from 0 to 1. Two maximal cones that share an internal wall of the fan are glued across it by $y \to 1/y$ in the coordinate transverse to that wall. The two half-line integrals then combine into one integral over the whole line. In the simplest instance,

$$\int_0^1 y^{a-1}(1+y)^{-a-b}\,dy \;+\; \int_0^1 y^{b-1}(1+y)^{-a-b}\,dy \;=\; \int_0^\infty y^{a-1}(1+y)^{-a-b}\,dy \;=\; B(a,b). \tag{91}$$

Each term on the left, expanded in a and b, produces integrals $\int_0^1 \log^m y\,\log^n(1+y)\,dy/y$ of weight $m+n+1$. From weight four on these involve $\log 2$ and $\mathrm{Li}_4(\frac{1}{2})$ as well as zeta values. The right-hand side produces no such constants, because the Beta function is an exponential of single zeta values,

$$B(a,b) \;=\; \frac{a+b}{ab}\,\exp\Big[\sum_{k\geq 2}\frac{(-1)^k\zeta_k}{k}\big(a^k+b^k-(a+b)^k\big)\Big]. \tag{92}$$

When a single polynomial factor depends on the coordinate transverse to an internal wall of the fan, the combined integral of the two adjacent cones over $0<y<\infty$ is the Beta function (91). Its expansion contains only zeta values at every order. In the full integrand several polynomial factors depend on the transverse coordinate and the combined integral is not a Beta function term by term, so (91) describes the mechanism by which the constants that are not zeta values can cancel between adjacent cones rather than proving that they do. The cancellation itself is established as follows: identically in s at weights three, four and five, granted finitely many numerically identified relations among the cone constants (Appendix A), the cancellation across internal walls of the fan being part of the proof at weight four; and at weight six, where the constants include the periods with quadratic letters of Section 5.3, at every rational point computed, to the accuracy of Appendix B. These relations are the weight-three and weight-four inputs named above and, at weight five, a lattice of integer relations among the cone constants, of height at most nine and rank 271, found numerically. Modulo this lattice, c_5 reduces exactly to its ζ_5 and $\zeta_2\zeta_3$ directions by the rank statement Eq. (141), which involves the fan alone. We make no such reduction at weight six, where the cancellation remains a numerical check at each rational point we evaluate.

The gluing also accounts for the first product of zeta values in the expansion. By Proposition 1 the residue of c_5 at each of the four cube walls is the weight-five term of the triple product of Beta functions (40). By (92) this term contains $\zeta_2\zeta_3$ with nonzero coefficient. That coefficient is the residue of R_5 at the same wall, where R_5 is the function that multiplies $\zeta_2\zeta_3$ in Table 2.

5.3 Weights five and six

Weight five is the first weight with two generators, and both appear:

$$c_5(s) \;=\; Q_5(s)\,\zeta_5 \;+\; R_5(s)\,\zeta_2\zeta_3, \tag{93}$$

with Q_5 and R_5 rational functions of the shape (26) and $R_5 \not\equiv 0$. At the generic rational point s^* of Eq. (28) we find

$$Q_5(s^*) = \frac{186226736168150016850683715530527178330620596354050107981300029}{51770921417837547188636567649418967034494981492654036316 9600}\,, \tag{94}$$

$$R_5(s^*) = -\frac{183620089923880183484570158441838917454467824603102322979640600 67}{336510989215944056726137689721223285724217379702251236060 24000}\,, \tag{95}$$

while $Q_5(s_0)$ and $R_5(s_0)$ are the two far shorter fractions in the $K(3,6)$ column of the weight-five row of Table 2. The two-term form (93) holds exactly, granted the integer relations among the C_r

(Appendix A.3). To conclude from it that a product of zeta values is forced at weight five one must assume in addition $\zeta_2\zeta_3 \notin \mathbb{Q}\,\zeta_5$, which is expected and open like every irrationality statement in this ring.

At weight six some of the cone constants C_r are no longer polylogarithms at roots of unity. Among them are periods whose letters involve the quadratic irrationalities $\sqrt{5}$, $\sqrt{-3}$ and $\sqrt{2}$, numbers that are not expected to be polynomials in zeta values and logarithms of rationals. These constants also cancel in the sum over cones, and

$$c_6(s) \;=\; A(s)\,\pi^6 \;+\; B(s)\,\zeta_3^2, \tag{96}$$

with A and B rational functions of the shape (26), which we determine from the values of c_6 at rational kinematic points rather than derive (Appendix B).

Concretely, we take A and B in the space of rational functions spanned by the weight-six W_r and solve exactly for their coordinates from linear conditions. Each condition comes from the value of c_6 at a rational point of $\mathcal{C}$, or from a Taylor coefficient of $c_6 \prod_a \kappa_a$ along a rational line through such points. The two-term integer relation among each such number, π^6 and ζ_3^2 gives one linear condition on the coordinates of A and one on those of B. The pair then reproduces c_6, to the accuracy of its values, at further rational points disjoint from those used. The ratio B/A varies with s, as R_5/Q_5 does, and at neither weight is the coefficient a single constant times a rational function. Table 2, including its weight-seven row, shows that every generator of (87) through weight seven is present in the expansion of $K(3,6)$.

5.4 Weight eight

At weight eight the multiple zeta values span at most four dimensions, conjecturally exactly four, and the polynomials in single zeta values account for only three. Write

$$c_8 \;=\; A_8\,\zeta_8 \;+\; B_8\,\zeta_3\zeta_5 \;+\; C_8\,\zeta_2\zeta_3^2 \;+\; D_8\,\zeta(3,5), \tag{97}$$

with the convention (88) for $\zeta(3,5)$. Numerically, from values of c_8 on two integration grids that agree to fifty-six significant figures (Appendix B), at the base point (29) we find

$$c_8(s_0) \;=\; -\tfrac{7173761239}{83980800}\,\zeta_8 \;+\; \tfrac{11628931}{145800}\,\zeta_3\zeta_5 \;-\; \tfrac{74762681}{437400}\,\zeta_2\zeta_3^2, \qquad D_8(s_0) = 0. \tag{98}$$

This is an integer relation of height about 10^{10} between $c_8(s_0)$ and the weight-eight elements of (87), in which the coefficient of $\zeta(3,5)$ is zero. It reproduces the value of $c_8(s_0)$ on the finer grid to sixty-two significant figures. We find the same three-term form, again with $D_8 = 0$, as a relation of height about 10^{11} that reproduces the finer-grid value to sixty-three significant figures, at a second rational point of the convergence region unrelated to the first,

$$\tilde{s}: \quad \begin{array}{lll} s_{135} = s_{136} = -\tfrac{2}{3}, & s_{146} = s_{235} = s_{236} = s_{245} = s_{246} = -1, & \\ s_{256} = s_{346} = s_{356} = -\tfrac{1}{3}, & s_{145} = \tfrac{1}{3}, & s_{156} = s_{345} = s_{456} = \tfrac{4}{3}. \end{array} \tag{99}$$

Appendix B.6 describes the identification, and the Supplementary Material tabulates the input values and the integer relations at both points. Two zeros of $D_8(s)$, a rational function of degree four in fourteen variables, do not show that it vanishes, and we have not computed D_8 as a function.

Let us nevertheless compare $K(3,6)$ at the base point with a second integral that differs from it only in the differential form. We keep the region $\mathbb{R}^4_{>0}$ of (23) and the Koba–Nielsen factor with the base-point exponents, and replace the canonical form $da\,db\,dc\,dd/(a\,b\,c\,d)$ of the positive part $X^+(3,6)$ by

$$\omega' \;=\; \mathrm{dlog}\frac{a}{1+a} \wedge \mathrm{dlog}\, b \wedge \mathrm{dlog}\, c \wedge \mathrm{dlog}\, d, \tag{100}$$

a dlog form with unit residues on a different set of divisors. The resulting integral pairs the positive chamber with a form other than its canonical form, an off-diagonal pairing of the kind $Z(\gamma|a)$, $a \neq \gamma$, of Section 4. Its expansion is the last column of Table 2, and at weight eight its coefficient contains $\zeta(3,5)$:

$$c_8(\omega') \;=\; -\tfrac{239364162391}{335923200}\,\zeta_8 \;+\; \tfrac{2795059}{36450}\,\zeta_3\zeta_5 \;-\; \tfrac{31228181}{437400}\,\zeta_2\zeta_3^2 \;-\; \tfrac{483731}{29160}\,\zeta(3,5). \tag{101}$$

The sign and size of the last coefficient depend on the convention (88). In terms of $\zeta(5,3) = \sum_{n>m\geq 1} n^{-5}m^{-3}$ the same number reads $-\frac{233791581271}{335923200}\zeta_8 + \frac{324503}{5400}\zeta_3\zeta_5 - \frac{31228181}{437400}\zeta_2\zeta_3^2 + \frac{483731}{29160}\zeta(5,3)$, by the stuffle relation. The denominators in (101) are $\{2,3,5\}$-smooth, and the $\zeta_2\zeta_3^2$ denominator 437400 coincides with that of $K(3,6)$ at the same point.

The region and the form are the two arguments of $Z(\gamma|a)$ in (57), which for disk integrals are two orderings of the points. Thus at a fixed region, here $X^+(3,6)$, the depth-two content of the α'-expansion depends on the second argument, the form. On the pentagon $\mathcal{M}^+_{0,5}$, charted by $(x,y) \in (0,1)^2$ with the marked points at $(0, xy, y, 1, \infty)$, we take the Koba–Nielsen factor $[\,xy(1-x)(1-y)\,]^{\varepsilon}(1-xy)^{-\varepsilon}$, that is, exponents $(1,1,1,1,-1)$ on its five letters x, y, $1-x$, $1-y$, $1-xy$. We expand as in (85) with ε^{w-2} in place of ε^{w-4}. Integrated against the canonical form of the pentagon, $\mathrm{dlog}\frac{x}{1-x} \wedge \mathrm{dlog}\frac{y}{1-y}$, which is the diagonal pairing, this factor gives

$$c_8 \;=\; \tfrac{43135}{576}\,\zeta_8 + \tfrac{75}{2}\,\zeta_3\zeta_5 - \tfrac{35}{2}\,\zeta_2\zeta_3^2, \tag{102}$$

with no $\zeta(3,5)$. Integrated instead against $\mathrm{dlog}\, x \wedge \mathrm{dlog}\, y$, which has unit residues on two of the five sides of the pentagon and is not its canonical form, it gives

$$c_8 \;=\; \tfrac{13603}{288}\,\zeta_8 - \tfrac{69}{2}\,\zeta_3\zeta_5 - 5\,\zeta_2\zeta_3^2 + 12\,\zeta(3,5). \tag{103}$$

On the pentagon this second pairing thus contains a depth-two zeta value that the diagonal pairing lacks. This agrees with the structure that Schlotterer and Stieberger found in the motivic organization of disk integrals [13, 39].

The last two columns of Table 2 show the same phenomenon at $k = 3$, on a space that is not a moduli space of punctured spheres. We conjecture that it is general: a bounded chamber of the arrangement integrated against its own canonical form has $D_8 \equiv 0$, and in particular $D_8(s) \equiv 0$ for $K(3,6)$. A chamber paired with another dlog form, such as the canonical form of a different chamber, obeys no such constraint. We base the conjecture on two rational kinematic points and one comparison form, and a single kinematic point at which $K(3,6)$ requires $\zeta(3,5)$ would refute it. We expect that the conjecture can be decided using the motivic coaction on the matrix of chamber integrals (Section 8.3).

5.5 All orders

A statement valid at every order follows from the positive parametrization (23) together with the results of Brown [45] and Panzer [68, 69].

Theorem 3. *For every $w \geq 0$ the coefficient $c_w(s)$ of the expansion* (85) *of $K(3,6)$ satisfies*

$$c_w(s) \;\in\; \sum_{k\leq w} \mathbb{Q}(s) \otimes_{\mathbb{Q}} \mathrm{MZV}_k, \tag{104}$$

with coefficients rational in the fourteen kinematic invariants, at most simple poles, located on the sixteen channel hyperplanes $\kappa_a(s) = 0$, and total degree $w - 4$.

The pole statement is that of (26), which we derived from the cone form (25). The arithmetic statement follows from linear reducibility in the sense of Brown [45]. Suppose a set of polynomials passes his compatibility-graph reduction with final alphabet in $\{0,-1\}$. Brown showed that every convergent integral over $\mathbb{R}^n_{>0}$ of a rational function with poles on those polynomials, times products of their logarithms, is then a rational combination of multiple zeta values.

The ten polynomials of (23) are linearly reducible in Brown's sense. We carried out his reduction for all twenty-four orders of integrating out a, b, c, d one at a time. In exactly eight of them the singularities of the partial integrals stay linear in the remaining variables at every stage and the final alphabet is $\{0,-1\}$. The eight orders are the four in which c and d are integrated first, in either order, and then a and b in either order, together with (a,c,d,b), (b,c,d,a), (c,a,d,b) and (c,b,d,a). In Brown's reduction [45] the sets of polynomials that remain after $\{c,d\}$, $\{c,d,b\}$ and $\{c,d,a\}$ have been integrated out are

$$S_{\{c,d\}} = \{\, a,\ 1+a,\ 1+a+ab,\ b,\ 1+b \,\}, \qquad S_{\{c,d,b\}} = \{\, a,\ 1+a \,\}, \qquad S_{\{c,d,a\}} = \{\, b,\ 1+b \,\}, \tag{105}$$

so that in each of the eight orders the last variable meets singularities at 0 and -1 only. The naive order (a,b,c,d) fails at the second step, where a polynomial quadratic in b appears.

