Irreducible multiple zeta values of depth three in four-edge modular graph functions

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October 1, 2026

This work is preliminary. Some content remains AI-generated and is still being validated.

Abstract

The promise of string theory is quantum gravity, and its concrete predictions are corrections to Einstein's equations whose one-loop coefficients are integrals of modular graph functions over the shape of the torus the string sweeps out. For a thin torus these functions reduce to Laurent polynomials, whose coefficients are conjectured to be single-valued multiple zeta values and to map into one another under the cosmic Galois group. D'Hoker and Green found only odd zeta values in the melon functions, D'Hoker and Duke in every three-edge function, and for odd zeta values both conjectured properties are automatic. We compute in closed form the Laurent polynomials of all forty-nine functions of the two-vertex four-edge family at weights seven through twelve. Multiple zeta values of depth three occur in this family, and we prove that in any two-vertex four-edge function they can appear in only one Laurent coefficient, fixed by the weight. At each weight these depth-three values span, modulo products, a space of the full dimension allowed by the Broadhurst–Kreimer conjecture. The generators of the cosmic Galois group can only send them to combinations of products of two odd zeta values, and every image lies in the span of the family's own lower-weight coefficients. Through transcendental weight twenty-one, past the reach of the explicit constructions of Dorigoni et al., every coefficient in the family is single-valued.

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1 Introduction

String theory is, before anything else, a proposal for quantum gravity. Fifty years in, the proposal has produced few concrete quantum-gravity predictions. Those it does produce are corrections to Einstein's equations, an infinite series of higher-derivative terms whose coefficients the theory

fixes, with no free parameters. At one loop each of these numbers is a period integral over the shape of the torus the string sweeps out. The integrands are combinations of modular graph functions [1], one for each Feynman graph of a scalar field on the torus. When the torus is long and thin, a modular graph function reduces to a finite Laurent expansion. Its coefficients involve rational numbers, values of the Riemann zeta function at odd integers and, for sufficiently complicated graphs, multiple zeta values that are not polynomials in these [1, 2].

The modular graph functions [1, 3] in the low-energy expansion of the genus-one four-graviton amplitude [4, 5] are $\mathrm{SL}(2,\mathbb{Z})$-invariant solutions of inhomogeneous Laplace equations [1, 6]. The two-vertex graph with exponents $a_1, \ldots, a_\ell$ on its ℓ edges gives

$$C_{a_1,\ldots,a_\ell}(\tau) \;=\; \sum_{\substack{p_1,\ldots,p_\ell \in \Lambda\setminus\{0\} \\ p_1+\cdots+p_\ell=0}} \prod_{r=1}^{\ell} \frac{(\mathrm{Im}\,\tau/\pi)^{a_r}}{|p_r|^{2a_r}}\,, \qquad \Lambda = \mathbb{Z}\tau + \mathbb{Z}\,, \tag{1}$$

of weight $w = \sum_r a_r$ and symmetric in the exponents. We write $C_{\mathbf{a}}$ with $|\mathbf{a}| = w$. Near the cusp a weight-w modular graph function is a Laurent polynomial in $y = \pi\,\mathrm{Im}\,\tau$ up to exponentially small terms [1, 2],

$$C(\tau) \;=\; \sum_{k=1-w}^{w} c_k\, y^k \;+\; O(e^{-2y})\,. \tag{2}$$

Integrated over the fundamental domain, a weight-w function enters the one-loop $D^{2w}R^4$ interaction [7, 8, 9].

The c_k are known in closed form, as polynomials in odd zeta values, for the melon graphs $C_{1,\ldots,1}$ [10] and for every three-edge function $C_{a,b,c}$ [11]. Zerbini [2] gave the Laurent polynomials of the two-vertex functions $C_{2,1,\ldots,1}$ with up to seven edges, the last with one coefficient undetermined, and of several three-vertex functions. Single-valued multiple zeta values of depth three already occur in the two-vertex $d_{1,1,5} = C_{2,1,1,1,1,1}$ and $d_{1,1,6}$ [1, eq. (6.6)]. Dihedral and trihedral graphs are tabulated through total modular weight 12 ($w \le 6$) [12], one coefficient of each $C_{k,1,1,1}$ is known [13], and at modular depth two the Laurent polynomials follow from Poincaré series [14, 15].

Zerbini proved that the coefficients c_k are cyclotomic multiple zeta values and conjectured that they are single-valued in Brown's sense [2, 16]. Motivic multiple zeta values carry derivations D_m, odd $m \ge 3$, which lower the weight by m and generate Cartier's cosmic Galois group [17, 18, 16]. Under this group the constants of a physical theory are expected to map into one another [19, 20]. The Broadhurst–Kreimer conjecture [21] gives the number of irreducible multiple zeta values of each weight and depth, and at depth three the cusp forms of $\mathrm{SL}(2,\mathbb{Z})$ lower it.

In the melon and three-edge families the coefficients are polynomials in odd zeta values, which are single-valued and mapped to such polynomials by every D_m. Hence Zerbini's conjecture and stability under the derivations hold there automatically. Neither family involves multiple zeta values of depth three, so the Broadhurst–Kreimer count and the Galois action have not yet been tested against a complete genus-one family in which such values occur.

In this paper we compute the Laurent polynomials of the two-vertex four-edge modular graph functions, all forty-nine functions of weights seven through twelve. We obtain all 740 coefficients, one for each Laurent power to which the sector decomposition described below contributes, in closed form in the ring of multiple zeta values. Poisson resummation of (1) splits each function into sectors given by signed Mordell–Tornheim sums, which we evaluate in

closed form for arbitrary exponents. The sector in which all four momenta p_r have a nonzero τ-component produces the depth-three values, for instance Brown's $\zeta_{\mathrm{sv}}(5,3,5)$ in the y^{-5} coefficient of $C_{2,2,2,2}$, Eq. (33). We call the resulting table of Laurent polynomials the atlas.

The forty-nine four-edge functions obey three laws, of which the first is the confinement of depth three: for every $\mathbf{a}$ we find

$$c_k(C_{\mathbf{a}}) \in \mathcal{P}_{w-k} = \mathbb{Q}[\zeta_3, \zeta_5, \zeta_7, \dots]_{w-k} \quad (k \neq 3-w)\,, \qquad c_{3-w}(C_{\mathbf{a}}) \in \mathcal{S}_{2w-3}\,, \tag{3}$$

where c_k has transcendental weight $w-k$ and $\mathcal{S}_n \supset \mathcal{P}_n$ is the weight-n part of the algebra generated over $\mathbb{Q}$ by the odd zeta values and Brown's single-valued multiple zeta values of depth three. Hence depth three occurs only in one slot, the deep coefficient c_{3-w} of transcendental weight $2w-3$. At four edges this follows from the sector decomposition (Proposition 1). In the Laplace equations [6] a solution's Laurent coefficients follow from those of its source except at a resonant eigenvalue $k(k-1)$. A transport theorem, proved for the invertible case, carries ring statements such as (3) through the Laplace equations to functions not computed directly, and the resonant functions the theorem cannot reach are covered by the atlas itself (Section 4).

The second law, saturation, concerns the depth-three parts $\bar{c}_{3-w}$ of the deep coefficients modulo products:

$$\dim_{\mathbb{Q}} \operatorname{span}\big\{\, \bar{c}_{3-w}(C_{\mathbf{a}}) \,:\, |\mathbf{a}| = w \,\big\} \;=\; D_{2w-3,\,3}\,, \qquad w = 7, \dots, 12\,. \tag{4}$$

Here $D_{n,3}$ is the conjectural Broadhurst–Kreimer number of depth-three irreducibles of weight n, equal to $1, 2, 2, 4, 5, 6$ at $n = 11, 13, \dots, 21$ by (14), and bounds the left-hand side a priori, since the classes $\bar{c}_{3-w}$ lie in the corresponding depth-three quotient.

The third law is closure under the derivations: for every $\mathbf{a}$, relative to a fixed basis of the irreducible generators,

$$D_m\, c_{3-w}(C_{\mathbf{a}}) \;\in\; \mathcal{A}_{2w-3-m}\,, \qquad m \text{ odd}, \quad 3 \le m \le 2w-5\,, \tag{5}$$

where $\mathcal{A}_n$ is the $\mathbb{Q}$-span of the weight-n Laurent coefficients of the forty-nine functions. Each image $D_m\, c_{3-w}(C_{\mathbf{a}})$ has depth at most two (Section 5.2) and is therefore a rational combination of products of two odd zeta values [16]. We find that every image lies in the span of the family's lower-weight coefficients alone, which is the content of (5).

The twenty-six deep coefficients at $w = 11, 12$ (transcendental weights 19 and 21) arise at total modular weight $2w > 20$, past the range in which the constructions of [22] account for single-valuedness. We find that they all belong to the single-valued ring, as the zeta-generator framework of that reference predicts conditionally.

When $C_{2,2,2,2}$ is integrated over the fundamental domain, its depth-three constant drops out of the divergent and logarithmic parts of the α'^8 coupling integrals. The two undetermined constants of [23], fixed there under a transcendentality assumption, are reproduced by a zero-mode matching that takes the printed ultraviolet zero modes as inputs, hence are consistent modulo that assumption. Uniform transcendentality at order α'^8 then requires the cancellation of two $\ln(2\pi)$ coefficients, rational multiples of $\pi\zeta_7$, (57).

Our results have the following status. (i) Computed exactly, by symbolic algebra: the forty-nine Laurent polynomials, by sector expansion and exact reduction, checked numerically and against [13, 24]; the images $D_m\, c_{3-w}$, which satisfy (5); and, by rational decompositions, the single-valuedness of the twenty-six deep coefficients at $w = 11, 12$. (ii) Proven: confinement of depth three to c_{3-w} for every four-edge exponent vector (Proposition 1); Laplace transport at

non-resonant eigenvalues in Zerbini's class of cusp expansions (Theorem 1). (iii) Conjectural or external input: the Broadhurst–Kreimer dimensions in (4), Zerbini's conjecture as the statement tested, and the assumptions of [23, 13] in the coupling results. (iv) Extracted numerically by integer relations: sixty-one of our formulas for the $\zeta_{\rm sv}(a,b,c)$, each reproduced exactly from Brown's single-valued map by the algebraic construction of Appendix C, which alone gives the other twenty-two; and the independence of the classes in (4) beyond the known relations. (v) Open or not done: why particular irreducibles are absent from particular deep coefficients (an identity of [24] explains one case); uniform transcendentality at order α'^8, for which (57) is only necessary; resonant Laplace systems, coupled [6] or three-vertex above weight eight [25]; depth four in five-edge functions beyond those computed in Appendix F.

Section 2 reviews the background, Section 3 gives the sector decomposition, Section 4 the Laurent polynomials, the first two laws and the single-valuedness of the deep coefficients, Section 5 the third law, and Section 6 the couplings at order α'^8. The appendices contain the proof of the transport theorem, the Broadhurst–Kreimer count, the $\zeta_{\rm sv}(a,b,c)$ formulas, numerical checks, the Laurent data and the five-edge functions.

2 Modular graph functions and single-valued periods

Throughout we set $y = \pi \,\mathrm{Im}\,\tau$, and w denotes the weight in (1).

