Supplementary material for "Factorization without clustering and the six-point Grassmannian string integral"

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This supplement accompanies the paper and its ancillary files. It gives in full the account of the computations that Appendix B of the paper summarizes: the tables of kinematic points, the conventions of each comparison, the values obtained, and the figures supplied to and the heights returned by the integer-relation searches. Sections S1 to S8 follow Appendices B.1 to B.8 of the paper in order and repeat their statements where the surrounding detail needs them. Equation, section, table, proposition and reference numbers without the prefix S are those of the paper; the two implementations, the named kinematic points s^*, s_0, $\tilde{s}$, $\hat{s}$, $\hat{s}'$ and the ancillary files are as defined there (Appendices B and B.9).

S1 Kinematic points

All kinematic points at which $K(3,6)$ itself is evaluated lie in the convergence region $\mathcal{C}$ of (17), where the sixteen channel forms are positive. In the chart (23) the six minors (123), (124), (125), (126), (134), (234) are constant, and momentum conservation (12) fixes their exponents. A point is thus specified by the remaining fourteen s_{abc}, and Table S1 lists the named points of the text in these coordinates. The expansion (30) and Table 2 refer to the base point s_0 of (29), which is s^* rounded to thirds. We make the finite-α' statements of Section 4 at the generic point $\hat{s}$ of (62), with all invariants in $\frac{1}{87}\mathbb{Z}$. There the sixteen channel forms run from 34/87 to 57/29 and the radius of convergence in ε is 29/57. At s^*, s_0, $\tilde{s}$, $\hat{s}$ and $\hat{s}'$ all ten polynomial exponents s_t of (23) are negative, and the positive critical point of the integrand is then unique (Section 3.6). Finally, the comparison form is the four-form ω' of (100), which in the off-diagonal pairing of Section 5 replaces the measure $da\,db\,dc\,dd/(abcd)$ of (23), with the region $\mathbb{R}^4_{>0}$, the Koba–Nielsen factor and the kinematics s_0 unchanged.

The six rows of the middle block of Table S1 are points outside $\mathcal{C}$ by construction. The residues of Section 3 are functions on the pole hyperplanes $\varepsilon\kappa_a = -n$, and we evaluate them there (Appendix B.4). For each residue statement of the text we list the first of the points used; the others differ from it by the steps named in the caption of the table. The last row is the integer point of Section 3.6 with three positive polynomial exponents, s_{136}, s_{256} and s_{346}. Its sixteen channel forms, in the order of (15), are $(5,4,1,1,1,3,3,1,2,3,4,7,1,8,4,7)$, so it lies in $\mathcal{C}$. With $S(x;s) = \sum_{a<b<c} s_{abc}\log(abc)$ and $y = (\log a, \log b, \log c, \log d)$, the gradient of S in y vanishes at

[*]Work done with Anthropic, but results and views are not endorsed by Anthropic.

point	s_{135}	s_{136}	s_{145}	s_{146}	s_{156}	s_{235}	s_{236}	s_{245}	s_{246}	s_{256}	s_{345}	s_{346}	s_{356}	s_{456}	used for
s^* (28)	$-\frac{2}{5}$	$-\frac{3}{7}$	$\frac{1}{4}$	$-\frac{4}{9}$	$\frac{2}{5}$	$-\frac{5}{11}$	$-\frac{6}{13}$	$-\frac{1}{3}$	$-\frac{2}{7}$	$-\frac{3}{8}$	1	$-\frac{4}{11}$	$-\frac{5}{13}$	$\frac{1}{2}$	(94), S_+, §6.4
s_0 (29)	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	(30), Table S3
$\tilde{s}$ (99)	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	-1	$\frac{4}{3}$	-1	-1	-1	-1	$-\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	(145), Table S3
$87\,\hat{s}$ (62)	-35	-26	38	-20	38	-38	-32	-38	-29	-20	87	-29	-23	58	(59), (65), (71), $\mathcal{M} > 0$
$840\,\hat{s}'$ (144)	-280	-315	252	-336	280	-420	-336	-252	-280	-280	720	-315	-240	525	$\mathcal{M} > 0$; (59) at $\gamma = \delta = I$
split point	0	0	$-\frac{17}{30}$	0	$\frac{2}{5}$	$-\frac{19}{12}$	0	0	0	0	$\frac{47}{30}$	$-\frac{47}{30}$	0	$\frac{1}{2}$	(118)
s_{156}, $\mathcal{L}_4$, $n = 1$	-2	0	-2	2	-1	-2	0	0	0	0	2	-2	0	$\frac{3}{2}$	(48), (51); 3 pts
s_{156}, $\mathcal{L}_4$, $n = 2$	-3	0	-3	3	-2	-3	0	0	0	0	3	-3	0	$\frac{5}{2}$	(48), (50); 4 pts
s_{156}, gen., $n = 1$	-2	$\frac{3}{2}$	-2	2	-1	-2	-2	-2	-2	2	2	-2	2	$\frac{3}{2}$	degree of $R^{(1)}$; 3 pts
s_{156}, gen., $n = 2$	-2	$\frac{3}{2}$	-1	2	-2	-2	-2	-2	-2	2	2	-2	2	$\frac{3}{2}$	degree of $R^{(2)}$; 4 pts
t_{4561}, $n = 1$	-2	2	$-\frac{3}{2}$	-2	2	-2	-2	$-\frac{1}{2}$	-2	2	2	-2	2	$\frac{1}{2}$	degree of $R^{(1)}$; 3 pts
$\tilde{R}$, $n = 1$	-2	2	$-\frac{1}{2}$	2	$\frac{1}{2}$	-2	-2	-2	2	-2	$\frac{1}{2}$	-2	2	0	degree of $R^{(1)}$; 3 pts
integer point	0	1	5	-4	1	-4	-2	0	-5	5	1	4	-4	1	§3.6

Table S1: Kinematic points used in the text, by their fourteen independent exponents in the chart (23); the six exponents of the constant minors follow from (12). For $\hat{s}$ and $\hat{s}'$ the entries are to be divided by the factor shown. The first six points lie in the convergence region $\mathcal{C}$; at the first five all ten polynomial exponents are negative. The six rows below them are the first points of the sequences on the pole hyperplanes $\varepsilon\kappa_a = -n$ (with $\varepsilon = 1$) at which the level-n residues of Proposition 1 were evaluated. Each sequence has three or four points in all, as indicated, and its remaining points are obtained from the first by steps of $(-\frac{1}{2}, +\frac{1}{2})$ in (s_{145}, s_{146}) for the s_{156} rows, of $+\frac{1}{2}$ in s_{135} for the t_{4561} row and of $(-\frac{1}{2}, +\frac{1}{2})$ in (s_{135}, s_{136}) for the $\tilde{R}$ row, each a direction transverse to the wall in the sense of (38). The $\mathcal{L}_4$ level-one row, with s_{136} changed to $\pm\frac{1}{6}$, is the point of (51). The split point is that of Section 7.1 and lies in $\mathcal{C}$. The last row is the integer point of $\mathcal{C}$ at which we find three critical points of the integrand in the positive part. The twelve points of the Gamma test are in Table S2.