Granting Brown's criterion and the hyperlogarithm integration and regularization of Panzer [68, 69], every term of the ε-expansion of (23) reduces to hyperlogarithms whose singularities in the last variable lie at 0 and -1 only. The divergences at $\varepsilon \to 0$, which produce the poles in the $\varepsilon\kappa_a$, are removed beforehand by the integration-by-parts regularization of [69]. This regularization writes (23) as a finite sum of absolutely convergent integrals of the same type, with coefficients in $\mathbb{Q}[s][(\varepsilon\kappa_a)^{-1}]$ and integrands built from the same ten polynomials. The reduction applies to each of these terms unchanged. The final integration over $(0,\infty)$ then produces multiple zeta values of weight at most w, and (104) follows. Theorem 3 is therefore conditional on the two cited results applying as stated to (23). The only input specific to $K(3,6)$ is the existence of a linearly reducible parametrization.

In every coefficient we have computed the weight is exactly w, and we conjecture that this uniform transcendentality holds at every order; a proof would need a manifestly pure regularization, which the reduction does not provide. The cone constants C_r that are not zeta values do not arise in the proof of Theorem 3. They come only from the subdivision of the region into fifty-two cones, which the hyperlogarithm reduction over the undivided region $\mathbb{R}^4_{>0}$ does not use. Conversely, since the multiple zeta values of weight eight include $\zeta(3,5)$, (104) places no constraint on D_8.

6 Counting solutions

The number twenty-six is the rank of the intersection form (58), and hence the number of independent chamber integrals in the double copy. It is also the number of saddle points of the integrand, among them the point x_+ that controls the high-energy limit (55). Here we obtain the same number in a third way, as $|\chi(X(3,6))|$, from the point count (80) and, by hand, from an arrangement of ten lines following [72]. We show that the Galois group of the twenty-six solutions is the full symmetric group. Using the low-energy expansion of $K(3,6)$ we exclude a closed form of the Veneziano type, a product of Gamma functions of linear forms in the kinematics.

The saddle points of the integrand of Eq. (11) are the solutions of the scattering equations of [14],

$$\frac{\partial \mathcal{S}}{\partial x_i} = 0, \qquad \mathcal{S}(x;s) \ = \sum_{1\le a<b<c\le 6} s_{abc}\, \log(abc), \tag{106}$$

with x_i any four coordinates on $X(3,6)$. The following three numbers, defined independently of one another, are equal:

$$\#\{\text{solutions of } (106)\} \;=\; \operatorname{rank} K(3,6) \;=\; \bigl|\chi(X(3,6))\bigr| \;=\; 26. \tag{107}$$

The first was obtained in [14]. The second is the holonomic rank of an A-hypergeometric system in the sense of Gel'fand, Kapranov and Zelevinsky [49, 50]. Regarded as a function of the coefficients of its letters, the integral (23) is an A-hypergeometric function, whose value at unit coefficients is $K(3,6)$. At generic kinematics that system is irreducible: we have evaluated its forty-three facet resonance conditions at generic rational points and none is met, so the A-hypergeometric function generates the full module and satisfies no system of smaller rank in the coefficients; at the arithmetically simple base point s_0 of Eq. (29) ten of the forty-three resonance conditions are met, and the irreducibility argument does not apply there.

The third number in (107) is a topological invariant of the configuration space. That the first and third agree for a generic master function on a smooth very affine variety is a theorem of Huh [70]. The second equals the other two because the twisted cohomology is concentrated in the middle degree [59, 66].

The Euler characteristic of $X(3,6)$ is its point count over finite fields at $q = 1$ [65]. For six points the count is classical, the number of ordered six-arcs in $\mathrm{PG}(2,q)$ modulo PGL_3 [64]. It is the polynomial $(q-2)(q-3)(q^2-9q+21)$ of Eq. (80), and

$$\bigl|X(3,6)(\mathbb{F}_1)\bigr| \;=\; (-1)(-2)(13) \;=\; 26, \tag{108}$$

whereas for n points on a line the count $|\mathcal{M}_{0,n}(\mathbb{F}_q)| = (q-2)(q-3)\cdots(q-n+2)$ splits into linear factors and gives $4! = 24$ at $n = 7$.

6.1 Charges on lines

The scattering equations (106) have an electrostatic interpretation of the kind Shapiro gave for the Koba–Nielsen integrand [71]. Let us single out the sixth point p_6 and hold the other five fixed. The part of $\mathcal{S}$ that depends on it is

$$\mathcal{S}_6(p_6) \;=\; \sum_{1\le a<b\le 5} s_{ab6}\, \log\,(ab6), \tag{109}$$

a two-dimensional logarithmic potential sourced on the ten lines through pairs of the five fixed points, with s_{ab6} the charge on the line $\overline{ab}$. The minor $(ab6)$ vanishes where p_6 meets that line. The scattering equations for p_6 are its equilibrium conditions in this arrangement. At positive kinematics every line repels p_6, and there is one equilibrium in each bounded chamber [14]. The number of equilibria is the Euler characteristic of the complement F_6 of the arrangement. For lines in the projective plane we compute this Euler characteristic elementarily: three for the plane, minus two for each line, plus $m-1$ for each point where m lines meet. Ten lines through five general points meet four at a time at the five points and in pairs at fifteen further points, which gives

$$\chi(F_6) \;=\; 3 - 2\cdot 10 + 5\cdot 3 + 15 \;=\; 13, \tag{110}$$

the thirteen bounded chambers drawn in Figure 1 of [14]. The five fixed points themselves have two equilibrium configurations, since $X(3,5) \cong X(2,5)$, and the map forgetting the sixth point is a fibration with constant fiber type. Therefore

$$26 \;=\; 2 \times 13. \tag{111}$$

This factorization is Example 5.2 of [72]. In the appendix to that paper Lam derives the same numbers from point counts.

The factorization (111) is the analog in the plane of the recursion for points on a line. There the nth puncture moves among $n-1$ fixed ones, the complement has Euler characteristic $3-n$ whatever their positions, and

$$\left|\chi(\mathcal{M}_{0,n})\right| \;=\; (n-3)\left|\chi(\mathcal{M}_{0,n-1})\right| \;=\; (n-3)!. \tag{112}$$

The product $(n-3)!$ is also the number of independent color-ordered amplitudes. In the plane the fiber type is constant over five fixed points in general position but not over six, and we derive the count at seven points, (122), in Section 7.3.

6.2 The Galois group of the scattering equations

Eliminating all but one generic linear coordinate, we reduce (106) to a single polynomial f of degree twenty-six with coefficients in $\mathbb{Q}(s)$, whose splitting field is generated by the coordinates of the solutions. By the exact computations of Appendix B.7 we find that f is irreducible over $\mathbb{Q}$ at two rational points of positive kinematics, at which all fourteen exponents in Eq. (23) are positive. For each of the two points we compute over $\mathbb{Q}$ the characteristic polynomial of multiplication by the linear coordinate that defines f, a separating element, on the twenty-six-dimensional quotient algebra of (106). This characteristic polynomial is f with s specialized to the chosen point, and at both points it has degree twenty-six and is squarefree and irreducible.

For the Galois group we reduce f at a third rational point modulo six primes close to 2^{30}. Each reduction is separable of degree twenty-six. By Dedekind's theorem the degrees of its irreducible factors give the cycle type of an element of the Galois group. The six reductions factor with cycle types

$$[1,3,8,14],\quad [1,1,3,6,6,9],\quad [1,1,2,5,17],\quad [1,2,3,3,3,3,11],\quad [5,21],\quad [4,6,16]. \tag{113}$$

No proper nonempty subset of $\{1,\dots,26\}$ has a size that is a sum of parts in all six patterns, and the group is therefore transitive. A power of the element with the third cycle type is a seventeen-cycle. A transitive group of degree twenty-six that contains a seventeen-cycle is primitive, because seventeen is a prime greater than half the degree. Seventeen is also no larger than the degree minus three, and by Jordan's theorem such a primitive group contains A_{26}. Since the third pattern is that of an odd permutation, the group is S_{26}. The twenty-six solutions therefore admit no block system and no proper subfamily defined over $\mathbb{Q}$. In particular no sum over a proper nonempty subset of the twenty-six solutions is defined over $\mathbb{Q}$, as a formula assembling $K(3,6)$ from two smaller solution systems would require. The same conclusions hold over $\mathbb{Q}(s)$, because the Galois group of f over $\mathbb{Q}(s)$ contains that of each of its separable rational specializations.

6.3 Closed forms of Gamma type

A product of Gamma functions, such as the Veneziano amplitude, satisfies a closed first-order system in the kinematics. Suppose $K=\prod_j \Gamma(\theta_j)^{\nu_j}$ is a finite product of Gamma functions of linear forms θ_j in the products εs_{abc}, with integer exponents ν_j. Then

$$\frac{\partial}{\partial s_i}\log K \;=\; \sum_j \nu_j\,\frac{\partial\theta_j}{\partial s_i}\,\psi(\theta_j), \tag{114}$$

with ψ the digamma function, is a closed first-order system whose solutions are the constant multiples of K. Now $K(3,6)$ is the unit-coefficient value of the A-hypergeometric function introduced after Eq. (107), of holonomic rank twenty-six. The rank concerns the system in the coefficients of the polynomials, whereas (114) is a system in the exponents at unit coefficients, so the rank by itself does not decide whether

$$K(3,6) \;\neq\; \prod_j \Gamma\big(\theta_j(\varepsilon s)\big)^{\nu_j} \tag{115}$$

for every finite collection of linear forms and integers. For example, the Gauss hypergeometric function has holonomic rank two in its argument, yet its value at unit argument is a quotient of Gamma functions of the parameters.

We establish (115) instead from the low-energy expansion of $K(3,6)$. As in Table 2 we write $c_2 = q_2\zeta_2$ and $c_3 = q_3\zeta_3$, and we denote by R_5 and B the $\zeta_2\zeta_3$ and ζ_3^2 coefficient functions of (93) and (96). If $K(3,6)$ were a single Gamma product, then since $c_1 = 0$ the terms in Euler's constant would have to cancel among its factors, as they do in Eq. (92). The full expansion of the product then follows by writing each factor as an exponential of a series whose remaining coefficients are single zeta values, and this forces

$$\frac{R_5}{c_0} = \frac{q_2}{c_0}\cdot\frac{q_3}{c_0}, \qquad \frac{B}{c_0} = \frac{1}{2}\Big(\frac{q_3}{c_0}\Big)^2, \qquad \text{and hence} \qquad 2\,B\,q_2 = R_5\,q_3. \tag{116}$$

Eliminating the normalization c_0 between the first two relations gives the third. A rational prefactor has an expansion in ε free of zeta values. In the expansion of the Gamma product, ζ_2, ζ_3, $\zeta_2\zeta_3$ and ζ_3^2 first occur at the orders defining q_2, q_3, R_5 and B. The prefactor therefore multiplies these four coefficients and c_0 by its leading term alone and leaves the three relations unchanged.

We test the three relations (116) in exact arithmetic at rational kinematics, where q_2, q_3, R_5 and B take rational values. All three fail at every one of twelve rational points in the convergence region, listed in Table S2 of the Supplementary Material, and at s_0 and $\tilde{s}$. For example, at s_0 the entries of Table 2 give $2c_0B - q_3^2 = \frac{7030049981}{972000}$, whereas the second relation requires zero. That the first relation fails already implies (115) if $\zeta_2\zeta_3 \notin \mathbb{Q} + \mathbb{Q}\,\zeta_5$, and that the second fails implies it if $\zeta_3^2 \notin \mathbb{Q} + \mathbb{Q}\,\pi^6$. Both irrationality statements are expected and open, like the one needed at weight five. The failure of (116) excludes a closed form for $K(3,6)$ as a finite product of Gamma functions of linear forms in εs with integer exponents, with or without a rational prefactor. Integer shifts of the arguments are absorbed by the prefactor. Gamma functions whose arguments are shifted by other rationals expand differently and are not tested by (116).

We find that the coefficient functions c_0 through c_6 do not factor: their numerators over the common denominator $\prod_a \kappa_a$ are irreducible polynomials over $\mathbb{Q}$. These constraints concern closed forms for $K(3,6)$ by itself. They are compatible with the twisted period relations and the intersection form (64), which relate the chamber integrals of the same integrand to one another without expressing any of them in closed form.

6.4 Real solutions

How many of the twenty-six solutions are real is a separate question, asked in general in [73]. At $(3,6)$ and positive kinematics all twenty-six are real, as observed in [14], and in the electrostatic picture they are the equilibria counted in (111), one in each bounded chamber over each of two real base configurations. At the two rational points of positive kinematics used in Section 6.2, at which all fourteen exponents in Eq. (23) are positive, we verify total reality exactly: the characteristic polynomial computed there is squarefree of degree twenty-six, and by Sturm's theorem all of its roots are real; at one of them we locate the twenty-six solutions in twenty-six distinct sign chambers of the fourteen letters, none of them the positive part. These points lie outside the convergence region $\mathcal{C}$, and inside $\mathcal{C}$ total reality fails: at the point s^* of Eq. (28) sixteen of the twenty-six solutions are real, by an exact root count of the degree-twenty-six eliminant there, and exactly one of them lies in the positive part of $X(3,6)$, as must be the case wherever in $\mathcal{C}$ the ten polynomial exponents are negative (Section 3.6): it is the point x_+ that controls the high-energy limit there.