2.1 Modular graph functions and their Laplace equations

The low-energy expansion of the genus-one four-graviton amplitude of type-II superstring theory [4] involves the *modular graph functions* of D'Hoker, Green, Gürdoğan and Vanhove [1], one for each Feynman graph of a conformal scalar field on the torus. Each is an $\mathrm{SL}(2,\mathbb{Z})$-invariant real-analytic function with the expansion (2) at the cusp. For two edges (1) reduces to the Eisenstein series E_w. We write the invariant Laplacian, which acts diagonally on the monomials of (2), as

$$\Delta = y^2\partial_y^2 + (y^2/\pi^2)\partial_x^2\,, \qquad x = \mathrm{Re}\,\tau\,, \qquad\qquad \Delta\, y^k = k(k-1)\, y^k\,. \tag{6}$$

The two-vertex functions of a given weight with three or more edges obey an inhomogeneous system

$$(\Delta - \lambda_i)C_i = S_i + \sum_l q_{il} C_l\,, \tag{7}$$

with rational λ_i, q_{il} and sources S_i polynomial in functions of lower weight [5, 1, 3, 24]. Basu's Laplace system for the $C_{a,b,c,d}$ [6] covers the four-edge family in principle but is not solved. For an uncoupled equation, (6) and (7) determine $c_k(C_i)$ from the Laurent polynomials of the sources unless $\lambda_i = k(k-1)$, as Theorem 1 makes precise.

For the melon graphs $D_N = C_{1,\dots,1}$, D'Hoker and Green [10] proved that every Laurent coefficient is a polynomial in odd Riemann zeta values with rational coefficients. D'Hoker and Duke [11] gave the Laurent polynomial of every three-edge function $C_{a,b,c}$ in closed form. Gerken [12] tabulated all dihedral and trihedral modular graph forms of total modular weight ($2w$ for a weight-w function) at most 12. The four-edge functions of weights 7 to 12 (total modular weight 14 to 24) lie above this range. D'Hoker and Geiser [13] computed the y^1 coefficient of each $C_{k,1,1,1}$, and Poincaré series yield Laurent polynomials for infinite families of modular depth two [14, 15].

The three-point functions d_{l_1,l_2,l_3} of [4] have weight $l_1+l_2+l_3$, and their Laurent polynomials below weight six are known [4, 5]. Zerbini [2] computed the complete Laurent polynomials at weight six, those of $d_{1,1,5}$ and $d_{1,3,3}$ at weight seven, and all but one coefficient of $d_{1,1,6}$ at weight eight. Beyond these results Laurent polynomials are known only in isolated instances [23]. Zerbini's $d_{1,1,5} = C_{2,1,1,1,1,1}$ and $d_{1,1,6}$, both two-vertex functions, contain irreducible single-valued multiple zeta values of depth three [2, 1].

2.2 Single-valued multiple zeta values

The coefficients c_k we obtain are rational combinations of multiple zeta values (MZVs),

$$\zeta(n_1,\dots,n_r) \;=\; \sum_{k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\cdots k_r^{n_r}}\,, \qquad n_r\geq 2\,, \tag{8}$$

of *weight* $n_1+\cdots+n_r$ and *depth* r, with $\zeta_n=\zeta(n)=\sum_{k\geq 1}k^{-n}$. We use this ascending-index convention throughout, and motivic MZVs [18, 16] for the coaction below.

Brown's single-valued map $\zeta\mapsto\zeta_{\rm sv}$ [16] annihilates ζ_2 (and with it every even zeta value) and doubles the odd ones,

$$\zeta_{\rm sv}(2n)=0\,, \qquad \zeta_{\rm sv}(2k+1)=2\,\zeta_{2k+1}\,, \tag{9}$$

and maps onto the subalgebra of single-valued MZVs. In Brown's model $\mathcal{U}\otimes\mathbb{Q}[f_2]$ of the motivic MZVs, with $\mathcal{U}=\mathbb{Q}\langle f_3,f_5,f_7,\dots\rangle$ the shuffle algebra on one letter per odd degree, the map sends f_2 to zero and acts on words by [16, eq. (7.3)]

$$\mathrm{sv}(w) \;=\; \sum_{uv=w} u \sqcup\!\sqcup \tilde{v}\,, \qquad \mathrm{sv}(f_af_bf_c)=2\,(f_af_bf_c+f_af_cf_b+f_cf_af_b+f_cf_bf_a)\,, \tag{10}$$

with $\tilde{v}$ the reversed word. On two letters (10) is a shuffle, and single-valued MZVs of depth two are therefore polynomials in odd zeta values, for instance $\zeta_{\rm sv}(3,5)=-10\,\zeta_3\zeta_5$. The first single-valued MZV that is not such a polynomial is $\zeta_{\rm sv}(3,5,3)$, at weight 11 [16, 1]. Closed forms for the depth-three values are given in [16, eq. (7.4)] at weights 11 and 13, and in [22] through weight 17; we tabulate them through weight 21 in Appendix C.

Zerbini [2] proved that the Laurent coefficients of modular graph functions are cyclotomic MZVs and conjectured the sharper statement that they are single-valued MZVs. The conjecture is the natural genus-one extension of what the closed string does at tree level, and it is open. At tree level the coefficients of closed-string amplitudes are single-valued MZVs, since these amplitudes are single-valued images of open-string amplitudes [26, 27, 28, 29].

2.3 Equivariant iterated Eisenstein integrals

Brown's class of real-analytic modular forms [30, 31] is generated by iterated integrals of holomorphic Eisenstein series with lower-depth completions whose coefficients are MZVs or further periods, and modular graph forms can be written in terms of such integrals [8, 9, 32]. A two-vertex function with ℓ edges has modular depth at most $\ell-1$, and for a weight-w four-edge function the part of modular depth three is a combination of equivariant iterated Eisenstein integrals of total modular weight $k_1+k_2+k_3=2w$ [32, 22]. At modular depth ≤ 2 these integrals have been constructed at all weights [30, 31, 32]. At modular depth three, Dorigoni et al. [22] construct every integral through total modular weight 20 from zeta generators, with coefficients

that are single-valued MZVs after specialization to modular graph forms. Periods beyond MZVs first enter such integrals at total modular weight $k_1{+}k_2{+}k_3 = 18$ [22], in combinations that drop out under that specialization, and no published construction reaches total modular weight 22.

2.4 Motivic derivations and the Broadhurst–Kreimer count

Motivic MZVs carry a coaction [18, 33, 16] encoding the action of the motivic Galois group of mixed Tate motives over $\mathbb{Z}$, Cartier's *cosmic Galois group* [17]. Its infinitesimal generators are derivations D_m, one for each odd $m \geq 3$,

$$D_m \;=\; (\mathrm{id} \otimes \pi_m) \circ \Delta^{\mathrm{mot}} \;:\; \mathcal{H}_N \longrightarrow \mathcal{H}_{N-m}\,, \qquad m = 3, 5, 7, \ldots\,, \tag{11}$$

where $\mathcal{H}_N$ denotes the motivic MZVs of weight N, Δ^{mot} is Goncharov's coproduct, and π_m projects the weight-m factor onto its coefficient of ζ_m, sending products and ζ_2 to zero. Hence

$$D_m\,\zeta_n = \delta_{mn}\,, \qquad D_m\,\zeta_2^{\,j} = 0\,, \qquad D_m(\xi\eta) = (D_m\xi)\,\eta + \xi\,(D_m\eta)\,, \tag{12}$$

and, for example, $D_5\,\zeta(3,5) = -5\,\zeta_3$ while $D_3\,\zeta(3,5) = 0$. For $m \geq 11$ the MZVs of weight m include depth-three irreducibles besides ζ_m, and π_m then depends on a choice of basis. We fix the depth-three generators (30) of Section 3.3, and every statement about the D_m below is relative to this choice. Moreover, the D_m preserve the subalgebra of single-valued MZVs [16]. Brown proposed, and Panzer and Schnetz observed for the ϕ^4 periods, that the periods of a quantum field theory span a space stable under the coaction [19, 20].

Broadhurst and Kreimer [21] conjectured (see also [34]) that the numbers $D_{n,k}$ of MZVs of weight n and depth k that are irreducible modulo products and lesser depth have the generating function

$$1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)} \;=\; \prod_{n\geq 3}\prod_{k\geq 1}\bigl(1 - x^n y^k\bigr)^{D_{n,k}} \tag{13}$$

(eq. (7) of [21] and eq. (3.5) of the Data Mine [35]). The left-hand side is $G(x,y) = 1 - O(x)\,y + S(x)\,y^2 - S(x)\,y^4$ with $O = x^3/(1{-}x^2)$ counting the odd zeta values and $S = x^{12}/((1{-}x^4)(1{-}x^6))$ counting cusp forms for $\mathrm{SL}(2,\mathbb{Z})$. At the odd weights $n = 11, \ldots, 21$, Möbius inversion of (13) (Appendix B) gives

$$\bigl(D_{11,3}, D_{13,3}, D_{15,3}, D_{17,3}, D_{19,3}, D_{21,3}\bigr) \;=\; \bigl(1,\,2,\,2,\,4,\,5,\,6\bigr)\,, \tag{14}$$

to be compared with the free count $1, 2, 3, 5, 7, 9$ of Lyndon words of length three in the letters of $\mathcal{U}$. Each pair of an odd zeta value and a cusp form whose weights sum to n lowers $D_{n,3}$ by one, the first being (ζ_3, Δ_{12}) at weight 15. The Data Mine tables [35] verify the counting at low weight; the conjecture itself is open.

We list which statements of this section are proved and which are conjectured.

- *Proved:* modular invariance and the Laplace systems (7); Zerbini's theorem (cyclotomic MZVs); the melon theorem [10]; Brown's single-valued projection (10); the motivic coaction and its derivations (11).

- *Conjectured:* Zerbini's single-valued conjecture; the Broadhurst–Kreimer dimension formula; and the cosmic Galois principle that the constants of a physical theory close under the derivations.

3 Sector decomposition of two-vertex lattice sums

Near the cusp we evaluate the lattice sum (1) by Poisson resummation. Its Laurent polynomial then splits into sector sums, labeled by the set S of edges with nonzero winding number, whose coefficients are rational combinations of signed Mordell–Tornheim sums. At four edges the sector with all four winding numbers nonzero contributes triple sums, which we reduce to multiple zeta values of depth at most three for arbitrary exponents.

3.1 The four-edge family

The function (1) has weight $w = a_1 + \cdots + a_\ell$ and is symmetric under permutations of its exponents. The distinct four-edge functions of weight w are thus labeled by the partitions of w into exactly four positive parts. There are $3, 5, 6, 9, 11, 15$ of them at $w = 7, \ldots, 12$, 49 functions in all. We use the weight-eight function $C_{2,2,2,2}$ as the running example.