$a = b = c = d = 1$, where the Hessian is

$$\left.\frac{\partial^2 S}{\partial y_i\,\partial y_j}\right|_{y=0} = \begin{pmatrix} -\frac{94}{45} & \frac{4}{5} & \frac{13}{15} & -\frac{22}{45} \\ \frac{4}{5} & -\frac{59}{45} & -\frac{16}{45} & -\frac{7}{45} \\ \frac{13}{15} & -\frac{16}{45} & -\frac{133}{90} & -\frac{1}{90} \\ -\frac{22}{45} & -\frac{7}{45} & -\frac{1}{90} & \frac{1}{30} \end{pmatrix},$$

with eigenvalues approximately -3.142, -1.042, -0.901 and 0.241. Searching from 400 starting points, we find two further critical points in the positive part, both local maxima. They lie near $(a, b, c, d) = (0.676, 0.391, 0.472, 12.397)$, with $S = -11.633$, and near $(1.865, 1.045, 1.403, 0.220)$, with $S = -12.118$, both above the value $S = -12.206$ at the critical point $a = b = c = d = 1$.

	s_{135}	s_{136}	s_{145}	s_{146}	s_{156}	s_{235}	s_{236}	s_{245}	s_{246}	s_{256}	s_{345}	s_{346}	s_{356}	s_{456}	c_0	$2c_0B - q_3^2$
1	-1	-1	0	$-\frac{1}{3}$	1	0	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{914157}{16000}$	$\frac{10060231447}{800000}$
2	0	-1	$-\frac{1}{3}$	$-\frac{1}{3}$	1	-1	0	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1128843}{11200}$	$\frac{12512114343}{2240000}$
3	$-\frac{2}{3}$	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	$\frac{1}{3}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$	1	$\frac{269631}{2800}$	$\frac{37179381}{14000}$
4	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	-1	$-\frac{2}{3}$	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{30861}{320}$	$\frac{19860273}{2560}$
5	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	-1	-1	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{4}{3}$	$\frac{13239}{64}$	$\frac{1255259}{256}$
6	$-\frac{2}{3}$	-1	1	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	-1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	-1	$\frac{4}{3}$	$\frac{351033}{22400}$	$\frac{95588343997}{53760000}$
7	$-\frac{2}{3}$	0	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	1	$\frac{345}{2}$	$\frac{13232}{3}$
8	$\frac{1}{3}$	-1	1	$-\frac{1}{3}$	$\frac{2}{3}$	-1	-1	$-\frac{2}{3}$	0	0	$\frac{2}{3}$	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2830167}{44800}$	$\frac{4581993987}{448000}$
9	-1	-1	$\frac{2}{3}$	-1	1	$-\frac{1}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{27954}{1225}$	$\frac{3149314219}{768320}$
10	-1	-1	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	$\frac{4}{3}$	-1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{70001}{2560}$	$-\frac{12933031}{98304}$
11	0	-1	0	$-\frac{2}{3}$	1	-1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	-1	$\frac{4}{3}$	$\frac{29331}{640}$	$\frac{748187579}{153600}$
12	-1	-1	$\frac{2}{3}$	$-\frac{1}{3}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$	-1	0	$-\frac{2}{3}$	1	-1	$\frac{1}{3}$	1	$\frac{256489}{19600}$	$\frac{152229801533}{98784000}$

Table S2: The twelve rational points of the convergence region $\mathcal{C}$ at which the three relations (116) were tested, in the coordinates of Table S1, with the exact value of c_0 and of the combination $2c_0B - q_3^2$, which a product of Gamma functions of linear forms would force to vanish. The test uses the exact rational values of c_0, q_2, q_3, R_5 and B at each point, with B the function determined in Section 5.3, and involves no further numerical integration; all three relations fail at every one of the twelve points, as they do at s_0 and $\tilde{s}$ (Section 6.3).

S2 The fan and the cone form

The polytope $P(3,6)$ is the Newton polytope of the integrand of (23), the Minkowski sum of the Newton polytopes of the four monomials and the ten polynomials P_t. It has 48 vertices, 98 edges, 66 two-faces and 16 facets. Its normal fan has sixteen rays, on which the tropical function of the integrand takes the values (15), and forty-eight maximal cones. Here forty-six of the maximal cones are simplicial. The other two are bipyramids over a triangle, one at each of the relations (16), with the five rays named in the relation. We split each bipyramid into three simplicial cones about the axis joining its two apexes, which gives the fifty-two cones C of (25). We checked in exact integer arithmetic that every one of the fifty-two cones is unimodular: the matrix of its four primitive ray vectors has determinant ± 1. The change of variables to the unit cube in (25) then has unit Jacobian and integer exponents. In each cone the ten polynomials become polynomials $Q_{C,t}(y)$ with unit coefficients and $Q_{C,t}(0) = 1$, once the monomial of lowest weight is factored out and absorbed into the exponents $\kappa_{C,i}$. We checked both properties for every cone and every t. On the 208 coordinate axes of the fifty-two cones every $Q_{C,t}$ restricts to 1 or to $1 + y$, which is the input to the weight-two statement (89). The D_4 cluster fan used in Section 4 is another unimodular simplicial refinement of the same fan on the same sixteen rays. It divides each bipyramid into two simplicial cones instead of three, and has fifty maximal cones and 233 cones in all, against 241 for the triangulation of (25). Several quantities are defined by the fan alone: the leading coefficient (27), the rational functions $W_r(s)$ of (86), the leading-order matrix m_B, and the trigonometric sums (60) and (64). We compute them in exact rational arithmetic, or as exact functions of the $e^{2\pi i\varepsilon\kappa_r}$, and they take the same values on both refinements. In particular c_0 from the fifty-two-cone sum (27) equals the 48-term formula for $m(3,6)$ of [1, 2] as a rational function.