7 Beyond $K(3,6)$

Beyond $K(3,6)$ one can keep six points and leave the family (3), as in the D_4 cluster string integral of [15], or stay within the family and add points. Since the duality $X(k,n) \cong X(n-k,n)$ identifies $(4,7)$ with $(3,7)$, the next members are $K(3,7)$ and $K(4,8)$. The first is a six-dimensional integral over seven points in the plane, with forty-two channel forms and 693 maximal cones, and the second a nine-dimensional integral over eight points in space.

7.1 The D_4 cluster deformation

In the D_4 cluster string integral of [15,29], whose two extra edge factors appeared in Eq. (37), every cluster variable of $\mathrm{Gr}(3,6)$ is raised to a power, including the two that are not Plücker coordinates, $X = (134)(256) - (234)(156)$ and its cyclic image $Y = (136)(245) - (126)(345)$. These two are absent from $K(3,6)$. Including them in the integrand of Eq. (11), evaluated in the chart (23), with exponents εc and $\varepsilon c'$ defines $I_{D_4}(c,c')$, and $K(3,6) = I_{D_4}(0,0)$.

For all kinematics and all (c,c') the weight-two and weight-three coefficients of I_{D_4} are rational multiples of ζ_2 and, given the identifications (138), of ζ_3, respectively. This follows as in Eq. (89) and Appendix A.2: every one-dimensional restriction of a letter is again 1 or $1+y$. The two-dimensional constants that occur are integrals (136) of pair polynomials and take only three values,

$$T_1 = \tfrac{3}{4}\,\zeta_3, \qquad T_2 = \tfrac{13}{24}\,\zeta_3, \qquad T_3 = -\tfrac{5}{12}\,\zeta_3, \tag{117}$$

of which the first is the alternating sum $\sum_{k\ge 1}(-1)^{k+1}k^{-3}$ and the other two are the constants E_2 and E_4 of Eq. (138), identified numerically there.

At weight four, the first weight at which constants other than a single zeta value could appear, we again find a rational multiple of ζ_4 at one kinematic point on the split locus of Section 3.4, the point with $s_{135} = s_{136} = s_{146} = s_{236} = s_{245} = s_{246} = s_{256} = s_{356} = 0$, $s_{145} = -\frac{17}{30}$, $s_{156} = \frac{2}{5}$, $s_{235} = -\frac{19}{12}$, $s_{345} = \frac{47}{30}$, $s_{346} = -\frac{47}{30}$ and $s_{456} = \frac{1}{2}$ in the chart (23), and at two values of the diagonal twist $c = c' = \gamma$,

$$c_4\big|_{\gamma=1/6} = -\frac{55913710999}{1244160000}\,\zeta_4, \qquad c_4\big|_{\gamma=-1/4} = -\frac{9936700447}{24885000}\,\zeta_4, \tag{118}$$

each relation holding at one hundred and twenty-seven digits, and an independent evaluation at the same point agreeing with both values to seven digits; the denominators $2^{13}\,3^5\,5^4$ and $2^3\,3^2\,5^4\,7\,79$ carry no prime above five that the exact lower-weight coefficients at the same twist do not already carry. These lower-weight coefficients are rational functions of γ, described in Appendix B.8.

We expand one twisted member of the family I_{D_4}, with $(c,c') = (\frac{1}{3}, -\frac{1}{3})$, through weight eight at the base point s_0 of Eq. (29). Its coefficients at weights two through seven are polynomials in single zeta values, with integer relations of height below 1.3×10^7, and at weight eight c_8 admits no integer relation in the basis (87), with or without $\zeta(3,5)$, of height below 3×10^{10} at the fifty-four digits on which two grids agree (Appendix B.6). Since the relation (101) has height 2.4×10^{11}, this is a bound and not a proof. We conjecture that this coefficient is not a multiple zeta value. If so, the ring of periods generated by the coefficients depends on the exponents (c,c'). The Grassmannian point $(c,c') = (0,0)$, whose coefficients are multiple zeta values to all orders by Theorem 3, is then distinguished arithmetically within the cluster family.

7.2 Residues and poles for general (k, n)

Our proof of Proposition 1 uses only two properties of the cone form (25). Every cone polynomial satisfies $Q_{C,t}(0) = 1$ and, having unit coefficients, is at least one on the closed cube; and at the points of the hyperplane $\varepsilon\kappa_a = -n$ where the residue is taken, the exponents of the coordinates other than y_a are positive. Neither is specific to six points. Indeed, the normalization $Q_{C,t}(0) = 1$ holds for the string integral of any polytope once its normal fan is refined to unimodular cones [15]. The second property need hold only on an open region of every wall, from which the residue is continued. Granted these properties, which we have verified in full only at $(3, 6)$, the same argument shows that the poles of $K(k, n)$ lie on the hyperplanes $\varepsilon\kappa_a(s) = -n$, $n = 0, 1, 2, \ldots$, of its channel forms. In the notation of Proposition 1 the residues are

$$\operatorname{Res}_{\varepsilon\kappa_a(s)=-n} K(k,n) \;=\; \sum_\lambda V_\lambda^{(n)}(s)\, F_\lambda(s_\parallel), \qquad \deg V_\lambda^{(n)} \le n, \tag{119}$$

a finite sum as in (38). For $n = 0$ this is the statement of [15, 29] that the leading residue is the string integral of the facet.

A residue of the type (119) has the bipartition form $\sum A_L A_R$ of Eq. (43) when the facet polytope is a product of two polytopes attached to complementary sets of labels, as every facet of the associahedron is. None of the sixteen facets (31) of $P(3, 6)$ is such a product. We expect that for every $k \ge 3$ the bipartition form is absent at generic kinematics on every wall, since k-label kinematics has no two-body invariant on which cluster decomposition could be stated. We have established this only at $(3, 6)$.

7.3 Solutions at seven and eight points

At six points the twenty-six solutions of the scattering equations factor as 2×13, Eq. (111). For seven points we fix six in general position and let the seventh move in the complement of the fifteen lines through pairs of the six. Over a generic configuration the fifteen lines meet five at a time at the six points and in pairs at $\binom{15}{2} - 6\binom{5}{2} = 45$ others, so

$$\chi(F_7) \;=\; 3 - 2 \cdot 15 + 6 \cdot 4 + 45 \;=\; 42. \tag{120}$$

If the fiber type were constant the count at seven points would be $42 \cdot 26 = 1092$. It is not constant: on the locus where the three lines of a perfect matching of the six points, say $\overline{12}$, $\overline{34}$, $\overline{56}$, pass through a common point, the complement of the arrangement has one bounded chamber fewer. There are fifteen perfect matchings, hence fifteen such concurrency divisors $D_M \subset X(3, 6)$. The stratified Euler characteristic of the fibration then gives the recursion of [72],

$$\operatorname{rank} K(3,n) \;=\; \chi(F)\,\chi(X(3, n{-}1)) \;-\; \sum_D \chi(\bar D), \tag{121}$$

with F the generic fiber and the sum running over the closed concurrency divisors of the base. The second term is present because the fiber's Euler characteristic drops by exactly one for each divisor through the base point (Lemma 5.5 of [72]). At seven points the fifteen divisors are permuted by relabeling, each with $\chi(\bar D_M) = -12$, and

$$\operatorname{rank} K(3,7) \;=\; 42 \cdot 26 \;+\; 15 \cdot 12 \;=\; 1092 + 180 \;=\; 1272, \tag{122}$$

the number of solutions found in [14, 74]. The split $1092 + 180$ of (122) also occurs in the soft limit of the seventh particle [72, 75], where 1092 of the solutions descend to a six-point solution times an equilibrium of the soft point. The remaining 180 move to the boundary in fifteen groups of twelve, the soft point approaching the common point of a concurrent matching.

We can iterate the recursion (121), because its output at n is the base Euler characteristic at $n+1$. At eight points the moving point lies in the complement of twenty-one lines. They meet in seven six-fold points and $\binom{21}{2} - 7\binom{6}{2} = 105$ double points, so $\chi(F) = 3 - 2\cdot 21 + 7\cdot 5 + 105 = 101$. The concurrency divisors are the 105 matchings of three disjoint pairs among seven points, each with $\chi(\bar{D}) = -568$ [72], and

$$\operatorname{rank} K(3,8) \;=\; 101\cdot 1272 \;+\; 105\cdot 568 \;=\; 188112, \tag{123}$$

the published count [72,75]. The value $\chi(\bar{D}_M) = -12$ entering (122) was likewise computed in [72], where it is denoted $\chi(3,6;3)$. We do not have a structural derivation of the -12. At nine points a new class of strata enters, on which four lines are concurrent, and (121) as written undercounts the known value 74,570,400 [72].

Incidence theorems constrain which matchings of the six points can be concurrent simultaneously. Two concurrent matchings sharing no pair imply a third, as in Pappus's configuration. For two concurrent matchings that share a pair, the third matching containing that pair can be concurrent as well only in characteristic two, and the corresponding stratum is empty over $\mathbb{C}$.

The two terms of the decomposition (122), which separate in the soft limit, also differ in how many of their solutions are real, and total reality at positive kinematics fails at $(3,7)$. We count real solutions there by numerical homotopy continuation. Inside the region of positive kinematics defined in [14] we find, at each of some thirty points, between 1188 and 1200 real solutions out of 1272. At the positive point of [73], which is positive in a different basis of invariants and lies outside that region, the count is 1210, as found there. For real kinematics of mixed sign the count falls to a few hundred. The 1092 solutions counted by the first term of (122) are all real at positive kinematics, as expected from the soft limit. There the six-point base solutions are real, and each bounded chamber of the fiber contains one real equilibrium. The 180 solutions associated with the concurrency divisors are not all real.

Scanning straight segments between random points of the region of positive kinematics, we find three points along them at which the real count changes by two. Two lie at fractions 0.0824 and 0.1730 along one segment and the third at 0.3441 along another. Each is a simple fold, certified by interval arithmetic, at which two real solutions coincide and become a complex-conjugate pair. None of the three lies on any of the 721 hyperplanes on which a ray coordinate of the tropical Grassmannian changes sign (Appendix B.7). These transitions, at least, are not where the invariants R and $\tilde{R}$ of [14] change sign, as had been suggested there. Neither the leading order $m(3,7)$, a sum over all 1272 solutions, nor $K(3,7)$ itself at two values of α' shows any feature across the first wall, numerically.

7.4 Point counts at seven points

We can obtain the Euler characteristics above from point counts over finite fields at $q=1$ [65], as for the six-point count (80), which is a polynomial in q. The seven-point count is not a polynomial: Glynn's theorem [64] gives

$$|X(3,7)(\mathbb{F}_q)| \;=\; (q-3)(q-5)\big(q^4 - 20q^3 + 148q^2 - 468q + 498\big) \;-\; 30\,[\,2\mid q\,], \tag{124}$$

where $[\,2\mid q\,]$ is 1 for even q and 0 otherwise. The correction counts labeled Fano planes: seven points of $\mathrm{PG}(2,2^h)$ can form a Fano subplane, in $7!/|\mathrm{PGL}_3(\mathbb{F}_2)| = 5040/168 = 30$ labeled ways up to projectivities [76,77], and such configurations are not in general position. Thus $X(3,7)$ is polynomial-count over $\mathbb{Z}[\frac{1}{2}]$ but not over $\mathbb{Z}$. The polynomial part at $q = 1$ is $(-2)(-4)(159) = 1272$, in agreement with (122). On smaller strata, where several matchings are concurrent, the point counts involve quadratic characters. The Hesse configuration, where the concurrency count reaches six, is defined over $\mathbb{Q}(\sqrt{-3})$ and contributes only for $q \equiv 1 \pmod 3$. A pentagonal stratum with ten concurrences, defined over $\mathbb{Q}(\sqrt{5})$, contributes only for $q \equiv \pm 1 \pmod 5$. These are the characters of conductors three and five in the counts of [72]. The count giving the divisor Euler characteristic -568 of (123) is already a polynomial plus a multiple of the conductor-three character. None of these characters has an analog for $\mathcal{M}_{0,n}$, whose counts are polynomials.

7.5 The low-energy expansion of $K(3,7)$

The leading coefficient of $K(3,7)$ is the canonical function of $P(3,7)$ [15,17], which is the generalized biadjoint amplitude [14,74], a sum over all 1272 solutions of the seven-point scattering equations:

$$c_0^{(3,7)} \;=\; m(3,7). \tag{125}$$

We compute $c_0^{(3,7)}$ as an exact cone sum over a unimodular refinement of the fan of $X(3,7)$, as in Eq. (27).

Let us write the higher coefficients as $c_w^{(3,7)}(s) = \sum_P R_P(s)\, P$. Here the R_P are rational functions, homogeneous of degree $w-6$, whose poles lie at the zeros of the forty-two channel forms, and the P are periods attached to faces of the fan. As at $(3,6)$, the individual periods are not all multiple zeta values and include polylogarithms at $\frac{1}{2}$ and $\frac{1}{3}$, but in every coefficient computed below, their sum involves only zeta values. At weights two and three this is again a consequence of the fan alone. Every restriction of a cone polynomial to a coordinate axis is 1 or $1+y$, and every two-face constant (136) that occurs is zero or one of T_1, T_2, T_3 of (117). Hence $c_2^{(3,7)}$ is a rational multiple of ζ_2 identically in s and, given the identifications (138), $c_3^{(3,7)}$ of ζ_3. We compute both as exact cone sums. For weights four through six we evaluate the coefficients at eleven integer kinematic points (Appendix B.8). In each case we identify the value by an integer relation as a polynomial in single zeta values.