3.2 Poisson resummation and the sector sum

On the imaginary axis $\tau = i\tau_2$ we write $p_r = m_r\tau + n_r$, with $|p_r|^2 = m_r^2\tau_2^2 + n_r^2$, and call m_r the winding number of edge r. The constraint $\sum_r p_r = 0$ splits into $\sum_r m_r = 0$ and $\sum_r n_r = 0$. We impose the second by integrating $e^{2\pi i u \sum_r n_r}$ over the relative position $u \in [0,1]$ of the two vertices, and for fixed winding vector m we Poisson resum the n_r-sum of every edge with $m_r \neq 0$. Since the Bessel function $K_{a-1/2}$ is elementary, the Fourier transform of $(n^2+y^2)^{-a}$ is a finite sum: for integer $a \geq 1$,

$$\Phi_a(y,\xi) \;:=\; \frac{2\pi^a}{\Gamma(a)} \left(\tfrac{|\xi|}{y}\right)^{a-\frac{1}{2}} K_{a-\frac{1}{2}}(2\pi|\xi|y) \;=\; e^{-2\pi|\xi|y} \sum_{j=0}^{a-1} d_{a,j}\, \pi^{a-j}\, |\xi|^{a-1-j}\, y^{-a-j}\,, \tag{15}$$

with rational $d_{a,j} = (a{-}1{+}j)!\big/[(a{-}1)!\,j!\,(a{-}1{-}j)!\,4^j]$. For an edge with $m_r = 0$ the sum over $n_r \neq 0$, with its factor $(\tau_2/\pi)^a$ from (1), is instead a Bernoulli polynomial,

$$P_a(u) \;=\; (-1)^{a+1} \tfrac{2^{2a}\pi^a}{(2a)!}\, \tau_2^a\, B_{2a}(u)\,. \tag{16}$$

We organize the sum over winding vectors by the sector $S \subseteq \{1, \ldots, \ell\}$, the set of edges with nonzero winding. The constraint $\sum_r m_r = 0$ excludes $|S| = 1$, and

$$C_{\mathbf{a}}(i\tau_2) \;=\; \sum_{|S|\neq 1} T_S(\tau_2) \;+\; O(e^{-2\pi\tau_2})\,, \qquad T_\emptyset = \int_0^1 \prod_r P_{a_r}(u)\,\mathrm{d}u\,. \tag{17}$$

The remainder is the exponentially small part of (2), and the Laurent polynomial is the sum of the T_S. For $|S| \geq 2$ the integrand carries the factor $\exp\big[-2\pi\tau_2\, u \sum_{r\in S} |m_r|\big]$ from (15). Hence the u-integral localizes at $u \to 0$ and, by the symmetry $u \to 1{-}u$, at $u \to 1$. Extending $\int_0^1$ to $\int_0^\infty$ changes T_S only by $O(e^{-2\pi\tau_2})$, and $\int_0^\infty u^p e^{-\lambda u}\mathrm{d}u = p!/\lambda^{p+1}$ evaluates every term:

$$T_S \;=\; 2 \sum_{\substack{m\in(\mathbb{Z}\setminus 0)^S \\ \sum m = 0}} \int_0^\infty \mathrm{d}u \prod_{r\in S} \big(\tfrac{\tau_2}{\pi}\big)^{a_r} \Phi_{a_r}(|m_r|\tau_2, u) \prod_{r\notin S} P_{a_r}(u)\,. \tag{18}$$

Collecting the powers of τ_2, π, u and $|m_r|$ from (15) and (16), we obtain every T_S as a rational combination of the signed Mordell–Tornheim sums

$$W_S(\mathbf{p};q) \;=\; \sum_{\substack{m\in(\mathbb{Z}\setminus 0)^S\\ \sum m=0}} \frac{1}{\prod_r |m_r|^{p_r}\,\big(\sum_r |m_r|\big)^q}\,, \tag{19}$$

with $p_r = a_r + j_r$ and $q-1$ the total degree in u of the term being integrated. Here j_r denotes the summation index of (15) for edge r. We write $W_{|S|}$ when only the size of S matters. In every term the power of τ_2 equals the power of π, which makes each T_S a Laurent polynomial in $y = \pi\tau_2$ with no free powers of π. Since no step above depends on the values of the exponents, the reduction to (19) holds for every $\mathbf{a}$.

For $S = \emptyset$ the Bernoulli integral $T_\emptyset$ is a rational multiple of y^w, the leading term. For $|S| = 2$ the constraint forces $m = (M, -M)$, and $W_2(\mathbf{p};q) = 2^{1-q}\zeta(p_1+p_2+q)$ is a single Riemann zeta value. For $|S| = 3$ one winding number m_s has sign opposite to the other two, and $|m_s| = |m_r| + |m_t|$ with $\{r,t\} = S\setminus s$, hence $\sum_{r'} |m_{r'}| = 2|m_s|$ and

$$W_3(\mathbf{p};q) \;=\; 2^{1-q}\sum_{s\in S} T(p_r, p_t;\, p_s+q)\,, \qquad T(a,b;c) = \sum_{m,n\geq 1} m^{-a}n^{-b}(m+n)^{-c}\,, \tag{20}$$

where $T(a,b;c)$ is the double Tornheim sum. Inserting $1 = (m+n)/(m+n)$ gives the recursion $T(a,b;c) = T(a{-}1,b;c{+}1) + T(a,b{-}1;c{+}1)$. For example, for $C_{1,1,1}$ we find

$$T_\emptyset = \tfrac{2}{945}y^3\,, \qquad \sum_{|S|=2} T_S = \zeta(3) + \tfrac{3}{4}\zeta(5)y^{-2} - \tfrac{3}{2}\zeta(4)y^{-1}\,, \qquad T_{\{1,2,3\}} = +\tfrac{3}{2}\zeta(4)y^{-1}\,, \tag{21}$$

the last from $W_3((1,1,1);1) = \pi^4/60$. The even zeta values cancel, and $C_{1,1,1} = \frac{2}{945}y^3 + \zeta(3) + \frac{3}{4}\zeta(5)y^{-2} + O(e^{-2y})$, that is, $C_{1,1,1} = E_3 + \zeta(3)$ in the D'Hoker–Green normalization [1].

3.3 Reduction to multiple zeta values

The recursion for $T(a,b;c)$ terminates at $a = 0$, where, in the convention of (8),

$$T(0,b;c) \;=\; \zeta(b,c) \;=\; \zeta(b)\zeta(c) - \zeta(b{+}c) - \zeta(c,b) \quad (b\geq 2)\,, \qquad T(0,1;c) \;=\; \sum_{k\geq 1}\frac{H_k}{k^c} - \zeta(c{+}1)\,, \tag{22}$$

with $H_k = \sum_{j\leq k} j^{-1}$. Euler's evaluation of the harmonic sum,

$$\sum_k H_k/k^s \;=\; (1+\tfrac{s}{2})\zeta(s{+}1) - \tfrac{1}{2}\sum_{i=1}^{s-2}\zeta(i{+}1)\zeta(s{-}i)\,, \tag{23}$$

then expresses $T(0,1;c)$ in Riemann zeta values. As a consequence, every W_3 is a rational combination of multiple zeta values of weight $|\mathbf{p}| + q$ and depth at most two.

The powers of y to which a sector contributes follow from (15)–(18): an edge $r \in S$ gives a factor y^{-j_r}, an edge $r \notin S$ a factor y^{a_r} times a polynomial in u of degree $2a_r$, and the integral of u^p a factor $\big(2y\sum_{r\in S}|m_r|\big)^{-p-1}$. For $|S| \geq 2$, with $a_S = \sum_{r\in S} a_r$, a term in which the Bernoulli factors supply the power u^i, where $0 \leq i \leq 2(w - a_S)$, is proportional to

$$y^k\, W_S\big((a_r+j_r)_{r\in S};\, w-k-a_S-|\mathbf{j}|\big)\,, \qquad k \;=\; |S| - 1 + w - 2a_S - i\,, \tag{24}$$

for every choice of the Bessel indices $\mathbf{j} = (j_r)_{r\in S}$, with $|\mathbf{j}| = \sum_r j_r$. The Mordell–Tornheim sum multiplying y^k thus has weight $w-k$, which is the transcendental weight of c_k. In particular, a sector containing every edge has $i=0$ and contributes to the single power $y^{|S|-1-w}$, which is y^{3-w} at four edges. For $C_{2,2,2,2}$ the ranges given by (24), together with $T_\emptyset \propto y^w$, are

$$|S|=0:\ y^8\,,\quad |S|=2:\ y^1\dots y^{-7}\,,\quad |S|=3:\ y^{-2}\dots y^{-6}\,,\quad |S|=4:\ y^{-5}\,, \tag{25}$$

so c_8, c_1, c_{-1}, c_{-7} come from $|S|\le 2$ alone and are rational or single zeta values, c_{-3} comes from $|S|\le 2$ as well (the $|S|=3$ contribution at y^{-3} vanishes because its Bernoulli coefficient is zero), and only c_{-5} receives an $|S|=4$ contribution.

For $|S|=4$ either one winding number m_s has sign opposite to the other three, in which case $|m_s| = \sum_{r\neq s}|m_r|$, or the winding numbers form two pairs of opposite sign with $|m_r|+|m_{r'}| = |m_t|+|m_{t'}| = N$. In either case $\sum_r |m_r|$ is twice $|m_s|$ or twice N, the overall sign of m gives a factor of two, and

$$\begin{aligned} W_4(\mathbf{p};q) &= 2^{1-q}\Big[\sum_{s\in S} T\big(\mathbf{p}_{S\setminus s};\,p_s+q\big) + \sum_{S=A\sqcup B}\sum_{N\ge 2}\frac{\sigma_A(N)\,\sigma_B(N)}{N^q}\Big]\,,\\ \sigma_{\{r,t\}}(N) &= \sum_{\substack{m+n=N\\ m,n\ge 1}}\frac{1}{m^{p_r}n^{p_t}}\,, \end{aligned} \tag{26}$$

where $T(p_1,p_2,p_3;d) = \sum_{m_1,m_2,m_3\ge 1} m_1^{-p_1}m_2^{-p_2}m_3^{-p_3}(m_1+m_2+m_3)^{-d}$ is the triple Mordell–Tornheim sum and $A\sqcup B$ runs over the three splittings of S into pairs. For the convolution σ we use the partial-fraction decomposition

$$\sigma_{\{r,t\}}(N) = \sum_{i\ge 1}\Big[\tbinom{p_r+p_t-i-1}{p_t-1} + \tbinom{p_r+p_t-i-1}{p_r-1}\Big]\frac{H^{(i)}_{N-1}}{N^{\,p_r+p_t-i}}\,,\qquad H^{(i)}_{N-1} = \sum_{0<m<N}\frac{1}{m^i}\,, \tag{27}$$

with $\binom{n}{k}=0$ for $k>n$. The triple sum obeys the recursion $T(p_1,p_2,p_3;d) = \sum_{r=1}^{3} T(\dots,p_r-1,\dots;d+1)$, which terminates at $T(0,p_2,p_3;d) = \sum_N \sigma_{\{2,3\}}(N)\sum_{M>N}M^{-d}$, and after (27) the remaining sums are multiple zeta values by definition,

$$\sum_{N\ge 2}\frac{H^{(i)}_{N-1}H^{(j)}_{N-1}}{N^c} = \zeta(i,j,c)+\zeta(j,i,c)+\zeta(i{+}j,c)\,,\qquad \sum_{2\le N<M}\frac{H^{(i)}_{N-1}}{N^bM^d} = \zeta(i,b,d)\,. \tag{28}$$

Hence every $W_4(\mathbf{p};q)$ is a rational combination of multiple zeta values of weight $|\mathbf{p}|+q$ and depth at most three, and with (20) and (22) the sums W_S of a sector S have depth at most $|S|-1$. In a four-edge function depth three can therefore arise only from $|S|=4$, in the coefficient c_{3-w} determined by (24). At unit exponents we can carry out the reduction by hand with the classical evaluations of $\zeta(1,1,3)$ and $\zeta(2,3)$, which gives

$$W_4((1,1,1,1);1) = 30\zeta_5 - 12\zeta_2\zeta_3\,, \tag{29}$$

as required by the Laurent polynomial of $D_4 = C_{1,1,1,1}$ [5] once the T_S with $|S|\le 3$ are subtracted.