S3 Values at finite string tension

Every value of $K(3,6)$ and of the chamber integrals $Z(\gamma|a)$ at finite ε is computed from the cone form (25). This is a sum of fifty-two integrals over the unit cube $(0,1)^4$, each with an integrand $\prod_i y_i^{\varepsilon\kappa_i - 1}$ times positive polynomials raised to powers. We integrate each cone by a tensor product of Gauss–Jacobi rules, one per axis, with the endpoint powers absorbed into the Jacobi weight; the remaining integrand is smooth on the closed cube. At $-\varepsilon$, and at relabeled kinematics outside $\mathcal{C}$, some endpoint exponents are negative and the integrals are defined by analytic continuation. We implement it by subtracting the leading endpoint behavior on each such axis and adding it back in closed form. Both implementations contain such an integrator, one on the fifty-two-cone fan and one on the D_4 cluster fan. The chamber integrals are stable to about thirteen significant figures at $+\varepsilon$ and about eleven at $-\varepsilon$, which sets the accuracy of the period-relation and kernel comparisons of Section 4. For $K(3,6)$ itself at points of $\mathcal{C}$ and at $+\varepsilon$ no continuation is needed, and two independent evaluations of $K(3,6)(s^*)$ at $\varepsilon = 1$ agree to about sixty significant figures, the accuracy quoted in Section 2.4.

The conventions of the 60×60 comparison of Section 4.2 are as follows. The sixty convex chambers are labeled by the sixty dihedral orderings of the six points. Each is the image of the positive chamber I under a relabeling $\pi_\gamma \in S_6$, so we compute all sixty integrals by the same quadrature with the labels permuted. Concretely, $Z(\gamma|a)(\pm\varepsilon)$ is the integral over I of the relabeled form $\Omega_{\pi_\gamma(a)}$ at the relabeled kinematics s^{π_γ}, $(s^\pi)_J = s_{\pi^{-1}(J)}$, multiplied by the product $\epsilon_\gamma \epsilon_a \epsilon_{\pi_\gamma(a)}$ of the signs incurred when each Parke–Taylor form is written with sorted minors. With this convention all twenty-six signs of the $Z(\gamma|I)$, $\gamma \in B$, are $+1$, and the leading coefficient of each equals the exact biadjoint amplitude $m(\gamma|I)$. On each real chamber u^ε is the positive branch, every minor replaced by its absolute value. At the relabeled points only the chamber I itself has all sixteen $\kappa_r > 0$, so the other integrals require the endpoint continuation just described. We find the face F_c of (64) constructively, by enumerating the cones of the D_4 fan that lie in the closures of both c and I. For the orderings $c \neq I$ with $m(I|c) \neq 0$ they form the closed star of exactly one face, with the dimension counts stated in Section 4.2, and for the fifteen orderings with $m(I|c) = 0$ there is no shared cone. We then assign the crossing factor $1/(2i \sin \pi\varepsilon\kappa_r)$ to the rays of F_c and the same-side factor $1/(e^{2\pi i\varepsilon\kappa_r} - 1)$ to the rays of its link. We do not tabulate the fifty-nine assignments here, since they follow from the fan by this rule. We evaluate the leading-order matrix m_B^{-1} and the cone sums over the D_4 fan in exact rational arithmetic. The coefficient of $(2\pi i\varepsilon)^{-4}$ in $H(\gamma, \delta)$ built from (64) equals the exact $m(\gamma|\delta)$ for all 3600 pairs.

We give one comparison in full: at $\hat{s}$ and $\varepsilon = 1/7$, with 24 Gauss–Jacobi nodes per axis, we computed all twenty-six basis forms over all sixty chambers at both signs of ε and formed the right side of (59) for every pair (γ, δ). The resulting matrix agrees with $H(\varepsilon)$ from (64) to 1.8×10^{-12} of the largest entry over all 3600 pairs, and to 6.8×10^{-13} on the 26×26 block. Entry by entry the relative agreement is 1.4×10^{-8} or better over the 2694 nonzero entries. The 906 entries that vanish by the rule come out below 4.9×10^{-13} of the largest, and the computed matrix is symmetric to 1.5×10^{-12}. For the rank of the 60×60 intersection matrix we compute its singular values carrying fifty significant figures. The twenty-sixth is of order 10^{-4} of the largest, and the twenty-seventh vanishes to the figures carried: $\sigma_{27}/\sigma_1 = 7.5 \times 10^{-51}$, 2.0×10^{-51} and 1.2×10^{-51} at $\varepsilon = 0.1$, 0.37 and 1/7. The matrix is the same on the 233-cone and 241-cone fans to the figures carried. We also formed the intersection matrix under other assignments of phases (Section 4.2): the half-monodromy factor written there, of either sign, in place of the crossing factor, or the opposite convention for the side on which a chamber lies. These alternatives fail all three tests above, of rank, of independence of the triangulation and of agreement with the period relations (59). They give matrices of full rank, with σ_{27}/σ_1 between 2.6×10^{-3} and 1.3×10^{-2}, two of them with imaginary parts comparable to their real parts. These matrices change at the per-mille to percent level between the two fans and, restricted to a basis of twenty-six chambers, differ from the matrix obtained from the right side of (59) by 0.14 to 1.8 of its largest entry. For the pair $\gamma = \delta = I$ alone we compared the two sides of (59) as functions of ε: at $\varepsilon = 1/20$, 1/7, 1/4 and 2/5 they agree to 2.5×10^{-15}, 3×10^{-14}, 3×10^{-14} and 3×10^{-9} in relative terms. The last value is limited by the continuation at $-\varepsilon$ near the radius of convergence. Order by order in ε, the comparison at $\gamma = \delta = I$ uses the exact coefficient functions of Section 5 through weight five and the pair A, B at weight six. The single-valued images (69) agree with the bilinear at $\hat{s}$ to 1×10^{-27} for $2Q_5\zeta_5$ and to 4×10^{-27} for $4B\zeta_3^2$.