At weight seven we have examined the dihedrally symmetric kinematics, at which s_{abc} depends only on the orbit of $\{a,b,c\}$ under the dihedral group of relabelings generated by $a \to a+1$ and $a \to -a$ modulo 7. The four orbits are those of s_{123}, s_{124}, s_{125}, and s_{135}. Momentum conservation, Eq. (12) with seven labels, leaves one relation among the four values, $s_{123} + 2s_{124} + s_{125} + s_{135} = 0$, and up to overall scale these kinematics form a two-parameter family. On this family the forty-two channel forms restrict to five linear forms. Expanded in partial fractions in these, $c_7^{(3,7)}$ is a combination of ninety-nine independent rational functions. We call the ninety-nine coefficients, each an exact rational combination of face periods, its coordinates. Numerically (Appendix B), every one of the ninety-nine coordinates is a combination of ζ_7, $\zeta_2\zeta_5$, and $\zeta_3\zeta_4$ with coefficients in $7\cdot\mathbb{Z}[\frac{1}{6}]$, that is, with numerators divisible by seven and denominators of the form 2^a3^b. Two of the coordinates, for example, are $\frac{63}{16}\,\zeta_7 - \frac{21}{16}\,\zeta_2\zeta_5 - \frac{63}{64}\,\zeta_3\zeta_4$ and $28\,\zeta_7$. Sixteen of the ninety-nine vanish, eighty are identified by integer relations found numerically, and three follow by exact linear algebra from the others together with the values of $c_7^{(3,7)}$ identified at integer points of the family. These three are therefore conditional on those identifications. At the integer point $(s_{123}, s_{124}, s_{125}, s_{135}) = (1, 0, -1, 0)$ of this family we find through weight seven

$$\begin{aligned}
&c_0^{(3,7)} = 119, \qquad c_2^{(3,7)} = -483\,\zeta_2, \qquad c_3^{(3,7)} = 616\,\zeta_3, \qquad c_4^{(3,7)} = \tfrac{8239}{4}\,\zeta_4,\\
&c_5^{(3,7)} = -245\,\zeta_5 - 2597\,\zeta_2\zeta_3, \qquad c_6^{(3,7)} = -\tfrac{10647}{8}\,\zeta_6 + \tfrac{3633}{2}\,\zeta_3^2,\\
&c_7^{(3,7)} = -\tfrac{4263}{2}\,\zeta_7 + 791\,\zeta_2\zeta_5 + \tfrac{21315}{2}\,\zeta_3\zeta_4.
\end{aligned} \tag{126}$$

The planar kinematic point of [80, 81] is the integer point with $s_{a\,a+1\,a+2} = 1$, $s_{a\,a+1\,a+3} = -1$ for every a modulo 7, and all other s_{abc} zero. It is invariant under the cyclic relabeling but not under the reflection, and so lies off the dihedral family, though inside the convergence region. There $c_0^{(3,7)} = m(3,7) = 462$, a three-dimensional Catalan number [80], and

$$c_7^{(3,7)} \;=\; 462\,\zeta_7 - 1561\,\zeta_2\zeta_5 + 6573\,\zeta_3\zeta_4. \tag{127}$$

At such symmetric kinematics the factor of seven and the denominators 2^a3^b occur at every weight, beginning with $c_0^{(3,7)} = 119 = 7\cdot 17$ in (126), and so are not specific to weight seven. Apart from $c_0^{(3,7)}$, $c_2^{(3,7)}$ and $c_3^{(3,7)}$, each coefficient in (126) and (127), like each coordinate, is expressed in zeta values of its weight by an integer relation accepted by the criterion of Appendix B; none of these relations is proved.

These data are partial in three respects, the first being that some of the integer points, among them points whose values were used to identify the coordinates, lie outside the convergence region, so that the values there are those of the continued coefficient functions; the points of (126) and (127) lie inside it. Second, at weight seven only the dihedral family and the one further point of (127) have been computed, and the three-parameter family of cyclically symmetric kinematics has not. Third, the coefficient functions at generic kinematics have not been computed, and weight eight, the first weight at which the depth-two value $\zeta(3,5)$ could appear (Section 5.4), has not been reached.

Unlike the parametrization (23) of $X(3,6)$, on whose linear reducibility in Brown's sense [45] Theorem 3 rests, the positive parametrization of $X(3,7)$ of [73] is not linearly reducible. Its minors have twenty-two distinct non-monomial factors, and these admit no order of integration satisfying Brown's criterion. After any single integration a letter quadratic in the remaining variables arises, and with it the square root $\sqrt{\lambda(p,q,r)}$ of the Källén function $\lambda = p^2+q^2+r^2-2pq-2qr-2rp$. Its arguments here are monomials in the chart coordinates, up to sign. Since λ is a perfect square when any argument vanishes, the root disappears on the $X(3,6)$ boundary of the chart, where reducibility is restored.

Linear reducibility can also fail for the individual face periods whose sum gives the weight-seven coefficient. One six-dimensional period, of the kind first occurring at weight seven, is the regularized integral over the cube of the logarithm of the single cone polynomial $f = 1 + y_6\big[(1+y_4)(1+y_5) + y_3y_5(1+y_4) + y_1y_2(1+y_5)\big]$, which is linearly reducible on its own. The regularization, however, involves the restrictions of f to twelve faces of the cube at once, and that set is not linearly reducible in any order of integration. A first obstruction is a quadratic letter of discriminant $y_3(y_3 - 4y_1y_2y_6)$, again a perfect square on the facet $y_6 = 0$. Linear reducibility is only a sufficient condition for multiple zeta values, and indeed we find that every coefficient computed above, in particular $c_7^{(3,7)}$ on the whole dihedral family, is a polynomial in single zeta values. We have examined only this parametrization of the integrand and one representative face period, and another parametrization of $X(3,7)$ may still be linearly reducible.

7.6 Higher points and higher k

The convex Parke–Taylor forms no longer span the cohomology at seven points (Section 4), so a seven-point double copy needs a basis of forms we have not identified. The polytope $P(3,7)$ has ninety-eight non-simple vertices among its 693 [15, 17]. The E_6 cluster fan on the same forty-two rays [19] resolves each of them, as D_4 resolves the two bipyramids at $(3,6)$. Each of the 140 contracted edges corresponds to a vanishing sum of cluster exponents, a property of the fan of $X(3,7)$ alone. The residues of $K(3,7)$ at finite α' have not been computed. The argument leading to (119) suggests their form, but we have not yet identified the facet integrals themselves, the seven-point analog of Table 1.

For $k \geq 4$, beginning with $(4,8)$, nothing beyond the leading order is known to us. The counting argument changes there, because the moving point in the fibration behind (121) lies in the complement of an arrangement of planes in space. Then the Euler characteristic of the fiber no longer drops by one for each concurrency divisor through the base point. The recursion (121) as written therefore does not apply, and we expect a suitably modified form of it to hold for $k \geq 4$. If the cone form of $K(4,8)$ has the two properties used in Section 7.2, its residues are again given by (119) and take the bipartition form at whichever facets of $P(4,8)$, if any, are products of two polytopes attached to complementary sets of labels.

8 Discussion

Table 3 lists the S-matrix properties transcribed to the fourteen-dimensional space of three-label invariants s_{abc}, of which those built from six momenta form only a subspace of codimension five. We prove Proposition 1, which gives the residue (38) at every pole and every level, by an elementary argument on the cone form and the fan refinement of Appendix A. We also prove the closed forms (40), (45), (47), (48) and (50), the double-pole coefficient $C_2 = A_4A_4$ of (54), and the identity $c_2 = q_2\zeta_2$. We prove that the critical point of the integrand in the positive part is unique wherever the ten polynomial exponents of (23) are negative, since there the logarithm of the integrand is strictly concave in logarithmic coordinates. Claim 2 is proven, with a finite enumeration as its last step. Theorem 3 is conditional on the theorems of Brown [45] and Panzer [68, 69] applying to (23), and we carry out the reduction (105) on which it rests. The closed forms of c_3, c_4 and c_5 hold identically in s assuming finitely many relations among cone constants, which we have identified numerically. We specify these relations in Appendix A and list them in full in the ancillary files. We compute in exact rational arithmetic the values (30) of the coefficient functions at s_0 through weight six. In the same arithmetic we find that the Gamma relations (116) fail at rational points and that the leading-order identity (42) holds at random rational points of the wall $t_{3456} = 0$. The eliminant whose Galois group is S_{26}, the six-point real-root counts, the Euler characteristics and the point counts are likewise exact. We count the seven-point real solutions by numerical path tracking certified by interval arithmetic. The other statements are numerical identifications, for each of which Appendix B gives the kinematic points, the significant figures and the height of the integer relation. In this way we identify the weight-six functions A and B from values of c_6, and also the coefficients c_7 and c_8 at the points s_0 and $\tilde{s}$, the ω' coefficient (101) and the seven-point and D_4 coefficients beyond weight three. The remaining numerical statements concern the rank of $H(\varepsilon)$ built from the face rule (64), its entrywise agreement with the twisted period relations (59) at $\hat{s}$, the single-valued image of the open expansion through weight six, the sign of the closed function (67) built from the face rule, and the facet identifications of Table 1 beyond leading order where no closed form exists. We state as conjectures the face rule for all s in the convergence region $\mathcal{C}$, the single-valued projection at every order, $D_8 \equiv 0$, and uniform transcendental weight. We also conjecture that the weight-eight coefficient of the twisted D_4 member of Section 7.1 is not a multiple zeta value. We leave open a general proof of fixed-angle softness on $\mathcal{C}$, that is, of $S_+ < 0$ in (55), and the determination of $D_8(s)$ and $c_8(s)$ as functions of s.

8.1 S-matrix properties of a three-label amplitude

On the nine-dimensional slice (52), where the invariants are those of six momenta, the three-particle poles of $K(3,6)$ are double, Eq. (54), with A_4A_4 as the exact leading coefficient and an orientation-

Property	For $K(3,6)$	Status	Section
Analyticity	meromorphic in the fourteen s_{abc}; at most simple poles, only on the sixteen families $\varepsilon\kappa_a(s) = -n$, $n \geq 0$; poles of higher order only where compatible walls meet; no cuts	proposition	2, Prop. 1
Channel duality	one function has all sixteen towers of poles; simultaneous poles only on the 66 two-faces, 98 edges and 48 vertices of $P(3,6)$, with leading-order sum $m(3,6)$	proposition	3
Crossing	the 372 real chambers are twisted cycles of one local system of rank $26 = \lvert\chi(X(3,6))\rvert$, in place of $(n-3)!$; intersection form of the sixty convex chambers in closed form, Eq. (64); its inverse on a basis is the α'-exact KLT kernel	theorem (rank of the local system); conjecture, verified at one point (kernel)	4, 6
Factorization on poles	$\mathrm{Res}_{\varepsilon\kappa_a=-n}\, K = \sum_\lambda V_\lambda^{(n)}(s)\, F_\lambda(s_\parallel)$: vertex polynomials of degree $\leq n$ in four, five or seven transverse directions, times integer-shifted facet integrals (one six-label function at the s and t walls, three Beta functions at the R walls)	proposition	3, Prop. 1
Momentum slice	on $s_{abc} = \frac{1}{2}(p_a{+}p_b{+}p_c)^2$ complementary s-walls coincide: three-particle poles are double, Eq. (54), with leading coefficient A_4A_4 and an orientation-dependent simple-pole remainder; two-particle poles simple, residue A_3 times a five-point function not yet identified	exact (C_2) and numerical (C_1), at the pole	3
Cluster decomposition	each pole a bipartition $I \sqcup \bar{I}$ with residue $\sum A_L A_R$; presupposes two-body invariants, and no boundary divisor of $X(3,6)$ is a product of complementary-label spaces	no bipartition invariant in three-label kinematics; not applicable	3
Positivity	in Weinberg's sense a statement about elastic residues, $A_L = \overline{A_R}$, and there are none; the rank-26 form $H(\varepsilon)$ is indefinite, but the closed function $\mathcal{M} = \sum Z H_B^{-1} Z = \int_{X(\mathbb{C})} \lvert u\rvert^{2\varepsilon}\Omega_I \wedge \overline{\Omega}_I$ and its residues along every wall are integrals of positive densities, as the Virasoro–Shapiro amplitude is in any d	no elastic residue (no-ghost condition has no object); positive as a measure, exactly; sign checked numerically at two points	3, 4.3
High energy	exponentially soft at fixed angle, at every kinematic point examined, from the dominant saddle among the twenty-six; Regge exponents are channel forms κ_a	numerical (softness); as in [15] (Regge)	3.6
Low energy	$\alpha' \to 0$ limit $m(3,6)$; coefficients in $\mathbb{Q}(s) \otimes \mathrm{MZV}_{\leq w}$ to all orders; uniform weight, and no $\zeta(3,5)$ at weight eight	exact; theorem; conjecture	5, Thm. 3
Four-Beta locus	four Beta functions whose eight visible walls are Veneziano poles; on s_{156} the level-two, $J=0$ coefficient is $\propto (26-d)$, Eq. (50)	exact	3

Table 3: S-matrix properties transcribed to the three-label kinematics of $K(3,6)$: for each property, the form it takes for $K(3,6)$, the status of that statement, and where it is established. Every property that refers only to poles, residues and the twisted homology and cohomology of $X(3,6)$ has an analog. The two that refer to a bipartition of the particles and a two-body subsystem, cluster decomposition and positivity of elastic residues, have no object to apply to. On the momentum slice the three-particle poles are double with leading coefficient A_4A_4. "Theorem" and "proposition" refer to statements proven in the text, modulo the cited results; "conjecture, verified at one point" to the face rule (64), checked against the twisted period relations at the kinematic point $\hat{s}$ of Eq. (62); "exact" to closed forms or exact rational arithmetic, and "numerical" to statements identified or checked numerically at the kinematic points listed in Appendix B, which gives the significant figures in each case.

dependent simple-pole remainder. Hence $K(3,6)$, read as a function of six momenta, is not a tree-level S-matrix element. Off the slice every pole is simple. In both cases the residue at every pole and every level has the boundary form of Proposition 1, established at leading level in [15]: a finite sum of vertex polynomials times integer-shifted amplitudes of the degenerate configuration. It does not take the bipartition form $\sum A_L A_R$. Complementary subsets of labels carry independent invariants, so cluster decomposition, from which that form follows [8], has no bipartition to refer to. Nor is any boundary stratum of $X(3,6)$ a product of two complementary-label configuration spaces. Positivity in Weinberg's sense, which is a statement about elastic residues of bipartition form, is correspondingly undefined for $K(3,6)$ rather than violated. The other properties in Table 3 are treated in the sections cited there: analyticity with at most simple poles, the homological analog of crossing with the intersection form (64), positivity of (72) as a measure, and high and low energy.