We carry out (17)–(28) for the whole family in exact rational arithmetic. By Gaussian elimination over $\mathbb{Q}$, the resulting multiple zeta values are reduced weight by weight modulo the

regularized double-shuffle relations [36] in depth at most three (the only depths that occur), namely the stuffle and shuffle families and Hoffman's regularization relation, as in [35]. We call the quotient the exact relation ring. Every Laurent coefficient thus has rational coordinates in a basis of this ring formed by products of odd zeta values, powers of ζ_2 and depth-two generators $\zeta(a,b)$, and by the depth-three generators

$$\begin{aligned}
&11:\ \zeta(3,3,5)\,,\\
&13:\ \zeta(3,3,7),\ \zeta(3,5,5)\,,\\
&15:\ \zeta(3,3,9),\ \zeta(3,5,7)\,,\\
&17:\ \zeta(3,3,11),\ \zeta(3,5,9),\ \zeta(5,3,9),\ \zeta(3,7,7)\,,\\
&19:\ \zeta(3,3,13),\ \zeta(3,5,11),\ \zeta(5,3,11),\ \zeta(3,7,9),\ \zeta(5,5,9)\,,\\
&21:\ \zeta(3,3,15),\ \zeta(3,5,13),\ \zeta(5,3,13),\ \zeta(3,7,11),\ \zeta(5,5,11),\ \zeta(7,3,11)\,,
\end{aligned} \tag{30}$$

labeled by transcendental weight. The numbers of generators at these weights, $1,2,2,4,5,6$, are the Broadhurst–Kreimer dimensions (14).

The sector sums, evaluated numerically to at least 290 digits (Appendix D, Table 4), agree with the exact coefficients of the forty-nine functions, and for the thirty-eight functions evaluated directly from (1), the lattice sum minus its Laurent polynomial decays as $e^{-2\pi\tau_2}$. The coefficients used below are those of the exact reduction. Every law stated in Sections 4 and 5 is consequently a statement about derived exact values, not about numerical fits.

4 Laurent polynomials at weights 7 to 12

By the reduction of Section 3, the Laurent coefficients c_k of the forty-nine four-edge functions of weights $7 \le w \le 12$ are elements of the ring of multiple zeta values. Depth three can occur only in the deep coefficient c_{3-w}, of transcendental weight $2w-3$ (Proposition 1).

4.1 The function $C_{2,2,2,2}$

For the weight-eight function $C_{2,2,2,2}$, the simplest member of the family with every exponent at least two, we obtain, in $y = \pi\,\mathrm{Im}\,\tau$,

$$\begin{aligned}
C_{2,2,2,2}(\tau) \;=\; &\tfrac{229}{3\,322\,873\,125}\,y^8 + \tfrac{1}{540}\,\zeta_7\,y - \tfrac{7}{180}\,\zeta_9\,y^{-1} + \tfrac{1}{12}\,\zeta_5^2\,y^{-2} + \tfrac{63}{80}\,\zeta_{11}\,y^{-3}\\
&- \tfrac{21}{4}\,\zeta_5\zeta_7\,y^{-4} + c_{-5}\,y^{-5} - \tfrac{1}{64}\big(810\,\zeta_5\zeta_9 + 567\,\zeta_7^2\big)y^{-6} + \tfrac{6825}{512}\,\zeta_{15}\,y^{-7} + O(e^{-2y})\,.
\end{aligned} \tag{31}$$

All other c_k with $-7 \le k \le 8$ vanish. No coefficient other than c_{-5} contains ζ_3: by (18) the smallest zeta argument in W_2 is $2\min(a_r)+3 = 7$, and the Tornheim recursion for $|S| = 3$ with all $p_r \ge 2$ terminates in $\zeta(b,c)$ with $b+c \ge 7$, whose odd-weight reductions start at ζ_5.

The coefficient c_{-5} of (31) has transcendental weight thirteen and the value

$$c_{-5} \;=\; 15.818917\ldots\,. \tag{32}$$

We find no expression for it as a rational combination of the odd-zeta monomials of that weight, ζ_{13}, $\zeta_3^2\zeta_7$ and $\zeta_3\zeta_5^2$. In terms of single-valued MZVs it has the closed form

$$c_{-5} \;=\; \frac{13803}{800}\,\zeta_{13} \;+\; \frac{153}{10}\,\zeta_3\zeta_5^2 \;+\; \frac{51}{400}\,\zeta_{\mathrm{sv}}(5,3,5)\,, \tag{33}$$

where $\zeta_{\mathrm{sv}}(5,3,5)$ is the depth-three single-valued multiple zeta value of [16, eq. (7.4)],

$$\zeta_{\mathrm{sv}}(5,3,5) \;=\; 2\,\zeta(5,3,5) - 22\,\zeta_5\,\zeta(3,5) - 120\,\zeta_3\zeta_5^2 - 10\,\zeta_5\zeta_8\,, \tag{34}$$

in the ascending-index convention of (8). In ordinary multiple zeta values alone,

$$c_{-5} \;=\; \frac{13803}{800}\,\zeta_{13} - \frac{153}{875}\,\zeta_2^4\zeta_5 - \frac{561}{200}\,\zeta_5\,\zeta(3,5) + \frac{51}{200}\,\zeta(5,3,5)\,. \tag{35}$$

The only ζ_2-dependence of (35), the term in $\zeta_2^4\zeta_5$, is the $-10\zeta_5\zeta_8$ piece of (34), and the irreducible part of c_{-5} is single-valued. Zerbini's single-valued conjecture therefore holds for $C_{2,2,2,2}$, but the D'Hoker–Green odd-zeta-only theorem does not extend past the melons.

Of the two independent depth-three single-valued MZVs at weight thirteen, $\zeta_{\mathrm{sv}}(5,3,5)$ and $\zeta_{\mathrm{sv}}(3,7,3)$, both present in Zerbini's $d_{1,1,6}$, only the first appears in c_{-5}, and the $\zeta_3^2\zeta_7$ coefficient also vanishes. Within the four-edge family the weight-seven functions $C_{3,2,1,1}$ and $C_{4,1,1,1}$ already contain $\zeta_{\mathrm{sv}}(3,3,5)$ in their y^{-4} coefficients (39), (40) below; modulo products it is a rational multiple of the $\zeta_{\mathrm{sv}}(3,5,3)$ of Zerbini's $d_{1,1,5}$, by (30). What $C_{2,2,2,2}$ supplies is the first such coefficient in a four-edge function with every exponent at least two, with the full Laurent polynomial derived in exact arithmetic (Section 3.3).

4.2 Structure of the Laurent polynomials

All forty-nine Laurent polynomials have the same form, (37) below. Recall that the ring of polynomials in odd zeta values is $\mathcal{P} = \mathbb{Q}[\zeta_3, \zeta_5, \zeta_7, \dots]$, with $\mathcal{P}_n$ its weight-n part, and $\mathcal{S}_n$ is the weight-n part of the ring generated over $\mathbb{Q}$ by the odd zeta values and the depth-three single-valued MZVs $\zeta_{\mathrm{sv}}(n_1,n_2,n_3)$ of [16],

$$\mathcal{S}_n \;=\; \mathcal{P}_n \;+\; \sum_{m\le n} \mathcal{P}_{n-m}\cdot\big\langle\, \zeta_{\mathrm{sv}}(n_1,n_2,n_3) \;:\; n_1+n_2+n_3 = m \,\big\rangle_{\mathbb{Q}}\,, \tag{36}$$

with the $\zeta_{\mathrm{sv}}(n_1,n_2,n_3)$ written as combinations of multiple zeta values as in (34). The common form is

$$\begin{aligned} &c_w \in \mathbb{Q}\,, && c_{1-w} = r\,\zeta_{2w-1}\,,\quad c_{2-w} = \sum_{\substack{a+b=2w-2\\ a,b \text{ odd}}} r_{ab}\,\zeta_a\zeta_b\,,\\ &c_k \in \mathcal{P}_{w-k} \quad (k \ne 3-w)\,, && c_{3-w} \in \mathcal{S}_{2w-3}\,, \end{aligned} \tag{37}$$

with rational r and r_{ab}; the second line is Proposition 1 below. At weight seven we obtain

$$\begin{aligned} C_{2,2,2,1}(\tau) \;=\;& \tfrac{2}{2764125}\,y^7 + \tfrac{1}{900}\,\zeta_5\,y^2 + \tfrac{7}{80}\,\zeta_9\,y^{-2} - \tfrac{11}{32}\,\zeta_{11}\,y^{-4} + \tfrac{63}{32}\,\zeta_5\zeta_7\,y^{-5} - \tfrac{825}{512}\,\zeta_{13}\,y^{-6}\\ &+ O(e^{-2y})\,, \end{aligned} \tag{38}$$

$$\begin{aligned} C_{3,2,1,1}(\tau) \;=\;& \tfrac{136}{174139875}\,y^7 + \tfrac{2}{42525}\,\zeta_3\,y^4 + \tfrac{43}{56700}\,\zeta_5\,y^2 - \tfrac{1}{1890}\,\zeta_3^2\,y + \tfrac{137}{15120}\,\zeta_7 + \tfrac{1}{360}\,\zeta_3\zeta_5\,y^{-1}\\ &- \tfrac{11}{480}\,\zeta_9\,y^{-2} + \tfrac{7}{48}\,\zeta_3\zeta_7\,y^{-3}\\ &+ \tfrac{1}{6720}\big(20860\,\zeta_{11} - 735\,\zeta_3^2\zeta_5 - 9450\,\zeta_2\zeta_9\\ &\qquad - 252\,\zeta_2^2\zeta_7 + 120\,\zeta_2^3\zeta_5 - 210\,\zeta(3,3,5)\big)y^{-4}\\ &+ \tfrac{1}{128}\big(27\,\zeta_3\zeta_9 - 75\,\zeta_5\zeta_7\big)y^{-5} + \tfrac{269}{512}\,\zeta_{13}\,y^{-6} + O(e^{-2y})\,, \end{aligned} \tag{39}$$

$$
\begin{aligned}
C_{4,1,1,1}(\tau) \;=\; & \tfrac{566}{638512875}\, y^{7} + \tfrac{1}{4725}\,\zeta_3\, y^{4} - \tfrac{1}{2700}\,\zeta_5\, y^{2} + \tfrac{1}{630}\,\zeta_3^{2}\, y - \tfrac{1}{1008}\,\zeta_7 - \tfrac{1}{120}\,\zeta_3\zeta_5\, y^{-1} \\
& + \tfrac{37}{192}\,\zeta_9\, y^{-2} - \tfrac{7}{16}\,\zeta_3\zeta_7\, y^{-3} \\
& + \tfrac{1}{17920}\big(-71015\,\zeta_{11} + 5880\,\zeta_3^{2}\zeta_5 + 75600\,\zeta_2\zeta_9 \\
& \qquad + 2016\,\zeta_2^{2}\zeta_7 - 960\,\zeta_2^{3}\zeta_5 + 1680\,\zeta(3,3,5)\big) y^{-4} \\
& - \tfrac{1}{128}\big(249\,\zeta_3\zeta_9 + 189\,\zeta_5\zeta_7\big) y^{-5} + \tfrac{2835}{1024}\,\zeta_{13}\, y^{-6} + O(e^{-2y})\,.
\end{aligned}
\tag{40}
$$

All coefficients not shown vanish. In the deep coefficients c_{-4} of $C_{3,2,1,1}$ and $C_{4,1,1,1}$ the weight-eleven analog of (34) combines $\zeta(3,3,5)$ and the powers of ζ_2 into a rational multiple of $\zeta_{\mathrm{sv}}(3,3,5)$ plus odd-zeta products. The remaining forty-five polynomials are in the ancillary files described in Appendix E, which displays five of them.