For the sign of the closed function $\mathcal{M}(\varepsilon)$ of (67) we evaluated the twenty-six basis chamber integrals and the block $H_B(\varepsilon)$ of (64) at $\varepsilon = \frac{1}{10}, \frac{1}{7}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}$ at both points $\hat{s}$ and $\hat{s}'$. At $\hat{s}$ we added eight values of ε between 0.39 and 0.45 bracketing the pole of H_B at 87/218, through which $\mathcal{M}$ is regular: $\mathcal{M} = 9.24 \times 10^6$, 9.20×10^6, 9.08×10^6 at $\varepsilon = 0.399$, 0.3995, 0.401. The inertias of H_B on the two sides of that pole and at $\hat{s}'$ are those given in Section 4.3, and the directions of negative inertia carry between one and six percent of $\mathcal{M}$ at every value sampled. Between $\varepsilon = \frac{1}{10}$ and $\varepsilon = \frac{1}{2}$ the value of $\mathcal{M}$ falls from 1.88×10^9 to 4.64×10^6 at $\hat{s}$ and from 2.30×10^9 to 5.64×10^6 at $\hat{s}'$. Evaluated at $\varepsilon = \frac{1}{7}$, the truncated Laurent series of (70) reproduces the finite-ε value 4.552×10^8 of $\mathcal{M}$ at $\hat{s}$ to the accuracy of the truncation. We fixed the normalization $(2i)^n$ in (72) by the one-dimensional case, where the bilinear of $B(a, b)$ with the Kita–Yoshida self-intersection number equals $2i \int_{\mathbb{C}} |z|^{2a-2}|1 - z|^{2b-2} d^2z$. We also find that at $\hat{s}'$ the period relation (59) at $\gamma = \delta = I$ holds to the accuracy of the integrals for $\varepsilon = \frac{1}{7}$ and $\frac{3}{10}$.

S4 Residues and the momentum slice

The cube entry of Table 1 and the s-facet entries at s_{156} and s_{345} hold in closed form at every order, and the other four s-facet entries and the six t-facet entries only at leading order (Section 3.2). At each of the twelve s and t walls we compare the facet integral $K_{\partial a}$ of (32) with a six-point disk integral defined as follows. The star of the ray a, projected along a, is a fan on nine rays (Appendix A.1). We map the nine rays to the nine chords of a hexagon so that the rays of every cone go to pairwise non-crossing chords. Such a map exists at all twelve walls, and under it the fourteen simplicial cones of the projected star correspond one to one to the fourteen triangulations of the hexagon. The map fixes a dihedral ordering β of six punctures and identifies each planar variable of β with the channel form on the corresponding ray, restricted to the wall; at the wall $s_{456} = 0$ this identification is the second line of (41). At an s wall the nine restricted forms are independent. At a t wall they span eight dimensions and obey one linear relation, the restriction (133) of (16), which equates two sums of two planar variables of β. By (27) applied to the facet, $c_0^{\partial a}$ is then, at all twelve walls, the fourteen-term amplitude $m(2, 6)$ in the ordering β evaluated at the restricted planar variables. At a t wall, by the linear relation (133), this amplitude equals the two-term form (42). We verified (42), as the residue of the exact c_0, in exact rational arithmetic at 25 random rational points of the wall $t_{3456} = 0$. Beyond leading order the disk integral is the six-point Koba–Nielsen integral $K(2, 6)$ in the ordering β at the same planar variables, continued in ε as $K_{\partial a}$ is. At the walls s_{156} and s_{345} a monomial change of the three facet coordinates, with integer exponents and determinant ±1, carries the facet integrand into the Koba–Nielsen integrand letter by letter, which establishes those two entries at every order. At the other four s walls and at the six t walls we compared $K_{\partial a}$ with $K(2, 6)$ in the ordering β numerically, as a function of ε, at rational points on each wall. The two agree to between 28 and 38 significant figures.

We computed the residues beyond the leading level from the formula (34) of Proposition 1. On the hyperplane $\varepsilon\kappa_a = -n$ the residue is a sum over the cones of the fan that contain the ray a. In each of them we take the nth derivative in y_a of the product of powers of the cone polynomials and set $y_a = 0$. The result is the product of the same powers of the facet's letters multiplied by a polynomial of degree n in the exponents, with coefficients that are rational functions of the three remaining coordinates. Integrating over those three coordinates gives the residue as a finite sum, (38), of three-dimensional integrals over the unit cube of the same type as (25), with positive endpoint exponents at the points used. We evaluate them as in Section B.3, by a tensor product of Gauss–Jacobi rules with the endpoint powers absorbed into the weight, at two orders, twenty-four and thirty-two nodes per axis, carrying forty significant figures. The difference between the two orders is the accuracy we assign: at least twenty significant figures at every point used, about twenty-five on and near the four-Beta locus, and more than thirty at the cube wall. The kinematic points are rational points on each hyperplane $\varepsilon\kappa_a = -n$, with denominator six, chosen so that the three endpoint exponents of the facet integral are at least one third. Table S1 gives the first point of each sequence. We also implemented (34) independently, in a single global chart adapted to each wall rather than cone by cone. It reproduces every residue value used to the accuracy just stated.

On the four-Beta locus $\mathcal{L}_4$ the residues of the s_{156} wall at levels one and two are known in closed form, (48), which gives a check independent of the quadrature. The cone sums reproduce them to better than thirty figures, and with them the level-two ratio $(26 - d)(d - 3)/50$ of (50) and its zero at $d = 26$. Away from the locus we test directly the consequence (38) of Proposition 1, that along the directions transverse to a wall the level-n residue is a polynomial of degree at most n. For the wall s_{156} these directions are (39). A polynomial of degree n fitted through $n + 1$ points of a sequence in Table S1 predicts the residue at the further point. On the wall s_{156} at levels one and two, and on the walls t_{4561} and $\tilde{R}$ at level one, the prediction is met to the accuracy of the quadrature, while a fit of degree $n-1$ fails. Thus the degree is exactly n at these points. The partial-wave coefficients (51), and the values $+\frac{1}{56}$, $-\frac{1}{64}$ for the second continuation $q' = q + \varepsilon s_{136}$ of the Veneziano variable, are ratios of coefficients of such degree-one polynomials at the $\mathcal{L}_4$ point of Table S1 displaced to $s_{136} = \pm\frac{1}{6}$. They are exact rationals, and the quadrature reproduces them to its accuracy.