8.2 Rigidity under boundary data

One can ask whether the sixteen residues of Table 1 determine $K(3,6)$. In the language of Section 4 an amplitude of this type is a triple: a twisted cocycle, a twisted cycle, and a normalization of the exponents. Consider on the compactified $(3,6)$ geometry any such triple that degenerates on every boundary stratum to the $k=2$ string data of Table 1, in residue form and with unit normalization. We find, by an argument that we do not reproduce in this paper, that it coincides with the triple of $K(3,6)$ up to the overall scale of α', granted three assumptions. The first is that the boundary data are imposed as residue forms on every boundary divisor, and not only as values of residues. Otherwise twenty-five of the twenty-six directions of the twisted cohomology remain undetermined. The second is that the structure of the residues at the faces and corners of $P(3,6)$, checked in that argument only at isolated kinematic points, persists throughout the convergence region. The third is that integer shifts of the exponents can be absorbed into the cocycle, as for the ordinary string. Since none of the sixteen residues is a product of two lower-point members of the family, this determination is not a recursion on the number of points. The ordinary string integrals, which have a worldsheet description, are rigid in the same sense, and this rigidity is distinct from that of the string spectrum under multiparticle factorization [82]. Thus rigidity under boundary data does not bear on whether a worldsheet theory producing $K(3,6)$ exists.

8.3 The weight-eight coefficient function

The $\zeta(3,5)$ coefficient of c_8 vanishes at the two rational points of Section 5.4, whereas that of one comparison four-form on the same integration domain does not. The statement one would like, that this coefficient function vanishes identically, $D_8(s) \equiv 0$ on the convergence region, is a conjecture. The most direct further test, not carried out here, is a symbolic evaluation of c_8 at one generic rational point by hyperlogarithm integration along one of the eight orders of (105). The vanishing of D_8 would most plausibly be proved through the motivic coaction on the matrix of chamber integrals, whose group-like form is itself conjectural beyond one moving puncture [39].

A sharper and more local question arises at weight five, where the coaction first acts nontrivially, on the $\zeta_2\zeta_3$ term. Proposition 1 suggests that at each wall

$$\mathrm{Res}_{\kappa_a=0}\, R_5(s) \;=\; [\zeta_2\zeta_3]\, c_5^{\partial a}, \tag{128}$$

the $\zeta_2\zeta_3$ coefficient of the facet's own weight-five term. At the four cube walls the right-hand side is the cross term of (40) in closed form, and we show in Appendix A.3 that (128) holds there. Through weight six the gluing mechanism of Section 5.2 accounts for the cancellation of the constants other than zeta values. A proof of (128) at every wall would extend this account from the constants to their motivic coaction. Like D_8, the other coefficient functions at weights seven and eight are known only at rational points. The weight-six pair $A(s)$, $B(s)$, which we have determined only from values, has not been derived either.

8.4 A worldsheet

The oldest question about the Grassmannian string integrals, raised in the papers that introduced them [15, 29], is whether any two-dimensional theory produces them, and we have not answered it. Claim 2 excludes the one candidate suggested by dimension counting, seven-point disk or sphere integrals, and only for constant coefficients and linear kinematic maps. A presentation by $N \geq 8$ disk integrals with kinematic prefactors [62], or a Z-theory combination at $N \geq 9$ [63], would contradict nothing in this paper. Either would have to reproduce every entry of Table 3, the residues of Proposition 1 at every level, and the expansion (30). The self-duality of $X(3,6)$ exchanging the six points with six lines is a further constraint, with no $k = 2$ analog. The split of the letters of $K(3,6)$ into five left, five right and four area letters in Eq. (84) suggests two copies of a five-point system coupled through the differences of left and right cross-ratios. The same rewriting applied to $K(3,5)$, however, produces an area letter although $K(3,5)$ is an ordinary disk integral (Section 4.5), so the split is not by itself evidence for such a structure.

8.5 Outlook

The first open question is whether $K(3,6)$ admits a positivity statement sharper than (72). A no-ghost condition would need either an elastic slice, which three-label kinematics lacks, or an inner product on the transverse excitations λ of (38), in effect a target space. The second concerns the momentum slice, where the residue at each two-particle pole $s_{a\,a+1}$ is A_3 times a five-point function that we have not identified (Section 3.5). The third is whether $K(3,6)$ is a combination of $N \geq 8$ disk integrals with kinematic prefactors [62, 63]. The fourth concerns $(4,8)$, the first member of the family with $k = 4$ that is not dual to an integral with $k \leq 3$ (Section 7.6). The fifth is the weight-eight coefficient function $D_8(s)$ and, one step before it, the wall-by-wall coaction identity (128). All five bear on the general question of which S-matrix properties of string amplitudes require a worldsheet and which follow from the twisted cohomology of the integration domain alone.

A Proofs

We proved Proposition 1 after Eq. (34) assuming two facts, which enter at level $n = 0$. The first is that setting $y_a = 0$ in each cone letter gives the corresponding letter of the facet integrand. The second is that the cones through the ray a, projected along a, form a unimodular refinement of the facet's normal fan. In Appendix A.1 we prove both facts from the geometry of the fan, and with them the statement that the residue of $K(3,6)$ at each wall is the facet's own string integral to all orders in ε. In Appendix A.2 we prove that c_3/ζ_3 and c_4/ζ_4 are rational functions of s with rational coefficients. The inputs to that proof are the values (138) of two double integrals and, at weight four, the expression of each cone constant as a rational combination of a common set of nineteen numbers, together with one relation among those numbers. The inputs that do not follow from classical results on polylogarithms are identified numerically (Appendix B.6). In Appendix A.3 we obtain the weight-five coefficient functions exactly from two ingredients: a rank statement about the fan, which we prove, and a lattice of integer relations among the cone constants, which we identify numerically. In Appendix A.4 we prove that the cone sum (60) for the self-intersection number of the positive chamber is even in ε.

Throughout, the fan is the normal fan of $P(3,6)$, complete in $\mathbb{R}^4$ with sixteen rays and forty-eight maximal cones, two of them bipyramids, refined to the fifty-two unimodular simplicial cones C of the cone form (25). We write $\kappa_a(s)$ for the channel form on the ray a. The facet's own string integral is a cone form of the shape (25) in three variables instead of four, evaluated at the kinematics induced on the wall $\kappa_a = 0$, and we write $c_w^{\partial a}$ for its weight-w coefficient.

A.1 Residues at the walls

The statement to be proved is

$$\mathrm{Res}_{\kappa_a=0}\, c_w(s) \;=\; c_w^{\partial a}(s) \qquad \text{for every wall } a \text{ and every } w \geq 0. \tag{129}$$

We prove it for all w at once by working with $K(3,6)$ as a function of ε and expanding in ε at the end. Each cone of (25) contributes

$$I_C(\varepsilon, s) \;=\; \int_{(0,1)^4} \prod_{i=1}^{4} y_i^{\varepsilon\kappa_{C,i}(s)-1} \prod_t Q_{C,t}(y)^{\varepsilon s_t}\, dy, \tag{130}$$

with the four $\kappa_{C,i}$ the channel forms on the rays of C and the $Q_{C,t}$ polynomials with unit coefficients and $Q_{C,t}(0) = 1$.

We first show that the pole at $\kappa_a = 0$ is simple and receives contributions only from the cones in the star of a. The sixteen channel forms are pairwise non-proportional, and κ_a is not among the exponents $\kappa_{C,i}$ of any cone outside the star of a. Hence near a generic point of the wall $\kappa_a = 0$ only the integrals I_C with $C \ni a$ are singular. Each such cone is unimodular, and in I_C the form κ_a appears exactly once, as the exponent of a single coordinate y_a. The pole is the endpoint singularity at $y_a = 0$ of the integral over that one coordinate, and we may take the residue cone by cone.

Next we show that the residue of each singular term I_C is a moment integral of the facet. For a cone $C \ni a$, with y' the three remaining coordinates,

$$\lim_{\kappa_a\to 0}\; \varepsilon\kappa_a \int_{(0,1)^4} y_a^{\varepsilon\kappa_a-1} \prod_{i\neq a} y_i^{\varepsilon\kappa_{C,i}-1} \prod_t Q_{C,t}(y)^{\varepsilon s_t}\, dy \;=\; \int_{(0,1)^3} \prod_{i\neq a} y_i^{\varepsilon\kappa_{C,i}-1} \prod_t Q_{C,t}^{\partial a}(y')^{\varepsilon s_t}\, dy', \tag{131}$$

where $Q_{C,t}^{\partial a}(y') = Q_{C,t}(y)|_{y_a=0}$. Setting $y_a = 0$ keeps the monomials on the face of each letter's Newton polytope that is extremal in the direction of a, and those restricted letters are the letters of the facet integrand. This establishes the first of the two facts assumed in the proof of Proposition 1. Since C is unimodular and a is one of its rays, the other three rays of C project to a basis of the quotient lattice $\mathbb{Z}^4/\mathbb{Z}a$. The restricted cone is therefore again unimodular, and no Jacobian is left over. The right-hand side of (131) is thus a three-dimensional moment integral of the shape (130), one dimension down.

Finally, we show that the sum of the restricted integrals (131) over the star of a is the facet's cone form. The projection of the star of a along a is a complete unimodular fan on the facet's rays, and we claim that it is a refinement of the facet's own normal fan. The sixteen channel forms span a fourteen-dimensional space, and their linear relations are generated by the two relations (16), one for each non-simple vertex of $P(3,6)$. In the incidence labeling of Section 3 these read

$$\begin{aligned} \kappa_{t_{1234}} + \kappa_{t_{3456}} + \kappa_{t_{5612}} &= \kappa_{R_{12,34,56}} + \kappa_{R_{12,56,34}}, \\ \kappa_{t_{2345}} + \kappa_{t_{4561}} + \kappa_{t_{6123}} &= \kappa_{R_{16,45,23}} + \kappa_{R_{16,23,45}}. \end{aligned} \tag{132}$$

There are three cases, one for each kind of wall. At the six s walls the projected star is the facet's normal fan itself, fourteen simplicial cones on nine rays for the fourteen vertices of the associahedron, and there is nothing to refine. At the six t walls the star has fourteen cones on nine rays, but the facet, an associahedron with one edge contracted, has thirteen vertices. The facet's normal cone at its non-simple vertex, where five facets of $P(3,6)$ meet, is a cone on four rays. By the first line of (132), read modulo κ_a at $a = t_{1234}$,

$$\kappa_{t_{3456}} + \kappa_{t_{5612}} \;\equiv\; \kappa_{R_{12,34,56}} + \kappa_{R_{12,56,34}} \quad (\mathrm{mod}\ \kappa_{t_{1234}}), \tag{133}$$

the two diagonals of that quadrilateral cone have the same sum, and the projected star subdivides it into two simplicial cones along one of them. At the four cube walls the star has ten cones on seven rays, while the normal fan of the cube has eight cones on six rays. Six of the seven rays fall into three pairs antiparallel modulo κ_a, one s and one t in each, and these are the cube's three axes. For $a = R_{12,34,56}$ the three pairs are $\{s_{123}, t_{5612}\}$, $\{s_{345}, t_{1234}\}$ and $\{s_{561}, t_{3456}\}$. The seventh ray is the opposite cube wall $R_{12,56,34}$, which shares the three cones of the split bipyramid with a. For this ray the first line of (132) reads

$$\kappa_{R_{12,56,34}} \;\equiv\; \kappa_{t_{1234}} + \kappa_{t_{3456}} + \kappa_{t_{5612}} \quad (\mathrm{mod}\ \kappa_{R_{12,34,56}}). \tag{134}$$

Hence the extra ray is the sum of the three rays of one maximal cone of the cube's normal fan, and the projected star is the stellar subdivision of that cone into three, $8 - 1 + 3 = 10$.