Our coefficients agree with the literature wherever the two can be compared (Table 8): with the $C_{k,1,1,1}$ coefficients of [13], the only individual coefficients published above Gerken's range [12], and, coefficient by coefficient, with the two weight-seven identities (71) of [24]. To our knowledge the full Laurent polynomials of $C_{k,1,1,1}$ for $k \geq 4$, and of every other atlas function above the Gerken range, appear here for the first time; at weight seven that of $C_{2,2,2,1}$ is fixed up to its constant term by a weight-seven identity of [24] together with the three-edge Laurent polynomials [11].

4.3 Confinement of depth three

The sector count of Section 3.3 and the exact reduction together give the first law, (3).

Proposition 1 (Confinement of depth three)**.** *(i) For every vector* $\mathbf{a}$ *of four positive integer exponents with* $|\mathbf{a}| = w$*, each* $c_k(C_{\mathbf{a}})$ *with* $k \neq 3-w$ *lies in the weight-*$(w-k)$ *part of the ring generated over* $\mathbb{Q}$ *by multiple zeta values of depth at most two, and* $c_{3-w}(C_{\mathbf{a}})$ *is a rational combination of multiple zeta values of weight* $2w-3$ *and depth at most three; in particular, depth three occurs only at transcendental weight* $2w-3$*. (ii) For each of the forty-nine four-edge functions* $C_{\mathbf{a}}$ *of weights* $7 \leq w \leq 12$*, moreover,*

$$
\begin{aligned}
c_k(C_{\mathbf{a}}) \;&\in\; \mathcal{P}_{w-k} \qquad \text{for every } k \neq 3-w\,, \\
c_{3-w}(C_{\mathbf{a}}) \;&\in\; \mathcal{S}_{2w-3}\,,
\end{aligned}
\tag{41}
$$

and the depth-three part of c_{3-w} *is nonzero for every* $\mathbf{a}$ *except* $(2,2,2,1)$*.*

Proof. (i) By (24), the sector $S = \{1,2,3,4\}$, which has no Bernoulli factor, contributes only to the power y^{3-w}, through rational multiples of $W_4(\mathbf{a}+\mathbf{j};\, w-3-|\mathbf{j}|)$ of weight $2w-3$, for every $\mathbf{j}$ and every four-entry $\mathbf{a}$. The sectors with $|S| \leq 3$ give depth at most two, each W_S having depth at most $|S|-1$ (Section 3.3). Hence depth three enters only through W_4, at y^{3-w} and weight $2w-3$. (ii) We read (41) off the exact reduction of all 740 coefficients. The 691 with $k \neq 3-w$ are odd-zeta polynomials, and the 49 deep coefficients are rewritten in $\mathcal{S}_{2w-3}$ with the formulas of Appendix C by exact elimination over $\mathbb{Q}$ (Section 4.5 treats transcendental weights 19 and 21 in detail). □

At four edges the confinement of depth three to that single slot follows from the sector expansion; that the other slots are free of even zeta values and of depth-two irreducibles is established function by function by the exact derivation. The nearest statement in print is that of Dorigoni et al. [22], who localize the new periods of their depth-three basis functions $F^{+(s)}$

at the single per-eigenvalue power y^{1-s}. The law here is of a different kind: a single-slot y^{3-w} statement for the functions $C_{\mathbf{a}}$ themselves, one slot per function across the whole family, rather than per eigenvalue for a basis. In Figure 1 we display all 740 coefficients aligned on $k = 3 - w$; at five edges the same sector count allows depth three in several coefficients and depth four in one (Appendix F).

4.4 Laplace transport and resonant eigenvalues

Matching power-behaved terms in the inhomogeneous Laplace equations has been used to fix Laurent data since [5, 1] (see also [6, 22]). The following theorem makes that step rigorous, with uniqueness and a criterion for coupled systems, in the cusp-expansion class of Zerbini's theorem [2], which contains every modular graph function, and so carries ring statements such as (3), once the sources' Laurent polynomials are known, to functions that we do not compute directly.

Theorem 1 (Laplace transport)**.** *Let $\lambda \notin \{k(k-1) : k \in \mathbb{Z}\}$ and let S be a finite $\mathbb{Q}$-combination of at most three-fold products of functions with Zerbini-type cusp expansions (Laurent-polynomial mode shells with exponents in $[1-w, w]$ and polynomially bounded coefficients). If C is a moderate-growth solution of $(\Delta - \lambda)C = S$ in the same class, then its Laurent coefficients are*

$$c_k \;=\; \frac{s_k}{k(k-1) - \lambda}\,, \tag{42}$$

where $\sum_k s_k y^k$ is the product of the sources' Laurent polynomials; the cross-mode parts of the source contribute nothing to any c_k, and no homogeneous ambiguity survives: any two such solutions have identical Laurent polynomials.

The proof (Appendix A) rests on the zero-mode equation $y^2u'' - \lambda u = E$ with exponentially small E: it has an exponentially small particular solution, and its homogeneous solutions y^α, $y^{1-\alpha}$ with $\alpha(\alpha-1) = \lambda$ have non-integer exponents, which the class excludes. Theorem 1 is proved exactly for the invertible case, and the family splits cleanly along its hypothesis. For a single equation the hypothesis is $\lambda \notin \{k(k-1) : k \in \mathbb{Z}\}$; for a coupled system with constant rational couplings q_{il},

$$(\Delta - \lambda_i)C_i = S_i + \sum_l q_{il}C_l\,, \qquad M_k = \operatorname{diag}\big(k(k-1) - \lambda_i\big) - (q_{il})\,, \tag{43}$$

it is the invertibility of every M_k. The Laurent data then follows power by power, and source coefficients in a $\mathbb{Q}$-subalgebra force every coefficient c_k of the solution into the same subalgebra. The Kleinschmidt–Verschinin tetrahedral systems [25] of weights six to eight satisfy the hypothesis, with eigenvalues $-6, -6, -4, -4, -10$, whereas at higher weight the tetrahedral spectra also contain resonant values (Appendix A).

At a resonant eigenvalue $\lambda = k(k-1)$ the coefficients c_k and c_{1-k} drop out of (42), which then imposes $s_k = s_{1-k} = 0$ and determines every other coefficient. In Basu's system for the four-edge functions [6] the diagonal eigenvalue of $C_{\mathbf{a}}$ is

$$\lambda_{\mathbf{a}} \;=\; \sum_r a_r(a_r - 1)\,, \qquad \text{resonant when } \lambda_{\mathbf{a}} = k(k-1) \text{ for some } 1 - w \le k \le w\,, \tag{44}$$

and by (44) twelve of the forty-six tuples $\mathbf{a}$ at weights 8–12 have a resonant eigenvalue. These are the seven tuples $(3,3,1,1)$, $(3,2,2,2)$, $(4,3,2,1)$, $(3,3,3,2)$, $(4,4,3,1)$, $(5,3,2,2)$, $(6,4,1,1)$

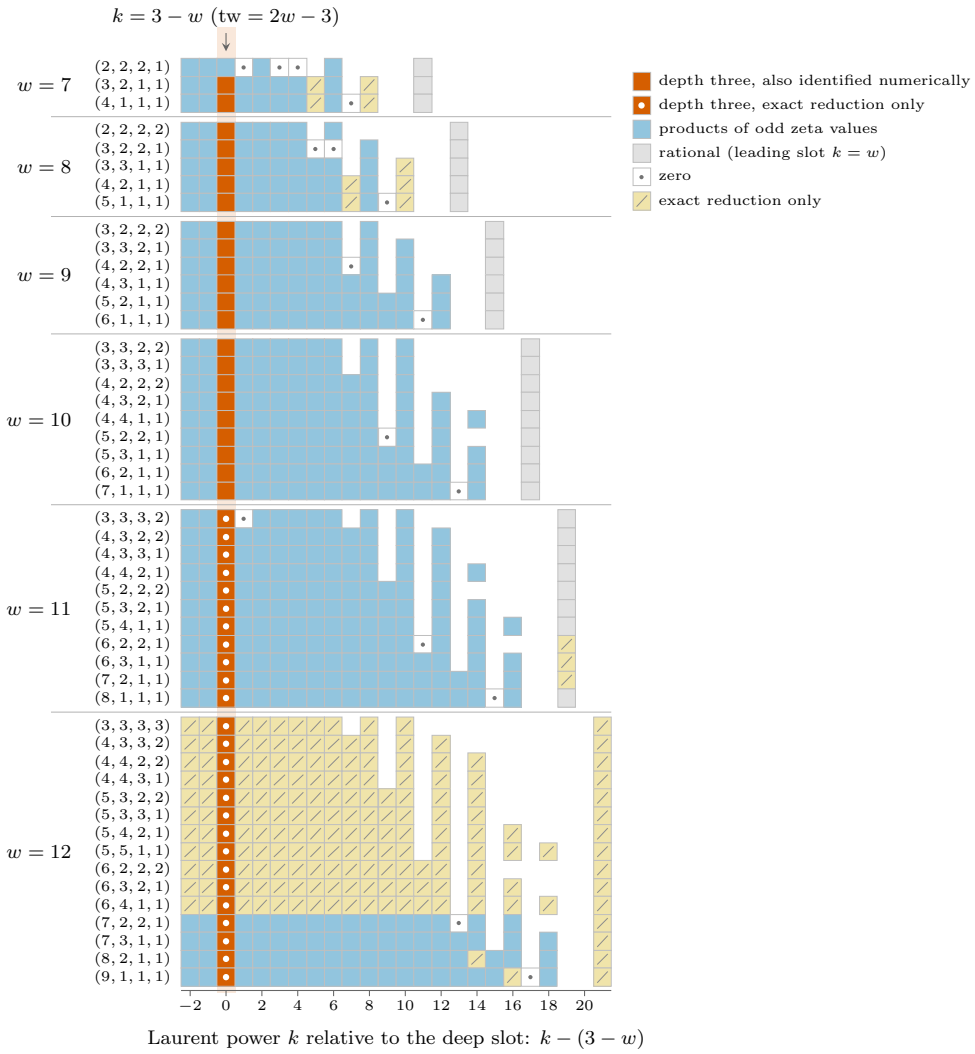

Figure 1: Every depth-three coefficient of the forty-nine four-edge functions sits in the single marked column $k = 3 - w$. Rows are the functions $C_{\mathbf{a}}$ of weights 7–12 and cells the Laurent powers y^k to which at least one term of the sector sums (18) contributes by the power count (24) (740 coefficients in all), with rows aligned on $k - (3 - w)$ so that c_{3-w}, of transcendental weight $2w - 3$, falls in the marked column. Shades give the type of coefficient: rational (leading power $k = w$), zero, polynomial in odd zeta values, or carrying depth three; forty-eight of the forty-nine deep slots carry depth-three constants (all by the exact reduction, twenty-two of them also identified numerically and twenty-six by the reduction alone, $C_{8,1,1,1}$ at weight 19 among them at 470 digits against its 700-digit evaluation), and the deep coefficient of $C_{2,2,2,1}$ is a polynomial in odd zeta values. A white dot on a depth-three cell, and elsewhere a pale cell with a diagonal stroke, mark coefficients obtained by the exact reduction alone and not also identified numerically (Appendix D). Gaps are powers to which no sector term contributes, and a cell marked zero is one whose contributions cancel exactly.

and the entire line $C_{k,1,1,1}$, whose eigenvalue is $k(k-1)$ identically. (For tuples with an exponent ≤ 2, all of the forty-six but $(3,3,3,3)$, the eigenvalue formula itself can shift, as Basu notes, and whether the coupled M_k is then singular is certain only for the $C_{k,1,1,1}$ line below.) We form no coupled M_k for these resonant tuples; since we compute all forty-six functions directly, no result here depends on the classification, which only identifies the coefficients a Laplace system might leave undetermined.