On the momentum slice of Section 3.5, where $s_{abc} = \frac{1}{2}(p_a + p_b + p_c)^2$ for six massless momenta, the coefficient of the double pole at $s_{156} = 0$ is the integral over the square face common to the two coincident facets, with the closed form $B(\varepsilon s_{23}, \varepsilon s_{34})\, B(\varepsilon s_{56}, \varepsilon s_{61})$ of Eq. (54). The closed form agrees with direct two-dimensional quadrature of the face integral over its four cones to nearly forty significant figures. To compare the double-pole coefficient with the full function, we evaluated $K(3,6)$ from the cone form along the line in the slice on which only s_{156} varies, at a sequence of values approaching the pole, and fitted the Laurent expansion. Its leading coefficient reproduces the closed form to about nine figures, the accuracy such a fit allows. The coefficient of the simple pole contains the finite parts of the two six-point facet integrals. These were computed at the pole, at three relative orientations of the triples $\{1,5,6\}$ and $\{2,3,4\}$ with the invariants within each triple held fixed, to about twenty significant figures. They change by about eleven percent from one orientation to the next, whereas the double-pole coefficient is the same at all three. At the two-particle wall $t_{4561} = s_{23}$ of the slice the facet data, evaluated in the same way, are independent of the relative orientation of the coalescing pair to forty figures.

S5 Expansion coefficients

The coefficient $c_w(s)$ is the finite sum (86) of exact rational functions $W_r(s)$, determined by the fan, times constants C_r that do not depend on s. The C_r are integrals, of dimension at most four, of products of logarithms of the cone letters over faces of the unit cube. We compute the W_r once, in exact arithmetic, as rational functions of the fourteen invariants, so that at a rational point they are rational numbers. The C_r were computed with both implementations. In the first, we evaluate each constant by quadrature rules adapted to its face and dimension, twice, at different quadrature orders and working precisions. At weight six the two totals $\sum_r W_r(s) C_r$ agree to between 64 and 76 significant figures at the 280 points of the weight-six data described below, and to 68 at s^*, s_0 and the two further reference points listed with those data (Appendix B.9). In the second, we expand the cone integrands in powers of ε at the kinematic point in question, which leaves integrals of products of logarithms of the cone letters over the unit cube. We evaluate them by tensor-product Gaussian quadrature on a sequence of nested grids, the finest with $(160, 120, 96, 88)$ nodes on the four axes, carrying about one hundred significant figures. The accuracy is limited by the node count, not by round-off, and we determine it by comparison with the coefficients c_w at the weights w for which they are known exactly. At a rational kinematic point the resulting values of c_w on the two finer grids are good to sixty significant figures or more through weight seven. The coarsest grid, of $(96, 80, 72, 64)$ nodes, agrees with the next, of $(128, 96, 80, 72)$ nodes, to between fifty-four and fifty-seven significant figures from weight five on, and these two grids carry the weight-seven and weight-eight values at s_0 and at $\tilde{s}$; the finest grid was used for the comparison form ω', and at weight eight it agrees with the second to sixty-three figures. Through weight six we compared the two implementations directly: at s_0 and $\tilde{s}$ both return $c_0, \ldots, c_6$ as the same rational multiples of the basis elements, (30) and (145). At two further rational points of $\mathcal{C}$ chosen for the purpose, independent evaluations of $c_2, \ldots, c_6$ agree to 48, 46, 36, 33 and 27 significant figures, the accuracy of the less precise evaluation at each weight.

By weight, the inputs are as follows. At weight two the statement $c_2 = q_2\zeta_2$ is the theorem of (89) and involves no numerical input; $q_2(s)$ is the exact sum over the 208 axes. At weight three $q_3(s)$ is exact given the identifications (138) of the two pair constants E_2, E_4 from their numerical values (Appendix B.6). The two implementations' independent symbolic forms of q_3 agree as rational functions. At weight four the seventeen identities (139) are proved by expansion of explicit polynomials and, independently, wall by wall at the 312 flags of the fan (Appendix A.2). The numerical input is the table of coordinates of the cone constants in the nineteen directions $B_{16} \cup \{\tau_1, \tau_2, \tau_3\}$, with the one relation among them. At weight five the rank statement (141) and the 217 dependences are proved by exact expansion, and the numerical input is the lattice of integer relations of height at most nine among the cone constants (Appendix A.3). The values (94), (95) at s^* are reproduced by a second evaluation, in which every weight-five constant is computed symbolically by hyperlogarithm integration and reduced modulo the same lattice of integer relations; the agreement therefore tests the evaluation of the constants and not the lattice. We compared the exact functions q_2, q_3, q_4, Q_5 and R_5 with values of $c_2, \ldots, c_5$ computed directly from the cone form at s^*, s_0, $\tilde{s}$, $\hat{s}$ and further rational points of $\mathcal{C}$, and they agree to the accuracy of those values.

At weight six some of the constants C_r are no longer polylogarithms at roots of unity, and $A(s)$ and $B(s)$ of (96) were determined differently. They are rational functions with the known denominator $\prod_a \kappa_a$ and numerator degree eighteen. We did not seek the numerators among all polynomials of that degree. By (86) c_6 is a combination of the constants C_r with coefficients $W_r(s)$, and at weight six the W_r span a space of rational functions of dimension 543. We took A and B in that space and solved exactly, for each of the two functions, for its 543 rational coordinates from 596 linear conditions. Each condition is the two-term integer relation in $\{\pi^6, \zeta_3^2\}$ satisfied by the value of c_6 at one of 190 rational points of the convergence region, or by a Taylor coefficient of $c_6 \prod_a \kappa_a$ along a rational line through such points. We solved the linear system modulo 96 primes and reconstructed the solution over $\mathbb{Q}$. The two functions reproduce c_6 at 90 further rational points, disjoint from those, to at least 67 significant figures, the accuracy of those values. The 190 points have denominators three and nine, and the 90 further points have denominators three, nine and six, thirty of each. In the same way we give Q_5 and R_5 as exact rational vectors in a 348-element basis of weight-five terms. The basis functions $W_r(s)$ are determined by the fan, and at weight six they are not tabulated; the ancillary files give their values at each listed kinematic point, so that A, B, Q_5 and R_5 are evaluated there by a finite rational computation. The functions A, B, Q_5 and R_5 in this form and the 280 points with their values of c_6 are in the ancillary files (Appendix B.9). We did not determine the weight-seven and weight-eight coefficients as functions of s. We computed their values at s_0 and $\tilde{s}$ on the two coarser grids of the second implementation and identified them as described next.