A cone form such as (25) is the integral over the positive orthant decomposed along the cones of the fan, with each unimodular cone mapped to a unit cube as in Section 2.3. Refining the fan into smaller unimodular cones only subdivides the domains of integration, and the cone form is therefore invariant under unimodular refinement. In all three cases the sum over the star is thus the facet's cone form, which is the second fact.

It remains to collect powers of ε. At $\kappa_a = 0$ the endpoint integral $\int_0^1 y_a^{\varepsilon\kappa_a - 1}(\cdots)\,dy_a$ has residue $\varepsilon^{-1}(\cdots)|_{y_a=0}$, and summing (131) over the star we obtain

$$\mathrm{Res}_{\kappa_a=0}\, K(3,6)(\varepsilon s) \;=\; \frac{1}{\varepsilon}\sum_{C\ni a} I_{C/a}(\varepsilon, s) \;=\; \frac{1}{\varepsilon}\, K_{\partial a}(\varepsilon s), \tag{135}$$

with $I_{C/a}$ the right-hand side of (131) and $K_{\partial a}$ the facet's three-dimensional string integral with its own normalization. Since $K(3,6) = \sum_w \varepsilon^{w-4} c_w$ and $K_{\partial a} = \sum_w \varepsilon^{w-3} c_w^{\partial a}$, matching powers of ε gives (129) at every w, with unit coefficient. Apart from the identification of a facet's string integral with its cone form, which is the same statement as for $K(3,6)$ itself in Section 2, the argument uses only the lattice geometry of the fan. The shorter statement, that the projected star equals the facet's normal fan, is false at ten of the sixteen walls, and the refinement step is needed there.

A.2 Weights three and four

Expanding each cone integral (130) in ε, we obtain the weight-w coefficient as the finite sum $c_w(s) = \sum_r W_r(s)\, C_r$ of Eq. (86). The $W_r(s)$ are rational functions of the fourteen invariants with rational coefficients, fixed by the fan. The C_r are logarithmic moment integrals over faces of the unit cube and do not depend on the kinematics. Restricted to the coordinate axes of the fifty-two cones, every letter $Q_{C,t}$ is either 1 or $1 + y$. Hence at weight two every nonzero constant is the integral $\zeta_2/2$ of Eq. (89), and $c_2(s) = q_2(s)\,\zeta_2$ identically in s with q_2 rational.

At weight three the new constants are attached to pairs of axes. For a pair polynomial $Q(u, v)$, the restriction of a letter to a two-dimensional coordinate face of a cone, we define

$$E(Q) \;=\; \int_0^1\!\!\int_0^1 \log\frac{Q(u,v)}{Q(u,0)\,Q(0,v)}\;\frac{du\,dv}{uv}. \tag{136}$$

Five pair shapes occur, up to transposition of u and v. We label their constants $E_1, \dots, E_6$, keeping separate labels E_2 and E_3 for the shape $1+v+uv$ and its transpose, and list the values or relations that hold by inspection:

$$\begin{aligned}
&Q = 1 + uv: && E_1 = \tfrac{3}{4}\,\zeta_3,\\
&Q = 1 + v + uv \ \text{ and } \ Q = 1 + u + uv: && E_2 = E_3,\\
&Q = 1 + u + v: && E_4,\\
&Q = (1+u)(1+v): && E_5 = 0,\\
&Q = 1 + v + uv + uv^2 = (1+v)(1+uv): && E_6 = E_1.
\end{aligned} \tag{137}$$

Four of the six constants are settled by hand or reduced to the other two. The constant $E_1 = \frac{3}{4}\zeta_3$ is the alternating sum $\sum_{k\ge 1}(-1)^{k+1}k^{-3}$ obtained by integrating the series for $\log(1+uv)$ term by term, and $E_6 = E_1$ because for $Q = (1+v)(1+uv)$ the integrand of (136) is the same as for $Q = 1+uv$. Likewise $E_5 = 0$, because its Q is a function of u times a function of v and the logarithm in (136) then vanishes, and $E_3 = E_2$ by transposition. The two constants that remain are

$$E_2 = \frac{13}{24}\,\zeta_3, \qquad E_4 = -\frac{5}{12}\,\zeta_3, \tag{138}$$

which we have identified numerically (Appendix B) and not derived. Granting (138), the $\zeta_2 \log 2$ and $\log^3 2$ components of $c_3(s)$ vanish identically in s and $c_3 = q_3\,\zeta_3$. Indeed, if (138) holds, every weight-three constant C_r is a rational multiple of ζ_3. Besides the E_i, the only weight-three constants are the axis integrals $\int_0^1 \log y \, \log(1+y)\, dy/y = -\frac{3}{4}\zeta_3$ and $\int_0^1 \log^2(1+y)\, dy/y = \frac{1}{4}\zeta_3$, which arise from expanding the left-hand side of Eq. (91). The $\zeta_2 \log 2$ and $\log^3 2$ components of c_3 thus vanish term by term in (86).

At weight four there are fifty-four constants C_r, integrals over faces of the unit cube of dimensions one, two and three. We express each as a rational combination of one common set of nineteen numbers, with coefficients determined as described below. Sixteen of these numbers lie in the ring generated by polylogarithms at sixth roots of unity, and we collect them in

$$\begin{aligned}
B_{16} = \{&\zeta_4,\ \zeta_3 \log 2,\ \zeta_2 \log^2 2,\ \log^4 2,\ \mathrm{Li}_4(\tfrac{1}{2}),\ \zeta_3 \log 3,\ \zeta_2 \log 2\, \log 3,\ \zeta_2 \log^2 3,\\
&\log^3 2\, \log 3,\ \log^2 2\, \log^2 3,\ \log 2\, \log^3 3,\ \log^4 3,\ \mathrm{Li}_4(\tfrac{1}{3}),\ \mathrm{Li}_4(-\tfrac{1}{3}),\ \mathrm{Li}_4(\tfrac{2}{3}),\ \mathrm{Li}_4(-\tfrac{1}{2})\}.
\end{aligned}$$

The other three, τ_1, τ_2, τ_3, are integrals over $(0,1)^3$ of the same type as the constants C_r attached to triples of axes (the three-dimensional analogs of the pair constants E_i), and we do not reduce them to polylogarithms. To each of the fifty-four constants we assign a coordinate vector $\gamma_r = (\gamma_r(g))$ over the nineteen numbers $g \in B_{16} \cup \{\tau_1, \tau_2, \tau_3\}$, with $C_r = \sum_g \gamma_r(g)\, g$. Some of these vectors follow from (92) and classical polylogarithm identities, and the rest are integer relations identified numerically from values of the constants (Appendix B.6). The nineteen numbers obey one further relation of the same kind, $\tau_1 + \tau_2 = \sum_{g \in B_{16}} r_g\, g$ with small rational r_g, also identified numerically. The integrands of τ_1, τ_2, τ_3, the table of vectors γ_r with the integral defining each constant, and the rationals r_g are given in the ancillary files (Appendix B.9).

Let $F_g(s) = \sum_r W_r(s)\, \gamma_r(g)$ be the total coefficient function of the direction g, so that $c_4 = \sum_g F_g\, g$. We show that, identically in s,

$$F_{\tau_3} \equiv 0, \qquad F_{\tau_1} \equiv F_{\tau_2}, \qquad F_g + r_g F_{\tau_1} \equiv 0 \quad (g \in B_{16},\ g \neq \zeta_4), \tag{139}$$

seventeen identities, one of which is a vanishing statement and sixteen of which are relations. Together with the one relation among the nineteen numbers, these identities reduce c_4 to a multiple of ζ_4:

$$c_4(s) = \sum_{g\in B_{16}} F_g\, g + F_{\tau_1}(\tau_1+\tau_2) + F_{\tau_3}\tau_3 = \sum_{g\in B_{16}} \left(F_g + r_g F_{\tau_1}\right) g = \left(F_{\zeta_4} + r_{\zeta_4} F_{\tau_1}\right)\zeta_4, \tag{140}$$

so $q_4 = F_{\zeta_4} + r_{\zeta_4} F_{\tau_1}$ and every component of c_4 outside ζ_4 vanishes. Since the coefficient function F_{τ_1} does not vanish by itself, the second equality in (140) uses the relation among the nineteen numbers, which contributes the term $r_{\zeta_4} F_{\tau_1}$ to q_4. The identities (139), whose coefficients are the rationals $\gamma_r(g)$ and r_g, are proved in exact arithmetic, as identities among rational functions of s. Read as the statement $c_4 = q_4\,\zeta_4$ about the integral, they are conditional only on the numerically identified γ_r and r_g being true relations among real numbers. In particular we assume no linear independence of the nineteen numbers over $\mathbb{Q}$.

We prove the identities (139) in two ways, a direct expansion and a structural argument by residues. In the direct proof we use the channel forms as linear coordinates on kinematic space. Fourteen of the sixteen channel forms are linearly independent and serve as the coordinates x, and the other two are the combinations of them fixed by (132). In these coordinates every W_r is a polynomial divided by the common denominator $\prod_a \kappa_a$ of degree sixteen, and the difference of the two sides of each identity in (139) clears to a polynomial. We expand it and find zero, whereas the same steps applied to the coefficient of ζ_4 leave a polynomial of some fourteen thousand terms.

The structural proof explains why the identities hold: each F in (139) is homogeneous of degree zero with poles at most on the sixteen walls, and we show that its residue at every wall vanishes; the wall-by-wall computation is recorded in ancillary files that accompany this paper. A degree-zero rational function with no poles is a constant, and we check at a single kinematic point that the constant is zero. The wall residues vanish for a geometric reason: they localize to the codimension-two flags of the fan at which two maximal cones meet along a common internal wall, 312 of them. Across each such wall the two adjacent cones are related by the gluing map $y \mapsto 1/y$ of Section 5.2 in the transverse coordinate. Under this map their two one-dimensional integrals combine into the Beta function of Eq. (91), whose expansion (92) contains only single zeta values. The map is an involution, and the monomials in the logarithms split into parts odd and even under it. Every direction other than ζ_4 is supported on odd combinations, which cancel pairwise across the internal walls, and the even combinations contribute only to ζ_4. The numerically identified γ_r satisfy these parity relations exactly, although the relations were not imposed when the γ_r were determined. At weight four this parity statement is exact and is part of the wall-by-wall computation just described; at higher weights the corresponding relations among the cone constants have not been derived, and there the cancellation of the constants other than zeta values rests on the results of Section 5, and the statement that every coefficient is a combination of multiple zeta values on Theorem 3.

A.3 Weight five

At weight five every multiple zeta value is a rational linear combination of ζ_5 and $\zeta_2\zeta_3$. Most of the individual constants C_r of (86) are now polylogarithms at rational arguments of large height, and we do not identify them one by one. We nevertheless determine the coefficient functions $Q_5(s)$ and $R_5(s)$ of (93) exactly, in two steps.

The first step is a rank statement about the fan alone. For every weight-five term r of (86), each cone C contributes to W_r a rational function $n_{C,r}(x)/P_C(x)$ in the coordinates $x = x(s)$ of Appendix A.2, with P_C the product of the four channel forms on the rays of C. Let $D(x) = \prod_a \kappa_a$ be the common denominator of all these contributions and $T_r(x) = \sum_C n_{C,r}(x)\, D(x)/P_C(x)$ the numerator polynomial of W_r. There are 364 such terms, and the polynomials T_r span a space of dimension 147 over $\mathbb{Q}$. Hence, identically in s,

$$c_5(s) \;=\; \sum_{p=1}^{147} \frac{T_p(x(s))}{D(x(s))}\, v_p, \tag{141}$$

with each v_p a fixed rational combination of the constants C_r. The 217 dependences $T_q = \sum_p a_{qp} T_p$ are polynomial identities and we prove them by expansion.

The second step uses linear relations with small integer coefficients among the constants C_r themselves. We find relations of height at most nine numerically (Appendix B); they form a lattice of rank 271 on the constants that are not individually identified. Reducing the v_p of (141) modulo this lattice leaves sixteen free generators among those constants, and every one of them enters c_5 with total coefficient exactly zero, whatever the values of the unidentified constants are. The identified constants span thirty directions outside $\{\zeta_5, \zeta_2\zeta_3\}$, and the total coefficient of each of these also vanishes as a rational function of s. The two directions that remain give

$$c_5(s) \;=\; Q_5(s)\,\zeta_5 + R_5(s)\,\zeta_2\zeta_3 \tag{142}$$

with Q_5 and R_5 rational functions of the shape (26), whose values at s^* are Eq. (94) and Eq. (95). The statement is exact granted the lattice of relations; it does not depend on the numerical value of any constant. The $\zeta_2\zeta_3$ term is present for any choice of basis for the constants C_r. Indeed, by Proposition 1 the residue of c_5 at each cube wall is the weight-five term of the triple Beta product (40) expanded by (92). The coefficient of $\zeta_2\zeta_3$ in that term is nonzero, and it is the residue of R_5 at the wall. The relation (128) therefore holds at the four cube walls. The polynomials T_p, the dependences a_{qp}, which fix the combinations v_p, and generators of the lattice are given in the ancillary files (Appendix B.9), together with the exact $Q_5(s^*)$ and $R_5(s^*)$.