For $C_{k,1,1,1}$ the obstruction is structural as well as spectral: no equation of the fixed-weight family produces $C_{k,1,1,1}$ on its right-hand side, so its column in M_k has no off-diagonal entries and the two slots float in every formulation; independent data is mathematically required, not merely convenient. We compute the resonant functions directly, which supplies in particular every coefficient a Laplace system leaves undetermined, and for the $C_{k,1,1,1}$ line, to our knowledge, those atlas Laurent polynomials are the only computation of that data. In Basu's weight-six consistency equation [6], with $\lambda = 12$, the conditions $s_4 = s_{-3} = 0$ hold, (42) gives every other coefficient of the solution, and its coefficient

$$c_{-3} \;=\; -\zeta_3^3/6 + 5\zeta_9/288 \tag{45}$$

follows only from the direct computation.

4.5 Single-valuedness of the deep coefficients

By Proposition 1 the coefficients with $k \neq 3-w$ are polynomials in odd zeta values, single-valued since $\zeta_{\rm sv}(2n+1) = 2\zeta_{2n+1}$. Hence Zerbini's conjecture [2] is at issue only for c_{3-w}, and each of the forty-eight deep coefficients with a depth-three part is potentially a test of it. At low weight the constructions in the literature already account for single-valuedness, and agreement there is a consistency check, not a test of the conjecture.

The equivariant iterated Eisenstein integrals that carry the depth-three part of c_{3-w} involve no period beyond multiple zeta values at $w = 7, 8$, and at $w = 9, 10$ such periods drop out of the constructions of [22] specialized to modular graph forms (Section 2.3). Through $w = 10$ these published constructions therefore account for a single-valued c_{3-w}, granted that the coefficients with which the integrals enter a modular graph function are themselves single-valued, which the authors of [22] note is not yet established in general. At $w = 11, 12$ (modular weights $22, 24$, transcendental weights $19, 21$) nothing in print forces the answer either way: the weight-19 and weight-21 slots are the first in the family not covered by an explicit published construction. The zeta-generator framework of [22], which deduces the products and higher-depth multiple zeta values of modular graph forms from the odd zeta values, does imply single-valued coefficients at these weights as well, but conditionally, on a group-like structure of its generating series that is established through modular weight 20 and conjectural beyond it.

For the eleven deep coefficients of transcendental weight 19 and the fifteen of weight 21, which the reduction gives in a basis containing powers of ζ_2, we determine directly whether they lie in $\mathcal{S}_{19}$ and $\mathcal{S}_{21}$. Such a coefficient lies in $\mathcal{S}_{2w-3}$ when it equals a rational combination of the weight-$(2w-3)$ generators of (36): the $\zeta_{\rm sv}$ values of that weight, written in the same basis, the odd-zeta monomials, and the products of lower-weight $\zeta_{\rm sv}$ values with odd-zeta monomials. The $\zeta_{\rm sv}$ values absorb the ζ_2-dependent terms as $\zeta_{\rm sv}(5,3,5)$ does in (33). We decide membership by Gaussian elimination over $\mathbb{Q}$: the coefficient lies in $\mathcal{S}_{2w-3}$ when its remainder vanishes identically.

With the extracted weight-19 and weight-21 $\zeta_{\rm sv}$ expressions of Appendix C, each also recovered there algebraically, we find that all eleven coefficients lie in $\mathcal{S}_{19}$ and all fifteen in $\mathcal{S}_{21}$, each by an exact rational decomposition given in the ancillary files. At weight 19 each decomposition

so obtained has eleven terms, all $\zeta_{\rm sv}$ values; at weight 21 each has fourteen, eleven $\zeta_{\rm sv}$ values and three monomials in odd zeta values. Such a decomposition is not unique, the $\zeta_{\rm sv}$ values of one weight being rationally related; Eq. (69) is the one obtained for $C_{8,1,1,1}$. The numerical re-evaluations of Appendix D serve only as checks of these decompositions. In this family, Zerbini's conjecture holds at every slot determined through weight 21, all eleven weight-19 slots among them, which lie past the modular weights that explicit published constructions reach (Section 2.3) and are therefore the first slots whose outcome no computation in print decides.

4.6 Ranks modulo products

The saturation law (4) concerns the rank of the deep coefficients modulo products. We write $\overline{\mathcal{S}}_n$ for $\mathcal{S}_n$ modulo $\mathcal{P}_n$ and products, spanned by the classes of the $\zeta_{\rm sv}(n_1,n_2,n_3)$ with $n_1{+}n_2{+}n_3=n$. In the exact relation ring these classes span a space of the Broadhurst–Kreimer dimension $D_{n,3}$ of (14), with basis the classes of the generators (30). The rank of the weight-w deep coefficients in $\overline{\mathcal{S}}_{2w-3}$, which we compute by exact elimination over $\mathbb{Q}$ on their (30) coordinates, is therefore at most $D_{2w-3,3}$. At each of the six weights the rank equals the conjectural Broadhurst–Kreimer depth-three dimension, derived here from the generating function rather than quoted (Appendix B):

$$\operatorname{rank}_{\mathbb{Q}}\big\{\,\overline{c_{3-w}(C_{\mathbf{a}})}\;:\;|\mathbf{a}|=w\,\big\}\;=\;D_{2w-3,3}\,,\qquad \begin{array}{c|cccccc} 2w-3 & 11 & 13 & 15 & 17 & 19 & 21\\ \hline D_{2w-3,3} & 1 & 2 & 2 & 4 & 5 & 6 \end{array} \tag{46}$$

from two, five, six, seven, eleven and fifteen of the functions respectively.[1] That the $D_{2w-3,3}$ classes of the $\zeta_{\rm sv}(n_1,n_2,n_3)$ are themselves independent, so that no relation outside the ring lowers these ranks, rests on integer-relation searches among their numerical values, which, with independence meaning no relation found to coefficient height 10^{15} in the ζ_2-free sector, give quotient rank $1,2,2,4,5,6$.

The Broadhurst–Kreimer dimensions themselves are conjectural, and the ranks are a priori bounded by them; the measurement finds equality at all six weights, which tests the family against the conjecture and does not prove the conjecture. What the measurement adds is that the string's coefficients reach the bound six times in a row, with no unexpected rational relation among them.

4.7 Irreducible parts and selection rules

Table 1 lists, for every function of weights seven to nine with depth-three content, the class of its deep coefficient in $\overline{\mathcal{S}}_{2w-3}$, which the ranks (46) do not determine. At transcendental weight 15 three of the six rows lie on one line and three lie off it, among them the $C_{k,1,1,1}$ member $(6,1,1,1)$, with coordinates proportional to $(-14,15)$. At transcendental weight 11 the deep coefficient of $C_{2,2,2,1}$ contains no irreducible, by (38), although Zerbini's $d_{1,1,5}=C_{2,1,1,1,1,1}$, a six-edge two-vertex function of the same weight, contains the weight-eleven irreducible in the same coefficient. The exception follows from an identity that D'Hoker and Kaidi [24] prove, the first identity in (71): a combination of $C_{2,2,2,1}$ with three-edge functions and Eisenstein series is independent of τ. Hence every non-constant Laurent coefficient of $C_{2,2,2,1}$ is a polynomial in odd zeta values, and none contains a depth-three value. The companion identity, the second in

[1] At transcendental weight 17, 7 of the 9 weight-ten functions already attain the maximal rank 4; $C_{3,3,2,2}$ and $C_{6,2,1,1}$ also contain depth-three values.

tw	**a**	irreducible part of c_{3-w}	line
11	$(3,2,1,1)$	$-\frac{3}{192}\,\zeta_{\rm sv}(3,3,5)$	
11	$(4,1,1,1)$	$+\frac{24}{512}\,\zeta_{\rm sv}(3,3,5)$	
13	$(2,2,2,2)$	$-\frac{204}{800}\,\zeta_{\rm sv}(3,5,5)$	
13	$(3,2,2,1)$	$+\frac{756}{19200}\,\zeta_{\rm sv}(3,5,5)$	
13	$(3,3,1,1)$	$\frac{1}{9600}\big(-225\,\zeta_{\rm sv}(3,3,7)-108\,\zeta_{\rm sv}(3,5,5)\big)$	
13	$(4,2,1,1)$	$\frac{1}{76800}\big(600\,\zeta_{\rm sv}(3,3,7)-624\,\zeta_{\rm sv}(3,5,5)\big)$	
13	$(5,1,1,1)$	$\frac{1}{6400}\big(75\,\zeta_{\rm sv}(3,3,7)-84\,\zeta_{\rm sv}(3,5,5)\big)$	
15	$(3,2,2,2)$	$\frac{1}{5120}\big(-168\,\zeta_{\rm sv}(3,3,9)+108\,\zeta_{\rm sv}(3,5,7)\big)$	on $(-14,9)$
15	$(3,3,2,1)$	$\frac{1}{61440}\big(672\,\zeta_{\rm sv}(3,3,9)-432\,\zeta_{\rm sv}(3,5,7)\big)$	on $(-14,9)$
15	$(4,2,2,1)$	$\frac{1}{30720}\big(224\,\zeta_{\rm sv}(3,3,9)-144\,\zeta_{\rm sv}(3,5,7)\big)$	on $(-14,9)$
15	$(4,3,1,1)$	$\frac{1}{15360}\big(14\,\zeta_{\rm sv}(3,3,9)-69\,\zeta_{\rm sv}(3,5,7)\big)$	off
15	$(5,2,1,1)$	$\frac{1}{122880}\big(-728\,\zeta_{\rm sv}(3,3,9)+828\,\zeta_{\rm sv}(3,5,7)\big)$	off
15	$(6,1,1,1)$	$\frac{1}{40960}\big(-280\,\zeta_{\rm sv}(3,3,9)+300\,\zeta_{\rm sv}(3,5,7)\big)$	off

Table 1: At transcendental weight 15 three of the six functions lie on one rational line. For each four-edge function $C_{\mathbf{a}}$ of weight $w=7,8,9$ with depth-three content, the depth-three part of its deep coefficient c_{3-w} ($y=\pi\,\mathrm{Im}\,\tau$; transcendental weight $\mathrm{tw}=2w-3$) is given by its rational coordinates on the classes of the single-valued MZVs $\zeta_{\rm sv}(n_1,n_2,n_3)$ of Appendix C modulo products, with the fractions left unreduced as computed. The coordinates depend on that basis; the last column does not, since it indicates only whether a row of transcendental weight 15 is proportional to $-14\,\zeta_{\rm sv}(3,3,9)+9\,\zeta_{\rm sv}(3,5,7)$. The $(2,2,2,2)$ row is the depth-three part of c_{-5} in (33), rewritten in this basis. At transcendental weight 11, $C_{2,2,2,1}$ (not shown) has no irreducible part at all.

(71), likewise forces the depth-three parts of $C_{4,1,1,1}$ and $C_{3,2,1,1}$ into the ratio $3:-1$, that of the $(4,1,1,1)$ and $(3,2,1,1)$ rows of Table 1.