S6 Identification of constants

Wherever a number is identified in the text as a rational combination of zeta values, the identification was made with an integer-relation algorithm (PSLQ) in a basis fixed beforehand. At weight five the basis is $\{\zeta_5, \zeta_2\zeta_3\}$, at weight six $\{\zeta_6, \zeta_3^2\}$, at weight seven $\{\zeta_7, \zeta_2\zeta_5, \zeta_3\zeta_4\}$, and at weight eight $\{\zeta_8, \zeta_3\zeta_5, \zeta_2\zeta_3^2, \zeta(3,5)\}$, with $\zeta(3,5)$ in the convention (147) of the paper. We state the convention because with the exponents exchanged the sum is $0.0377076729\ldots$, and (101) holds with the first and not the second; the two are related by the stuffle identity $\zeta(3,5) + \zeta(5,3) = \zeta_3\zeta_5 - \zeta_8$. By the height of an integer relation we mean the largest absolute value of the coefficients of the primitive relation. We accept a relation only if (i) it is returned unchanged at several numbers of supplied figures below the accuracy of the input, (ii) its residual on the grid used is at the level of that grid's own error, as calibrated by the exactly known lower weights, and (iii) it continues to reproduce

coefficient	value
$c_7(s_0)$	$-9.149929299019655236652131195676826139299490365137654$
$c_8(s_0)$	$-392.614385840461859967131651673400033138246385815426973$
$c_7(\tilde{s})$	$1868.5573159151915254315437461020681878743341043416646$
$c_8(\tilde{s})$	$-10858.0482604663239738878860798735425661771240949164233$
$c_8(\omega')$	$-792.970650098009420861949959362894972743972179642158750435254$

	grids agree	figures supplied	residual (finer)	residual (coarser)	height	relation
$c_7(s_0)$	54.1	50, 46, 42	3.76×10^{-60}	7.07×10^{-54}	133437637	(30)
$c_8(s_0)$	56.4	52, 48, 44	6.89×10^{-61}	1.45×10^{-54}	14354434752	(98)
$c_7(\tilde{s})$	55.6	51, 47, 43	2.76×10^{-59}	4.85×10^{-53}	12169360122	(146)
$c_8(\tilde{s})$	56.8	52, 48	7.87×10^{-60}	1.85×10^{-53}	79042085321	(146)
$c_8(\omega')$	62.9	58, 60, 62	3.0×10^{-73}	1.1×10^{-60}	239364162391	(101)

Table S3: Inputs and results of the weight-seven and weight-eight identifications. Upper part: the value of each coefficient, computed on the grids of Appendix B.5, truncated two figures short of the number of leading figures on which the two grids used agree. Lower part: that number of figures; the numbers of significant figures supplied to the integer-relation algorithm, at each of which the same relation was returned; the absolute difference between the relation and the value on the finer and on the coarser grid; the largest coefficient of the primitive integer relation among the value and the basis elements of its weight; and where the relation is printed. For the four $K(3,6)$ entries the grids are those of $(96, 80, 72, 64)$ and $(128, 96, 80, 72)$ nodes and the basis is (87) without $\zeta(3,5)$, the coefficient of $\zeta(3,5)$ being zero when it is included in the basis (see text); for $c_8(\tilde{s})$ the relation, of height near 10^{11}, is not resolved from 44 figures. For ω' the grids are those of $(128, 96, 80, 72)$ and $(160, 120, 96, 88)$ nodes and the basis includes $\zeta(3,5)$ with the convention (147).

the value on a finer grid than the one it was found on. The relations for the weight-seven and weight-eight coefficients of $K(3,6)$ at s_0 and at $\tilde{s}$ have integer heights between 10^8 and 10^{11}. Each was found on the coarser of the two grids from about fifty significant figures, is unchanged when a few more or fewer figures are supplied, and reproduces the value on the finer grid to at least sixty significant figures, beyond the fifty-four to fifty-seven figures to which the two grids agree. Table S3 covers these four coefficients and the weight-eight coefficient of the comparison form ω'. It gives each input value, the numbers of figures supplied to the algorithm, the residuals and integer height of the accepted relation, and the equation in which that relation is printed.

We describe the weight-eight searches in more detail, because the claim that $\zeta(3,5)$ is absent from (98) rests on them. At s_0, in the three-element basis of products of single zeta values, the search returns (98) unchanged at 52, 48 and 44 supplied figures. Its residual is 6.89×10^{-61} on the finer grid and 1.45×10^{-54} on the coarser, the latter equal to the difference of the two grids. In the four-element basis with $\zeta(3,5)$ adjoined, the search at the same three numbers of supplied figures returns a different relation at each, with residuals between 10^{-45} and 10^{-37} and heights between 10^7 and 10^{10}, and none of these meets the acceptance criterion. When relations with residual above 10^{-42} are excluded, the search returns (98) itself, with coefficient zero for $\zeta(3,5)$. At $\tilde{s}$ the relation for c_8 in (146) is returned unchanged at 52 and 48 figures, with residuals 7.87×10^{-60} and 1.85×10^{-53} on the two grids. It is not resolved from 44 figures, because its height is close to 10^{11}, and in the four-element basis it is returned with coefficient zero for $\zeta(3,5)$ when relations with residual above 10^{-46} are excluded. In a wider basis with $\zeta_7 \log 2$ and $\mathrm{Li}_8(\frac{1}{2})$ adjoined no stable relation appears at either point, and we state no exclusion height for a relation with $D_8 \neq 0$ beyond what these searches imply. The relation (101) for ω' has height 2.4×10^{11}. It first appears when about fifty-eight significant figures are supplied and reproduces the value to about seventy; in the basis with $\zeta(3,5)$ removed no relation of comparable height exists. For this coefficient we used the finest grid, whose own error at weight eight, calibrated by the weights four to seven, is near 10^{-72}. Both PSLQ and, independently, the LLL algorithm return the relation at 58 to 72 supplied figures, with or without an unrelated sixth constant added to the basis.