A.4 Parity of the cone sum

The self-intersection number (60) of the positive chamber in Section 4.1 is a sum over the cones of a complete simplicial fan in $\mathbb{R}^4$ of products of $g(x) = \left(e^{2\pi i x} - 1\right)^{-1}$ over the rays of each cone, evaluated at $x = \varepsilon\kappa_r$. Two facts make it even in ε for any kinematics. First, $g(-x) = -1 - g(x)$. Second, the face poset of a complete simplicial fan in $\mathbb{R}^4$ is that of a triangulated three-sphere, so the link of a face S is a sphere of dimension $3 - |S|$ and

$$\sum_{C \supseteq S} (-1)^{|C|-|S|} \;=\; 1 - \chi(\operatorname{link} S) \;=\; (-1)^{|S|}. \tag{143}$$

We substitute $g(-x) = -1 - g(x)$ into the cone sum at $-\varepsilon$, expand the products, and resum with (143). This gives back the cone sum at $+\varepsilon$. The right-hand side of the twisted period relation (59) is even for a different reason. For $\gamma = \delta$ it pairs the vector of chamber integrals at ε with the same vector at $-\varepsilon$ through the symmetric matrix m_B^{-1}, and is unchanged when the two are exchanged. Every odd order of the relation therefore holds identically, whatever the integrals are, and only the even orders constrain them.

B Numerical methods and reproducibility

The numbers quoted in the text come from four kinds of computation: exact rational arithmetic on the normal fan of $P(3,6)$, numerical quadrature of the cone form (25) and of the residue formula (34), integer-relation searches on the values so obtained, and computations with the scattering equations (106), by exact polynomial algebra or by certified numerical path tracking. For each computed object we give the method, the kinematic points and the accuracy reached. The Supplementary Material, whose sections and tables we cite as S1, S2, and so on, gives the full account, and exact data too long to print are in the ancillary files (Appendix B.9). We wrote two independent numerical implementations of the cone form, with different quadrature rules and working precisions, and used both for the coefficient functions and their values through weight six, the residues of Section 3 and the chamber integrals of Section 4. The weight-seven and weight-eight values at s_0 and $\tilde{s}$, the ω' column of Table 2, the D_4 coefficients beyond weight six and the $K(3,7)$ coefficients beyond weight three come from one implementation only.

B.1 Kinematic points

All kinematic points at which $K(3,6)$ itself is evaluated lie in the convergence region $\mathcal{C}$ of (17), where the sixteen channel forms are positive. Table S1 gives, for each point named in the text, the fourteen exponents s_{abc} of the non-constant minors in the chart (23). Individual values of the integral and of its coefficients are quoted at the generic point s^* of (28); there, with ε absorbed into the exponents, $K(3,6)(s^*) = 127.3994148\ldots$ to about sixty significant figures. We check the sign of the closed function (67), built with the face rule (64), at the point $\hat{s}$ of (62) and at a second generic point of the convergence region,

$$840\,\hat{s}' : \quad \begin{aligned} &(s_{135}, s_{136}, s_{145}, s_{146}, s_{156}) = (-280, -315, 252, -336, 280),\\ &(s_{235}, s_{236}, s_{245}, s_{246}, s_{256}) = (-420, -336, -252, -280, -280),\\ &(s_{345}, s_{346}, s_{356}, s_{456}) = (720, -315, -240, 525),\\ &(s_{123}, s_{124}, s_{125}, s_{126}, s_{134}, s_{234}) = (599, -901, -305, 1297, -291, 878), \end{aligned} \tag{144}$$

where the channel forms run from 1/3 to 1703/840 and the radius of convergence is 840/1703. At the point $\tilde{s}$ of (99) the leading coefficient (27) and the coefficient functions q_2, q_3, q_4, Q_5, R_5, A and B give

$$\tilde{s}: \quad \begin{aligned} &c_0 = \tfrac{22331}{5600}, \quad c_2 = -\tfrac{659399}{25200}\,\zeta_2, \quad c_3 = \tfrac{3855109}{75600}\,\zeta_3, \quad c_4 = \tfrac{2375089}{14175}\,\zeta_4,\\ &c_5 = -\tfrac{42609131}{680400}\,\zeta_5 - \tfrac{3331471}{9450}\,\zeta_2\zeta_3, \quad c_6 = \tfrac{1161690737}{3857868000}\,\pi^6 + \tfrac{132129581}{340200}\,\zeta_3^2, \end{aligned} \tag{145}$$

the analog at $\tilde{s}$ of the expansion (30) at s_0. We identify the weight-seven and weight-eight coefficients at $\tilde{s}$ as in Appendix B.6. They are again polynomials in single zeta values,

$$\tilde{s}: \quad \begin{aligned} &c_7 = -\tfrac{7373851979}{6123600}\,\zeta_7 + \tfrac{74388967}{255150}\,\zeta_2\zeta_5 + \tfrac{2028226687}{1020600}\,\zeta_3\zeta_4,\\ &c_8 = -\tfrac{790420853321}{18370800}\,\zeta_8 - \tfrac{7490791}{109350}\,\zeta_3\zeta_5 - \tfrac{4156024139}{1530900}\,\zeta_2\zeta_3^2, \qquad D_8(\tilde{s}) = 0. \end{aligned} \tag{146}$$

Table S1 also gives the points on the pole hyperplanes $\varepsilon\kappa_a = -n$, on and off the four-Beta locus $\mathcal{L}_4$, at which the residues were evaluated (Appendix B.4) and the integer point of Section 3.6, at which a search from 400 starting points finds the three critical points described there. Section S1 contains the Hessian of $\log u$ in $y = (\log a, \log b, \log c, \log d)$ at the critical point $a = b = c = d = 1$, its eigenvalues and the positions of the two maxima.

B.2 The fan and the cone form

The polytope $P(3,6)$ has 48 vertices, 98 edges, 66 two-faces and 16 facets, and its normal fan has sixteen rays and forty-eight maximal cones: forty-six simplicial and two bipyramids over a triangle, one at each of the relations (16). Splitting each bipyramid into three simplicial cones gives the fifty-two cones C of (25). We checked in exact integer arithmetic that every one of the fifty-two cones is unimodular, that is, the matrix of its four primitive ray vectors has determinant ± 1. We also checked that in each cone the ten polynomials P_t factor as monomials in y times polynomials $Q_{C,t}(y)$ with unit coefficients and $Q_{C,t}(0) = 1$. On the 208 coordinate axes of the fifty-two cones every $Q_{C,t}$ restricts to 1 or to $1 + y$, which is the input to the weight-two statement (89). The D_4 cluster fan of Section 4 is another unimodular simplicial refinement on the same rays. The quantities defined by the fan alone, namely the leading coefficient (27), the $W_r(s)$ of (86), the matrix m_B and the trigonometric sums (60) and (64), are computed in exact arithmetic and take the same values on both refinements. In particular, c_0 from (27) equals the 48-term formula for $m(3,6)$ of [14, 46] as a rational function (Section S2).

B.3 Values at finite string tension

Every value of $K(3,6)$ and of the chamber integrals $Z(\gamma|a)$ at finite ε is computed from the cone form (25) by a tensor product of Gauss–Jacobi rules with the endpoint powers absorbed into the weight. For $-\varepsilon$, and for relabeled kinematics outside $\mathcal{C}$, we continue the integrals analytically in both implementations, subtracting the leading endpoint behavior on each axis with a negative exponent and adding it back in closed form. The chamber integrals are stable to about thirteen significant figures at $+\varepsilon$ and about eleven at $-\varepsilon$, which sets the accuracy of the period-relation and kernel comparisons of Section 4. Two independent evaluations of $K(3,6)(s^*)$ at $\varepsilon = 1$, where no continuation is needed, agree to about sixty significant figures.

In the 60×60 comparison of Section 4.2 the sixty chamber integrals are obtained by the same quadrature with the labels permuted, and the face F_c of (64) is found constructively from the D_4 fan. In exact arithmetic the coefficient of $(2\pi i\varepsilon)^{-4}$ in $H(\gamma, \delta)$ equals $m(\gamma|\delta)$ for all 3600 pairs. At $\hat{s}$ and $\varepsilon = 1/7$, with 24 nodes per axis, the right side of (59) formed from all twenty-six basis forms over all sixty chambers agrees with $H(\varepsilon)$ to 1.8×10^{-12} of the largest entry over all 3600 pairs. Section S3 gives the conventions, the full comparison and the alternative phase assignments. For the rank of the 60×60 intersection matrix we compute its singular values carrying fifty significant figures. The twenty-sixth is of order 10^{-4} of the largest, and the twenty-seventh vanishes to the figures carried: $\sigma_{27}/\sigma_1 = 7.5 \times 10^{-51}$, 2.0×10^{-51} and 1.2×10^{-51} at $\varepsilon = 0.1$, 0.37 and $1/7$. Then, order by order in ε at $\gamma = \delta = I$, we compare the single-valued images (69) with the bilinear (67) at $\hat{s}$, using the exact coefficient functions through weight five and the pair A, B at weight six. The two sides agree to 1×10^{-27} for $2Q_5\zeta_5$ and to 4×10^{-27} for $4B\zeta_3^2$. At $\hat{s}'$ we also find that the period relation (59) holds at $\gamma = \delta = I$, to the accuracy of the chamber integrals, for $\varepsilon = \frac{1}{7}$ and $\frac{3}{10}$ (Section S3). Section S3 tabulates the values of $\mathcal{M}(\varepsilon)$ of (67) used in the positivity statements of Section 4.3 at $\hat{s}$ and $\hat{s}'$, including eight values of ε bracketing the pole of H_B at $87/218$. It also gives the inertias of H_B and the normalization check of (72).

B.4 Residues and the momentum slice

At each of the twelve s and t walls we compare the facet integral $K_{\partial a}$ of (32) with the disk integral $K(2,6)$ in a dihedral ordering β. The ordering is fixed by a map from the nine rays of the projected star of a to the nine chords of a hexagon, under which the fourteen cones of the star become the fourteen triangulations (Section S4). Then $c_0^{\partial a}$ is $m(2,6)$ in the ordering β, and we find that its t-wall form (42) equals the residue of the exact c_0, in rational arithmetic, at 25 random rational points of the wall $t_{3456} = 0$. At the walls s_{156} and s_{345} a monomial change of facet coordinates with determinant ± 1 carries the facet integrand into the Koba–Nielsen integrand letter by letter. At the other ten walls the two integrals agree numerically, as functions of ε at rational points on each wall, to between 28 and 38 significant figures. Beyond the leading level we compute the residues from (34) of Proposition 1, as finite sums (38) of three-dimensional integrals of the type (25) with positive endpoint exponents at the points used. We evaluate them as in Section B.3, by a tensor product of Gauss–Jacobi rules with the endpoint powers absorbed into the weight, at two orders, twenty-four and thirty-two nodes per axis, carrying forty significant figures. The difference between the two orders is the accuracy we assign: at least twenty significant figures at every point used, about twenty-five on and near the four-Beta locus, and more than thirty at the cube wall. The kinematic points are rational points on each hyperplane $\varepsilon\kappa_a = -n$, with denominator six, chosen so that the three endpoint exponents of the facet integral are at least one third.

On the four-Beta locus the closed forms (48) are reproduced to better than thirty figures. Away from the locus we confirm the degree-n property (38) by polynomial fits through the sequences of Table S1, and the degree-one fits reproduce the exact rational partial-wave coefficients (51) (Section S4). On the momentum slice of Section 3.5 the double-pole coefficient has the closed form Eq. (54). The closed form agrees with direct two-dimensional quadrature of the face integral over its four cones to nearly forty significant figures. A Laurent fit to $K(3,6)$ along the line in the slice on which only s_{156} varies reproduces the coefficient to about nine figures, which is the accuracy that such a fit allows. The coefficient of the simple pole contains the finite parts of the two six-point facet integrals. These were computed at the pole, at three relative orientations of the triples $\{1,5,6\}$ and $\{2,3,4\}$ with the invariants within each triple held fixed, to about twenty significant figures. These finite parts change by about eleven percent from one orientation to the next, whereas the double-pole coefficient is the same at all three. Section S4 describes an independent implementation of (34) in a global chart and the analogous comparison at the t wall of the slice.

B.5 Expansion coefficients

The coefficient $c_w(s)$ is the finite sum (86) of rational functions $W_r(s)$, determined by the fan and computed once in exact arithmetic, times constants C_r. The C_r are integrals, of dimension at most four, of products of logarithms of the cone letters over faces of the unit cube. The C_r were computed with both implementations. In the first, each constant is evaluated by quadrature rules adapted to its face, twice, at different orders and precisions. The two resulting values of c_6 agree to between 64 and 76 significant figures at the 280 points of the weight-six data of Appendix B.9 and to 68 at the reference points listed there. In the second, we expand the cone integrands in powers of ε, which leaves integrals of products of logarithms of the cone letters over the unit cube. We evaluate them by tensor-product Gaussian quadrature on a sequence of nested grids, the finest with $(160, 120, 96, 88)$ nodes on the four axes, carrying about one hundred significant figures. At a rational kinematic point the resulting values of c_w on the two finer grids are good to sixty significant figures or more through weight seven. The coarsest grid, of $(96, 80, 72, 64)$ nodes, agrees with the next, of $(128, 96, 80, 72)$ nodes, to between fifty-four and fifty-seven significant figures from weight five on, and these two grids carry the weight-seven and weight-eight values at s_0 and at $\tilde{s}$; the finest grid was used for the comparison form ω', and at weight eight it agrees with the second to sixty-three figures. Through weight six both implementations give the same rational multiples (30) and (145) at s_0 and $\tilde{s}$, and they agree at two further points (Section S5). The exact coefficient functions through weight five reproduce the values of $c_2, \ldots, c_5$ computed from the cone form at s^*, s_0, $\tilde{s}$, $\hat{s}$ and further rational points of $\mathcal{C}$.