We refer to such patterns as selection rules, and at higher weight we read them off Appendix E in the same way. At transcendental weight 19, for instance, the deep coefficient of $C_{8,1,1,1}$ has nonzero coordinates on all five weight-19 depth-three generators (30),

$$\begin{aligned} c_{-8}(C_{8,1,1,1}) \;=\; & -\tfrac{79497}{191426560}\,\zeta(3,3,13)+\tfrac{369}{382853120}\,\zeta(3,5,11)-\tfrac{20187}{47856640}\,\zeta(5,3,11) \\ & +\tfrac{81963}{191426560}\,\zeta(3,7,9)+\tfrac{164097}{382853120}\,\zeta(5,5,9) \\ & +\big(\text{a rational multiple of }\zeta_{19}\big) \\ & +\big(\text{products of zeta values of depth at most two}\big)\,, \end{aligned} \tag{47}$$

and the complete expression is (76). We have not examined whether identities of the kind (71) at higher weight account for the weight-15 line. Apart from the weight-eleven exception explained above, the mechanism behind these selection rules is open; Sections 5 and 7 discuss it.

5 Motivic derivations of the depth-three coefficients

The derivations D_m of (11) send each depth-three value $\zeta_{\rm sv}(a,b,c)$ of the family to single-valued constants of lower weight. We write $n=a+b+c\in\{11,13,\dots,21\}$ for the weight of such a value, and the deep coefficient $c_{3-w}(C_{\mathbf{a}})$ of a weight-w function contains values of weight $n=2w-3$.

5.1 Derivation images of the single-valued multiple zeta values

A value $\zeta_{\mathrm{sv}}(a,b,c)$ of weight n has one image $D_m\,\zeta_{\mathrm{sv}}(a,b,c)$ for each odd m with $3 \le m \le n-2$, an element of the single-valued ring of weight $n-m$ [16]. We compute each image by applying D_m term by term to a formula for $\zeta_{\mathrm{sv}}(a,b,c)$ in ordinary multiple zeta values and writing the result as a rational vector in the basis of the exact relation ring. Here D_m is defined on all motivic multiple zeta values through the Goncharov coproduct [18] and normalized by (12). We therefore need formulas for all 83 values $\zeta_{\mathrm{sv}}(a,b,c)$ with odd $a,b,c \ge 3$ at weights 11–21. Appendix C lists them: we extracted 61 numerically by integer relations, and the algebraic construction described there reproduces these exactly and alone gives the other 22.

For example, the weight-11 value $\zeta_{\mathrm{sv}}(3,3,5)$, whose rational multiples are the irreducible parts of $c_{-4}(C_{3,2,1,1})$ and of $c_{-4}(C_{4,1,1,1})$ (Table 1), has

$$D_3\,\zeta_{\mathrm{sv}}(3,3,5) = -10\,\zeta_3\zeta_5\,, \qquad D_5\,\zeta_{\mathrm{sv}}(3,3,5) = -10\,\zeta_3^2\,, \qquad D_7\,\zeta_{\mathrm{sv}}(3,3,5) = D_9\,\zeta_{\mathrm{sv}}(3,3,5) = 0\,. \tag{48}$$

The last two images vanish because the single-valued ring is zero in weights 4 and 2. The two nonzero ones are rational multiples of four-edge coefficients of the same transcendental weights, for instance of $c_1(C_{6,1,1,1}) = -\zeta_3\zeta_5/12600$ and $c_1(C_{4,1,1,1}) = \zeta_3^2/630$ [13].

5.2 Closure of the family

The form of the images $D_m\,\zeta_{\mathrm{sv}}(a,b,c)$ follows from Brown's model (10), in which a derivation removes one letter and the image of a depth-three value therefore has at most two letters. Since a single-valued combination of at most two letters is a polynomial in odd zeta values [16], the image at the even weight $n-m$ has the form

$$D_m\,\zeta_{\mathrm{sv}}(a,b,c) \;=\; \sum_{\substack{r+s=n-m \\ r,\,s \text{ odd}}} q_{rs}\,\zeta_r\zeta_s \tag{49}$$

with rational q_{rs}.

Let $\mathcal{C}_v$ denote the set of Laurent coefficients $c_k(C_{\mathbf{b}})$ of transcendental weight v over the forty-nine functions, and $\mathcal{A}_v = \mathrm{span}_{\mathbb{Q}}\,\mathcal{C}_v$ the space entering (5). At every even weight $v = n-m$ that occurs, the elements of $\mathcal{C}_v$, the c_{2-w} of (37) included, are rational combinations of products of two odd zeta values, and we find that in this family they span the whole space of such products. The form (49) then gives

$$D_m\,\zeta_{\mathrm{sv}}(a,b,c) \;\in\; \mathcal{A}_{n-m} \qquad \text{for all odd } m \text{ with } 3 \le m \le n-2, \tag{50}$$

for each of the 83 values.

We have computed the 627 images of the 83 values in rational arithmetic, relative to the fixed basis of irreducible generators that defines π_m for $m \ge 11$ (Section 2.4). Each of the 382 nonzero images (256 from the extracted and 126 from the derived formulas) is a rational combination of the coefficient values in $\mathcal{C}_{n-m}$ alone (zero and nonzero counts by weight in Table 7).

The deep coefficient $c_{3-w}(C_{\mathbf{a}})$ is a rational combination of values $\zeta_{\mathrm{sv}}(a,b,c)$ of weight $2w-3$ and of products of odd zeta values, on which D_m acts by the Leibniz rule. In the forty-nine deep coefficients these products have at most three factors. By (12) their images are again rational combinations of products of two odd zeta values, of weight $2w-3-m$, and lie in $\mathcal{A}_{2w-3-m}$ because the family's even-weight coefficients span all such products. Hence (50) gives

$D_m\, c_{3-w}(C_{\mathbf{a}}) \in \mathcal{A}_{2w-3-m}$ for every function of the family and every odd m with $3 \le m \le 2w-5$, which is (5). This closure would be automatic if the $\mathbb{Q}$-span of all Laurent coefficients of the family were a comodule under the derivations. A span computation on a finite family establishes membership, not structure: the 627 images are consistent with a comodule structure; they do not demonstrate one.

The formulas of Appendix C express the $\zeta_{\mathrm{sv}}(a,b,c)$ through ordinary multiple zeta values and powers of ζ_2, and an incorrect formula would in general leave π^2 in its images, which a single-valued image cannot contain [16]. The 627 images are free of π^2 as computed, with no single-valued projection applied. The agreement is what the structure requires; it checks the 83 formulas and the implementation of the derivations rather than adding a property of the family.

5.3 Coaction structures and the selection rules

Two structures in the literature could restrict which depth-three constants appear in the Laurent polynomial of an individual graph, beyond what the closure (50) implies. The first is the motivic coaction on the generating series of modular graph forms of Kleinschmidt, Porkert and Schlotterer [37]. By (10), whose right-hand side is twice the reversed word modulo shuffles, the single-valued map is twice the identity on three-letter words modulo shuffles. Hence the single-valued images of the depth-three classes of that series span the entire depth-three quotient at every weight. A principle that admits the whole space cannot cut it, so coaction membership alone yields no selection rule; any graph-by-graph prediction would need the per-graph coefficient data of the generating series, which is not available at these weights.

The second structure is the coaction principle of Brown and of Panzer and Schnetz [19, 20], that the constants of a physical theory close under the motivic Galois action. Our results are consistent with this principle, and at depth three it imposes nothing beyond the closure (50). Indeed, each cusp form paired with an odd zeta value of complementary weight lowers the dimension of the depth-three quotient by one (cusp-form censorship), leaving the Broadhurst–Kreimer dimension at each weight. Since by (46) the deep coefficients of the family already span this whole quotient, the principle cannot single out directions inside it. Neither structure explains the selection rules of Table 1 at transcendental weights 11 and 15. The first of them is accounted for by the weight-seven identity (71) of [24], which lies outside both structures; the weight-15 rule and the patterns read off Appendix E at higher weight sit below the resolution of coaction and censorship, and remain the main open question of the atlas (Section 7).

6 The couplings at order α'^8

At order α'^8 (the $D^{16}R^4$ terms) the genus-one four-graviton amplitude involves two independent weight-eight combinations of modular graph functions, the coefficients of $\sigma_2\sigma_3^2$ and σ_2^4, with $\sigma_n \propto s_{12}^n + s_{13}^n + s_{23}^n$ the power sums of the Mandelstam invariants s_{ij} in the normalization of [4, 23]. Each combination is a sum of products of modular graph functions (its strata), which we integrate separately over the fundamental domain. In the representation of [23] the two combinations carry undetermined constants c_1 and c_2,[2] fixed there by assuming both are rational multiples of $\zeta_3\zeta_5$, as uniform transcendentality requires [13].

We determine which Laurent coefficients of the integrand enter the cutoff dependence of the integrated couplings and reproduce c_1 and c_2 by zero-mode matching. Uniform transcendentality

[2] This notation of [23] is unrelated to the Laurent coefficients c_k of (2).

then requires the cancellation of the two $\ln(2\pi)$ coefficients (57), each a rational multiple of $\pi\zeta_7$.

6.1 Laurent coefficients in the regularized integral

For a weight-w modular graph function C with Laurent polynomial (2) written as $\sum_k \hat{c}_k\,\tau_2^k$, $\hat{c}_k = \pi^k c_k$, and $\mathcal{M}_L$ the fundamental domain cut off at $\tau_2 = \mathrm{Im}\,\tau = L$, the integration formula of [13] is

$$\int_{\mathcal{M}_L} C\,d\mu \;=\; I_C \;+\; \hat{c}_1\,\ln L \;+\; \sum_{k\neq 1} \frac{\hat{c}_k}{k-1}\,L^{k-1} \;+\; O(e^{-2\pi L})\,, \qquad d\mu = \frac{d\tau_1\,d\tau_2}{\tau_2^2}\,, \tag{51}$$

where I_C is independent of L. The growing powers L^{k-1}, $k \geq 2$, cancel against the contribution of the region $\mathrm{Im}\,\tau > L$ [4, 13] and are dropped, while for $k \leq 0$ one has $L^{k-1} \to 0$. Hence the coupling retains only I_C and $\hat{c}_1 \ln L$.

In the integrands of both weight-eight combinations the depth-three constants of weight thirteen (cf. (33)) appear only in the coefficient of y^{-5}. The representation of [23] in equivariant iterated Eisenstein integrals β^{eqv} leaves further constants undetermined, a weight-nine constant U and higher-weight constants of modular depth four, and these appear in the coefficients of y^{-1} through y^{-4}. None of these constants enters the coefficient of $\ln L$ or a growing power of L in (51); they can affect the couplings only through I_C, which c_1 and c_2 enter as the volume terms $\pi c_1/3$ and $\pi c_2/3$. This is slot arithmetic: it follows from counting Laurent powers in (51) rather than from a computed cancellation, and needs no transcendentality input; in particular $\zeta_{\mathrm{sv}}(5,3,5)$ enters neither a growing power of L nor the coefficient of $\ln L$. For instance, on the strip $|\tau_1| \leq \frac{1}{2}$, $1 \leq \mathrm{Im}\,\tau \leq L$, the y^{-5} term alone integrates to

$$\int_1^L \hat{c}_{-5}\,\tau_2^{-5}\,\frac{d\tau_2}{\tau_2^2} \;=\; \frac{\hat{c}_{-5}}{6}\,\big(1 - L^{-6}\big)\,, \tag{52}$$

whose L-independent part $\hat{c}_{-5}/6$ can cancel only against the rest of the integral.