The exact statements at weights three to five also take identified constants as input. The two weight-three constants (138) and the small-height relations among the weight-five constants used in Appendix A were found in the same way, the former from values good to about two hundred significant figures. For E_2 an independent evaluation exists in which one intermediate constant is itself identified numerically, while E_4 rests on the two-hundred-figure identification alone. At weight four we determined the coordinates of the fifty-four cone constants in the nineteen directions of Appendix A.2 once, some in closed form and the rest by integer-relation searches. The relation $\tau_1 + \tau_2 = \sum_g r_g\, g$ has coefficients of height at most 91/24. At weight five the 768 relations of height at most nine that generate the lattice of Appendix A.3 were each found at two working precisions and checked against a lower-precision evaluation of the constants independent of the one searched. For the D_4 and seven-point coefficients of Sections 7.1 and 7.5 we used the bases and the acceptance criterion given above. The D_4 coefficients at $(c, c') = (\frac{1}{3}, -\frac{1}{3})$ were computed at s_0 on two grids that agree to at least fifty-four significant figures; at weight eight no relation of height below 3×10^{10} exists at that accuracy, and none below 2×10^{12} in the roughly sixty figures of the finer grid alone. Through weight six we recomputed these D_4 coefficients with the other implementation on its own fan and re-identified them from its own values. The weight-seven relation at this value of (c, c') has height 1.0×10^{6} and residual 2.2×10^{-61} on the finer grid. For the weight-seven coefficient of $K(3,7)$ on the dihedral family, the eighty coordinates identified directly were supplied to the algorithm with between twenty-one and forty significant figures, according to the dimensions of the faces that contribute to each. The three coordinates that are neither zero nor among these eighty follow by linear algebra from the others and from the values of $c_7^{(3,7)}$ at integer points of the family. Each of the three was recovered from four or more independent choices of those points. The weight-four to weight-six values of $K(3,7)$ at the eleven points of Appendix B.8 carry between 29 and 33 significant figures, and the relations found there have heights up to 2.2×10^{9}. The weight-six relation at the dihedral point $(3,-1,-1,0)$, whose height is largest relative to the figures available, is the least secure of them.

S7 Algebraic and path-tracking computations

The statements of Section 6 about the twenty-six solutions rest on exact computations with the scattering equations (106). We write them as polynomial equations by introducing one auxiliary variable λ_J for each of the fourteen non-constant minors, with $\lambda_J \cdot (J) = s_J$ and $\sum_J \lambda_J\, \partial(J)/\partial x_i = 0$. At the point s^* we eliminate all variables but one by a Gröbner-basis computation with the program msolve. This gives a rational univariate parametrization of the solutions whose eliminating polynomial has degree 26 with rational coefficients. It is squarefree, its greatest common divisor with its derivative being constant, and exact real-root isolation by Sturm sequences gives exactly 16 real roots, one of which corresponds to a point of the positive part. At two rational points at which all fourteen exponents in (23) are positive we computed instead the twenty-six-dimensional quotient algebra of the ideal, saturated by the product of the fourteen minors, and in it the matrix of multiplication by the separating element $t = 2a + 3b + 5c + 11d$. We reconstructed its characteristic polynomial over $\mathbb{Q}$ from reductions modulo primes by the Chinese remainder theorem and rational reconstruction. The result is stable after 43 primes and verified against 53 further primes at one point, after 70 and against 26 more at the other. Its coefficients have up to 182 digits at the first point and up to 147 at the second. At both points it has degree 26, is squarefree and irreducible over $\mathbb{Q}$, and has 26 real roots by Sturm sequences. At one of the two points we refined the twenty-six solutions and certified them by interval arithmetic. They lie in twenty-six distinct sign chambers of the fourteen minors, none of them the positive part. The coordinates of these two points are not reproduced here. For the Galois group we used a third point, specified by a different choice of fourteen independent exponents from that of Table S1, with positive integer exponents $(s_{125}, s_{126}, s_{135}, s_{136}, s_{145}, s_{146}, s_{156}, s_{245}, s_{246}, s_{256}, s_{345}, s_{346}, s_{356}, s_{456}) = (59, 73, 61, 59, 67, 77, 26, 25, 67, 62, 82, 80, 25, 14)$ and the remaining six fixed by (12), and computed the eliminating polynomial directly modulo the six primes 1073741783, 1073741789, 1073741827, 1073741831, 1073741833 and 1073741839. Each reduction has degree 26 and is squarefree. It is therefore the reduction of the characteristic-zero eliminant, and its factorization pattern is the cycle type of a Frobenius element. The six patterns, in the order of the primes, are $[4, 6, 16]$, $[5, 21]$, $[1, 3, 8, 14]$, $[1, 1, 3, 6, 6, 9]$, $[1, 1, 2, 5, 17]$ and $[1, 2, 3, 3, 3, 3, 11]$, the list (113).

The forty-three resonance conditions of the A-hypergeometric system quoted in Section 6 come from the matrix A whose columns are the exponent vectors of the monomials of the fourteen nonconstant minors, with one homogenizing row per minor. The cone spanned by the columns has forty-three facets, and the system is resonant when the linear functional defining a facet takes an integer value on the vector of exponents. We evaluated the forty-three functionals in exact arithmetic at generic rational points of $\mathcal{C}$, where none is an integer, and at s_0, where ten are. The Gamma test (116) is likewise exact: at each point of Table S2 and at s_0 and $\tilde{s}$ the values of c_0, q_2, q_3, R_5 and B are rational numbers computed from the exact coefficient functions, with B the function determined in Section 5.3, and the three combinations are nonzero rationals. The irreducibility over $\mathbb{Q}$ of the numerators of $c_0, \ldots, c_6$ over the common denominator $\prod_a \kappa_a$ was decided by exact polynomial factorization.

Three statements about solutions of the scattering equations rest on floating-point computation: the first two on path tracking with the package HomotopyContinuation.jl, certified where stated, and the third on a search for local maxima of S from many starting points. A certified solution is one enclosed in a region proven by interval arithmetic to contain exactly one solution. First, at each of 300 points of positive kinematics for $(3, 6)$, with the fourteen exponents drawn from three distributions on the positive reals, all 26 solutions are certified real and distinct. This is the numerical counterpart of the exact statement at two points above. Second, for the $(3, 7)$ counts of Section 7.3, we tracked all 1272 paths at some thirty random points of the region of positive kinematics of [1], and at each of them between 1188 and 1200 of the solutions are real. We scanned three straight segments $u(t) = (1-t)u_A + tu_B$ between such points and located by bisection, on two of them, the three transitions at which the number of real solutions changes, to brackets of width 4×10^{-4}. At both ends of each bracket all 1272 solutions are certified distinct and the certified real counts differ by two. We then compared the position of each transition with the 721 hyperplanes on which a positive tropical ray coordinate changes sign, and none of the three lies on any of them. The three segments, given by their endpoints u_A, u_B in the space of the twenty-eight exponents, are in the ancillary files. Across the first transition we evaluated $m(3, 7)$, as the sum over the 1272 solutions, at a sequence of points bracketing it, and $K(3, 7)$ from its cone form on the 952-cone fan for $\alpha' = 1/64$ and $1/16$, at 26 points of the segment. Neither shows a feature at the transition beyond the error of the evaluation. Third, the softness sample of Section 3.6 consists of 300 random integer points of $\mathcal{C}$ with coordinates of absolute value at most six. At each we maximized S in the coordinates y from 40 starting points. The largest maximum found is $S_{\max} = -7.585$, at a point with one positive polynomial exponent, and at none of the 300 points did we find more than one local maximum. This last computation is a numerical search and proves nothing about other points of $\mathcal{C}$.