At weight six we take $A(s)$ and $B(s)$ of (96), with denominator $\prod_a \kappa_a$ and numerator degree eighteen, in the 543-dimensional span of the weight-six W_r. We solve for their coordinates exactly from 596 linear conditions, working modulo 96 primes and reconstructing over $\mathbb{Q}$. Each condition is the two-term relation in $\{\pi^6, \zeta_3^2\}$ obeyed by c_6 at one of 190 rational points of $\mathcal{C}$ or by a Taylor coefficient along a rational line through them. The two functions reproduce c_6 at 90 further rational points, disjoint from those, to at least 67 significant figures, the accuracy of those values. We give Q_5 and R_5 likewise in a 348-element basis. The weight-six $W_r(s)$ are not tabulated, but the ancillary files give their values at each of the 280 listed points, so that A, B, Q_5 and R_5 are evaluated there by a finite rational computation (Appendix B.9).

B.6 Identification of constants

Wherever a number is identified in the text as a rational combination of zeta values, we made the identification with an integer-relation algorithm (PSLQ) in one of the following bases. At weight five the basis is $\{\zeta_5, \zeta_2\zeta_3\}$, at weight six $\{\zeta_6, \zeta_3^2\}$, at weight seven $\{\zeta_7, \zeta_2\zeta_5, \zeta_3\zeta_4\}$, and at weight eight $\{\zeta_8, \zeta_3\zeta_5, \zeta_2\zeta_3^2, \zeta(3,5)\}$, with

$$\zeta(3,5) \;=\; \sum_{n>m\geq 1} \frac{1}{n^3 m^5} \;=\; 0.2046611369\ldots. \tag{147}$$

We state the convention because with the exponents exchanged the sum is $0.0377076729\ldots$, and (101) holds with the first and not the second.

By the height of an integer relation we mean the largest absolute value of the coefficients of the primitive relation. We accept a relation only if it is returned unchanged at several numbers of supplied figures, has residual at the level of the quadrature error of the grid as calibrated by the exactly known lower weights, and reproduces the value on a finer grid. The relations for the weight-seven and weight-eight coefficients of $K(3,6)$ at s_0 and at $\tilde{s}$ have integer heights between 10^8 and 10^{11}. Each was found on the coarser of the two grids from about fifty significant figures, is unchanged when a few more or fewer figures are supplied, and reproduces the value on the finer grid to at least sixty significant figures, beyond the fifty-four to fifty-seven figures to which the two grids agree. Table S3 gives each input value, the figures supplied, the residuals and the height, for these and for c_8 of the comparison form ω'. In the four-element basis with $\zeta(3,5)$ adjoined, the accepted relation is returned at both points with coefficient zero for $\zeta(3,5)$ once relations with residual above the error of the grid are excluded. We state no exclusion height for a relation with $D_8 \neq 0$ beyond what these searches imply (Section S6). The relation (101) for ω' has height 2.4×10^{11}. It first appears when about fifty-eight significant figures are supplied and reproduces the value to about seventy; in the basis with $\zeta(3,5)$ removed no relation of comparable height exists. The exact statements at weights three to five also take identified constants as input. The two weight-three constants (138) and the small-height relations among the weight-five constants used in Appendix A were found in the same way, the former from values good to about two hundred significant figures. For E_2 we also have an independent evaluation, in which one intermediate constant is itself identified numerically, whereas for E_4 the two-hundred-figure identification is the only source.

For the D_4 and seven-point coefficients we used the same bases and acceptance criterion as for $K(3,6)$. The D_4 coefficients at $(c,c') = (\frac{1}{3}, -\frac{1}{3})$ were computed at s_0 on two grids that agree to at least fifty-four significant figures; at weight eight no relation of height below 3×10^{10} exists at that accuracy, and none below 2×10^{12} in the roughly sixty figures of the finer grid alone. For the weight-seven coefficient of $K(3,7)$ on the dihedral family, the eighty coordinates identified directly were supplied to the algorithm with between twenty-one and forty significant figures, according to the dimensions of the faces that contribute to each. The remaining nonzero coordinates follow by linear algebra from these and from values of $c_7^{(3,7)}$ at integer points of the family. Section S6 gives the checks on the weight-three to weight-five inputs and on the D_4 coefficients, and the figures and heights of the $K(3,7)$ identifications. The least secure of these is the weight-six relation at one integer point of the dihedral family: its height is the largest relative to the figures available.

B.7 Algebraic and path-tracking computations

We derive the statements of Section 6 about the twenty-six solutions from exact computations with the scattering equations (106). At s^* a Gröbner-basis computation with the program msolve gives a rational univariate parametrization whose eliminating polynomial has degree 26, is squarefree, and has exactly 16 real roots by Sturm sequences, one of them in the positive part. At two rational points with all fourteen exponents in (23) positive we computed the characteristic polynomial of multiplication by the separating element $t = 2a + 3b + 5c + 11d$ in the twenty-six-dimensional quotient algebra saturated by the product of the minors. We reconstructed this polynomial over $\mathbb{Q}$ from reductions modulo primes. The reconstruction is stable after 43 primes and verified against 53 more at one point, after 70 and against 26 more at the other, with coefficients of up to 182 and 147 digits. At both points the polynomial has degree 26, is squarefree and irreducible over $\mathbb{Q}$, and has 26 real roots. For one of the two points we certified the solutions by interval arithmetic and found them in twenty-six distinct sign chambers, none of them the positive part (we omit the coordinates of the two points). For the Galois group we reduce the eliminating polynomial at a third point, with positive integer exponents, modulo six primes. Each reduction is squarefree of degree 26, and the six factorization patterns are the cycle types (113) (point, primes and patterns in Section S7). The forty-three resonance conditions of Section 6 correspond to the forty-three facets of the cone spanned by the columns of the matrix A, which are the exponent vectors of the monomials in the fourteen non-constant minors. The A-hypergeometric system is resonant when a facet functional takes an integer value on the vector of exponents. We evaluated the functionals in exact arithmetic at generic rational points of $\mathcal{C}$, where none is an integer, and at s_0, where ten are. We established the irreducibility over $\mathbb{Q}$ of the numerators of $c_0, \ldots, c_6$ by exact polynomial factorization.

By path tracking with the package HomotopyContinuation.jl, certified by interval arithmetic, we find all 26 solutions real and distinct at each of 300 points of positive kinematics for $(3, 6)$. For the $(3, 7)$ counts of Section 7.3 we tracked all 1272 paths at some thirty random points of the positive region of [14]. On segments between these points we located by bisection the three transitions at which the number of real solutions changes, with all 1272 solutions certified distinct at both ends of each bracket. None of the three lies on any of the 721 hyperplanes on which a positive tropical ray coordinate changes sign, and neither $m(3, 7)$ nor $K(3, 7)$ shows a feature across the first transition (Section S7). We checked fixed-angle softness in Section 3.6 at 300 random integer points of $\mathcal{C}$, at each of which a search from 40 starting points found one local maximum of $\log u$ (Section S7). This search is numerical and confined to the sample, so it establishes nothing about other points of $\mathcal{C}$. Section S7 also describes the enumeration and the exact linear algebra on the on-shell space that together establish Claim 2, repeated at eight, nine and ten points, and the linear-reducibility computations. In these we apply Brown's compatibility-graph reduction to the ten P_t with a, b, c, d in all twenty-four orders, eight of which end with the alphabet $\{0, -1\}$. We carry out the same reduction for the twenty-two non-monomial factors of the chart of $X(3, 7)$ of [73] and for the cone polynomial f of Section 7.5 with its twelve face restrictions.

B.8 Seven-point and D_4 computations

For $K(3, 7)$ we use the positive parametrization analogous to (20), with seven constant, six monomial and twenty-two polynomial minors. The normal fan of $P(3, 7)$ has forty-two rays and 693 maximal cones, refined to 952 unimodular simplicial cones in one implementation and 944 in the other. The leading coefficient is an exact cone sum, unchanged under re-refinement and equal at a random rational point to $m(3, 7)$ as a sum of 1064 planar diagrams [74]. The coefficients $c_2^{(3,7)}/\zeta_2$ and, given (138), $c_3^{(3,7)}/\zeta_3$ are exact cone sums as well (Section 7.5). For instance, $c_0^{(3,7)} = 119$,

$c_2^{(3,7)}/\zeta_2 = -483$, $c_3^{(3,7)}/\zeta_3 = 616$ at the dihedral point of (126) and $c_0^{(3,7)} = 462$ at the planar point of (127), and both implementations give these rationals there and agree with each other at seven further dihedral points. We computed the coefficients of weights four to six with the second implementation only, by expanding at eleven integer points and using the parity $(-1)^w$ for the values at $-s$. Eight of the points are dihedral and three have cyclic symmetry, one of them outside $\mathcal{C}$, where the values are those of the continued coefficient functions (Section S8).

The D_4 integral $I_{D_4}(c, c')$ of Section 7.1 is (23) with the integrand multiplied by $X^{\varepsilon c} Y^{\varepsilon c'}$, where $X = c\,(b + d)$ and $Y = a\,(1 + a + c + ab + bc)$ in the coordinates a, b, c, d of the chart (20) (this coordinate c is not the twist). Its fan is the D_4 cluster fan with fifty unimodular cones, and the twists enter the exponents linearly. At the split point on the diagonal $c = c' = \gamma$ the coefficients c_0, c_2/ζ_2 and c_3/ζ_3 are exact rational functions of γ, of numerator degrees four, six and seven over a common denominator of degree seven. The zeros of the denominator, $\gamma \in \{-\frac{3}{2}, -\frac{9}{10}, -\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{16}{15}\}$, are the twists at which a channel exponent vanishes, and the three functions are given in the ancillary files. We obtain the weight-four values (118) from the twenty-seven weight-four cone constants evaluated to at least 128 significant figures and a two-term integer relation with ζ_4. The independent evaluation to seven figures quoted in Section 7.1 is a fit of finite-ε values of I_{D_4} (Section S8).

B.9 Ancillary files

Two sets of ancillary files accompany this paper. The first, the weight-six data, contains the functions A and B as exact rational vectors in the 543-coordinate basis, and Q_5, R_5 in the 348-coordinate basis, of Appendix B.5. The same set holds the 190 rational kinematic points used to determine A and B and the 90 further points. Every point is listed in the coordinates of Table S1 with its value of c_6 from each of the two quadrature evaluations to 80 printed figures, of which between 64 and 76 are common to both. With each point we also list the vector of exact rational values $W_r(s)$, whose contraction with the coordinate vectors of A, B, Q_5 and R_5 gives these four functions there. The first set further contains the reference points s^*, s_0 and two neighbors of s_0 with their exact A, B, Q_5 and R_5. For the count of real solutions at $(3, 7)$ in Section 7.3 it contains the endpoints of the three segments scanned and the three brackets enclosing the transitions. For the D_4 family it gives the twist conventions, with the letters X, Y written out, and the exact functions $c_2(\gamma)/\zeta_2$, $c_3(\gamma)/\zeta_3$ at the split point. A program included with it recomputes the c_6 table at the reference points exactly from the coordinate vectors of A and B and repeats the comparison at all 280 points.

The second set is the proof data for weights three to five. It contains the table of the five pair shapes of (136) with their constants, the exact values of c_0, q_2 and q_3 at s^* and at a second generic point and of q_4 at s^*, and the exact $Q_5(s^*)$ and $R_5(s^*)$ with the counts of the lattice reduction of Appendix A.3. It gives, as tables of exact rationals, the sixteen channel forms and the fifty-two cones of the refined fan with the exponents and letters of every cone integral (130), and the 312 flags. The functions $W_r(s)$ of (86) at weights two to five are given as polynomials in the coordinates x of Appendix A.2 over the products of channel forms. The weight-three and weight-four tables hold the fifty-four constants with their integrands, their coordinate vectors γ_r over $B_{16} \cup \{\tau_1, \tau_2, \tau_3\}$ and numerical values, the integrands of τ_1, τ_2, τ_3, the rationals r_g and the seventeen identities (139). The weight-five tables hold the 364 numerator polynomials T_r, the 217 dependences a_{qp}, the numerically evaluated constants with the coordinates of those identified individually, and the integer relations that generate the lattice. For the D_4 family the set gives the exact function $c_0(\gamma)$ at the split point with its poles and residues. Three documents describe the computations step by step, and two programs are included. One recomputes the closed forms from the exact rationals and compares them with the stored numerical values. The other derives from the tables the exact values of $c_0, \dots, c_4$ at s^*, expands the seventeen identities and the 217 dependences to zero, reduces c_5 modulo the lattice to (94) and (95), checks the parity relations of Appendix A.2 at the 312 flags, and confirms the poles of $c_0(\gamma)$. Neither set contains the weight-six functions $W_r(s)$ in closed form, the chamber integrals or the matrix $H(\varepsilon)$ of Section 4, or the seven-point expansion data. In particular the twelve points of Table S2 are not among the 280, and the files do not by themselves reproduce the values of B used there. The second set also contains the Supplementary Material, in Sections S1–S8 with Tables S1–S3, which gives the details summarized in Appendices B.1–B.8.

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