For the proof of Claim 2 we enumerate the fifty-six planar channels of seven massless particles modulo $s_I = s_{\bar{I}}$, the 1260 pairwise compatible triples and, for each triple, the three channels compatible with all three of its members. Exact linear algebra on the on-shell space then shows that each of the 3780 quintuples (79), a triple together with two of its three further channels, is linearly independent. The search for an injective map from the sixteen facets to the fifty-six channels respecting (77) is exhaustive over assignments, pruned facet by facet. We repeated the enumeration of quintuples at eight, nine and ten points and again found none of them linearly dependent. For the linear-reducibility statements we apply Brown's compatibility-graph reduction to the ten polynomials P_t of (22) together with a, b, c, d, in all twenty-four orders of integration. Eight orders end with the alphabet $\{0, -1\}$ and give the sets (105), while the order (a, b, c, d) produces two polynomials quadratic in b at the second stage. We apply the same reduction to the twenty-two non-monomial factors of the chart of $X(3, 7)$ of [3], and separately to the cone polynomial f of Section 7.5 together with its restrictions to the twelve faces entering its regularization. For both sets, in every order of integration some polynomial becomes nonlinear in the variable to be integrated next, and neither set is linearly reducible.

S8 Seven-point and D_4 computations

For $K(3, 7)$ we use the positive parametrization analogous to (20), with seven constant minors, six monomial minors and twenty-two polynomial minors. The normal fan of $P(3, 7)$ has forty-two rays and 693 maximal cones, ninety-eight of them not simplicial. We refine it to 952 unimodular simplicial cones, and the second implementation uses an independent refinement into 944. At a random rational kinematic point the leading coefficient $c_0^{(3,7)}$, computed in exact rational arithmetic, is unchanged under a different refinement of the fan and equals $m(3, 7)$ evaluated as the sum over the 1064 planar $k = 3$ Feynman diagrams of [4]. The cone form, the leading coefficient and the weight-two and weight-three coefficients are then exact as at $(3, 6)$. Every restriction of a cone polynomial to an axis is 1 or $1 + y$, and every two-face restriction is one of the five shapes of (136) or a product of two of them. Hence $c_2^{(3,7)}$ and, given (138), $c_3^{(3,7)}$ are exact rational multiples of ζ_2 and ζ_3 at every point. In this way $c_0^{(3,7)} = 119$, $c_2^{(3,7)}/\zeta_2 = -483$ and $c_3^{(3,7)}/\zeta_3 = 616$ at the dihedral point of (126), and $c_0^{(3,7)} = 462$ at the planar point of (127), are exact cone sums, and both implementations give the same rationals at these and at seven further dihedral points. The weight-four to weight-six values come from the second implementation's expansion, with the accuracy stated in Appendix B.6. The eleven integer points at which they were identified are eight points of the dihedral family of Section 7.5, $(s_{123}, s_{124}, s_{125}, s_{135}) = (1, 0, -1, 0)$, $(2, -1, -1, 1)$, $(3, -1, -1, 0)$, $(1, 1, -2, -1)$, $(1, 0, 0, -1)$, $(2, -1, 0, 0)$, $(1, -2, 1, 2)$ and $(2, 0, -1, -1)$, and three points with cyclic but not dihedral symmetry. At these three points s_{abc} is constant on the five orbits of the cyclic group, with values $(s_{123}, s_{124}, s_{125}, s_{135}, s_{126}) = (1, 1, -1, 0, -1)$, $(1, 1, 0, -1, -1)$ and $(2, 0, -1, 0, -1)$. Of the eleven, the dihedral point $(1, -2, 1, 2)$ lies outside the convergence region, and the values there are those of the continued coefficient functions. The other ten and the planar point lie in $\mathcal{C}$. Values at kinematics of the opposite overall sign follow from these: the coefficient $c_w^{(3,7)}$ is homogeneous in s and changes by the factor $(-1)^w$ under $s \to -s$. At weight seven the ninety-nine coordinates of $c_7^{(3,7)}$ on the dihedral family were identified as stated in Section 7.5 and Appendix B.6.

The D_4 cluster integral $I_{D_4}(c, c')$ of Section 7.1 is the integral (23) with its integrand multiplied by $X^{\varepsilon c} Y^{\varepsilon c'}$. In the coordinates a, b, c, d of the chart (20), whose third coordinate c is not to be confused with the twist exponent c, the two compound cluster variables are $X = c\,(b + d)$ and $Y = a\,(1 + a + c + ab + bc)$. The fan of I_{D_4} is the D_4 cluster fan on the sixteen rays, with fifty unimodular maximal cones, and its cone form is (25) with the two extra letters included. The exponents c, c' enter the $\kappa_{C,i}$ linearly, as the s_{abc} do. At the split point of Table S1 and on the diagonal $c = c' = \gamma$, the coefficients c_0, c_2/ζ_2 and c_3/ζ_3 are exact rational functions of γ, of numerator degrees four, six and seven over a common denominator of degree seven. The zeros of this denominator, $\gamma \in \{-\frac{3}{2}, -\frac{9}{10}, -\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{16}{15}\}$, are the twists at which a channel exponent vanishes. The three functions are in the ancillary files (Appendix B.9). For the weight-four values (118) we evaluated the twenty-seven weight-four cone constants of this fan to at least 128 significant figures, formed c_4 with exact rational weights at 130-digit working precision, and searched for a two-term relation with ζ_4. The residuals of the relation found are 7.2×10^{-130} and 2.9×10^{-128} at $\gamma = \frac{1}{6}$ and $-\frac{1}{4}$, and a continued-fraction expansion of c_4/ζ_4 returns the same fractions. The independent evaluation quoted in Section 7.1 is, at each of the two twists, a fit of finite-ε values of I_{D_4} at the ten values $\varepsilon = \frac{1}{24}, \frac{1}{28}, \ldots, \frac{1}{60}$, with the exact orders through ε^{-1} subtracted. The ε^0 coefficient so obtained is stable to seven figures across the fits used. The twisted member $(c, c') = (\frac{1}{3}, -\frac{1}{3})$ was expanded at s_0 through weight eight on the grids of $(96, 80, 72, 64)$ and $(128, 96, 80, 72)$ nodes. Its leading coefficient there, $807/8$, is again an exact cone sum.

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