The constant terms I_C depend on the full modular graph function, and slot arithmetic determines neither them nor whether these constants enter them. Deciding whether U enters I_C would require the regularized integral of the modular function that multiplies U, which we have not evaluated. No cusp-form L-value enters them, since the rapidly decaying cuspidal components of any modular graph function are orthogonal to the constant; that the I_C of the thirty-nine strata not yet integrated (Section 6.3) carry no depth-three value of any weight follows under the weight-nine assignment of [13], no irreducible depth-three value existing below weight eleven, and is otherwise part of that open computation.

6.2 The constants c_1 and c_2

Assembling the y^0 coefficient $\mathcal{I}_0$ of each combination from the Laurent polynomials of its equivariant iterated Eisenstein integrals [23, 22], we find

$$\mathcal{I}_0\big|_{\sigma_2\sigma_3^2} = c_1 + \tfrac{154781}{2639952}\,\zeta_3\zeta_5\,, \qquad \mathcal{I}_0\big|_{\sigma_2^4} = c_2 + \tfrac{55229}{2933280}\,\zeta_3\zeta_5\,, \tag{53}$$

in agreement with [23]. Matching (53) to the ultraviolet zero-mode values of [23] (286/2835 and 11/420, in units of $\zeta_3\zeta_5$), taken as inputs, fixes

$$c_1 = \Big(\tfrac{286}{2835} - \tfrac{154781}{2639952}\Big)\zeta_3\zeta_5 = \tfrac{79673}{1885680}\,\zeta_3\zeta_5\,, \qquad c_2 = \Big(\tfrac{11}{420} - \tfrac{55229}{2933280}\Big)\zeta_3\zeta_5 = \tfrac{617}{83808}\,\zeta_3\zeta_5\,. \tag{54}$$

This matching reproduces the c_1 and c_2 of [23], with rational parts $79673/(97 \cdot 3^3 \cdot 720)$ and $617/(97 \cdot 6^2 \cdot 24)$. The printed values are thereby shown to be consistent, modulo their $\zeta_3\zeta_5$-proportionality assumption; that assumption is exactly the weight-nine uniformity which the uniform-transcendentality pattern of [13] requires of the amplitude, and nothing in the unconditional data here contradicts it.

6.3 The $\ln(2\pi)$ terms and three corrected formulas

At weight eight, in the strata whose integrals are known in closed form, uniform transcendentality can fail only through a term $\ln(2\pi)$, which has no consistent transcendental weight and enters through exactly one regularized integral, namely $\int_{\mathcal{M}_L} E_2^{*2} E_4^*$ [13]. Here $E_k = C_{1,k-1}$ is the Eisenstein series in the normalization (1) and $E_k^* = \frac{1}{2}\Gamma(k)\, E_k$. The two partitions of 8 into three integers ≥ 2 give

$$\int_{\mathcal{M}_L} E_2^* E_3^{*2}\, d\mu \approx \frac{\pi\zeta_7}{1728}\,, \tag{55}$$

$$\int_{\mathcal{M}_L} E_2^{*2} E_4^*\, d\mu \approx -\frac{\pi\zeta_7}{4320}\Big[\ln(2\pi) - \ln(2L) + \mathcal{Z}(7) - 2\mathcal{Z}(4)\Big]\,, \tag{56}$$

where $\approx$ denotes equality up to positive powers of L and $O(e^{-2\pi L})$. The symbol $\mathcal{Z}(n)$ denotes the weight-one combination of $\zeta'(n)/\zeta(n)$ and harmonic numbers of [13, eq. (A.44)]. In (56) we have used the corrected sign (58), derived below.

By Section 6.1 and (54), c_1 and c_2 contribute no $\ln(2\pi)$ term. Summing over the strata whose integrals are known in closed form [5, 13], we find that their contributions to the two $\ln(2\pi)$ coefficients are

$$\tfrac{11}{60}\, u \quad (\sigma_2\sigma_3^2)\,, \qquad -\tfrac{99}{640}\, u \quad (\sigma_2^4)\,. \tag{57}$$

Here $u = -\pi\zeta_7/3240$ denotes the coefficient of $\ln(2\pi)$ in the regularized integral $\int_{\mathcal{M}_L} E_2^2 E_4\, d\mu$.

There remain twenty-nine strata in the first combination and ten in the second whose integrals are not known in closed form, including $\int E_2^2 C_{2,1,1}$, $\int E_2^4$ and $\int C_{2,1,1}^2$, and those of the connected trihedral, box, kite and tetrahedral graphs at weight eight. This count and (57) exclude the one term of the $\sigma_2\sigma_3^2$ combination of [23] that carries seven Green functions, item (ii) below. As a consequence, uniform transcendentality at order α'^8 requires these strata together to contribute exactly $-\frac{11}{60}\, u$ and $+\frac{99}{640}\, u$ to the $\ln(2\pi)$ coefficients of the two couplings respectively, and holds if, further, their finite parts I_C are otherwise uniform. The requirement follows because the $\ln(2\pi)$ coefficient of each coupling is additive over its strata. Conversely, sufficiency would need $\pi\zeta_7 \ln(2\pi)$ to be the only term of non-uniform weight that a finite part I_C can carry, which is known only for the strata integrated in closed form [13]. We therefore do not decide here whether the couplings at order α'^8 are uniformly transcendental, and (57) remains a requirement on the strata not yet integrated.

Three formulas of [13, 23] enter the computations above in corrected form (Table 8). (i) For $w = s_1 + s_2 + s_3$ even, the triple-Eisenstein integration formula for $\int E_{s_1}^* E_{s_2}^* E_{s_3}^*$, eq. (5.7) of [13], carries the overall sign $(-1)^{\lfloor (w+1)/2 \rfloor}$. By analytic continuation of the exact eq. (5.4), along two independent directions at $(2,2,4)$, $(2,3,5)$, $(3,3,6)$ and $(2,4,6)$, we find instead

$$(-1)^{\lfloor (w+1)/2 \rfloor} \;\longrightarrow\; (-1)^{\lfloor w/2 \rfloor + 1} \qquad (w \text{ even})\,, \tag{58}$$

the sign of eq. (5.6). Direct quadrature of $\int E_2^{*2} E_4^*$ and the $\ln L$ coefficient of $\int E_4 C_{2,1,1}$ from (51) agree with (58), and the odd-w branch is unaffected.[3]

The other two corrections concern [23]: (ii) in their $\sigma_2\sigma_3^2$ combination the term $+\frac{1}{45360}\, C^+{}_{[1111]}^{[1111]}$ Tri[1|1|1] carries seven Green functions (edges) against eight in every other term, and we exclude it. (iii) Their $C_{2,1,1}$ example reproduces the Laurent polynomial of $C_{2,1,1}$ [5] only with $-18\,\beta^{\mathrm{eqv}}[\begin{smallmatrix}2&0\\4&4\end{smallmatrix}]$ in place of their $-18\,\beta^{\mathrm{eqv}}[\begin{smallmatrix}0&2\\4&4\end{smallmatrix}]$. With their label the y^4 and y^{-1} coefficients differ from those of $C_{2,1,1}$ by $-1/1125$ and $-5\zeta_5/6$, in the conventions fixed by the weight-six example of [23] and by (53). Both are transcription-level defects; neither propagates into any value used in this paper: the seven-edge term is excluded from the strata split above, (57) is stated with that exclusion, and (53) and the β^{eqv} route do not involve it.

7 Conclusions and outlook

We have computed the Laurent polynomials of the forty-nine two-vertex four-edge modular graph functions of weights seven through twelve. These are all the two-vertex four-edge functions at those weights, and they obey the three laws (3)–(5). More specifically, the depth-three constants occur only in the coefficient c_{3-w} and are single-valued through transcendental weight twenty-one. At each weight their classes modulo products span a space of the dimension that the Broadhurst–Kreimer conjecture predicts. Their images under every motivic derivation D_m (rational combinations of products of two odd zeta values, as Brown's model (10) requires [16]) lie in the span of the family's lower-weight coefficients.

At five edges the power counting of (18) shows that the four-fold Mordell–Tornheim sums of the sector $|S| = 5$, the only possible source of depth four, enter a single coefficient, c_{4-w}. In all twenty-nine five-edge functions computed through weight 11 that coefficient is free of depth four. Whether depth four appears beyond weight 11 is open. The three coefficients below c_{4-w}, where determined, have the same form as the corresponding four-edge coefficients (Table 9). By contrast, depth three is no longer confined to c_{3-w} and appears in several coefficients above it (Appendix F).

Beyond two vertices, the tetrahedral functions of Kleinschmidt and Verschinin [25] satisfy Laplace systems whose eigenvalues, through weight eight, are not of the resonant form $k(k-1)$. Theorem 1 applies there, and (42) yields their Laurent polynomials once those of the sources are supplied. We have set up these Laplace systems and enumerated their eigenvalues (Appendix A), but the sources are the obstruction: the required kite Laurent polynomials are not in print, and the sector expansion of Section 3.2 is a two-vertex method, so a three-vertex analog would have to be built before the transport can be run.

For the couplings at order α'^8 we have derived a necessary condition for uniform transcendentality: the thirty-nine strata not yet integrated must contribute the negatives of the two $\ln(2\pi)$ coefficients in (57). Among these strata, the two-vertex products $\int E_2^2 C_{2,1,1}$, $\int E_2^4$ and $\int C_{2,1,1}^2$ can be evaluated by quadrature with their $\ln L$ coefficients exact, which does not fix their $\ln(2\pi)$ parts; those parts and the connected weight-eight graphs require methods not developed here, so the target of Section 6.3 remains open.

For the cosmic Galois symmetry, a first problem is to extend our computation of the images under D_m from depth three, the maximal depth of the four-edge family, to depth four. A second question is whether the Laurent coefficients of the family form a comodule under the derivations,

[3] In [38], Doroudiani independently prints the corrected value of the affected triple-Eisenstein integral without remarking on the conflict; to our knowledge the sign error itself has not been reported before.

which would explain (5). Part of the fine structure remains unexplained: that $C_{2,2,2,1}$ contains no irreducible while its weight-seven siblings do follows from the first identity in (71), due to [24], but no principle we tested says why three of the six weight-nine functions share a single rational line. It would be interesting to see whether these selection rules have a world-sheet origin.

Acknowledgments

Funding for this project was provided by Anthropic. This is not an official Anthropic project and results and views are not endorsed by Anthropic.

A Proof of the transport theorem

Throughout, $y = \pi\tau_2$, the Laplacian is $\Delta = y^2\partial_y^2 + (y^2/\pi^2)\partial_x^2$ on functions of $\tau = \tau_1 + i\tau_2$ with $x = \tau_1$, and on x-independent monomials $\Delta\, y^k = k(k-1)\, y^k$. The Zerbini cusp-expansion class $\mathcal{M}_w$ of Theorem 1 consists of the functions F that are smooth, 1-periodic in x, and have an absolutely and uniformly convergent, termwise differentiable double expansion on $\tau_2 \geq \epsilon$,

$$F \;=\; \sum_{\mu,\nu\geq 0} d_F^{(\mu,\nu)}(y)\, q^\mu \bar{q}^\nu\,, \qquad d_F^{(\mu,\nu)}(y) = \sum_{k=1-w}^{w} a_k^{(\mu,\nu)} y^k\,, \tag{59}$$

with $q = e^{2\pi i\tau}$ and polynomially bounded coefficients $|a_k^{(\mu,\nu)}| \leq C_F(1+\mu+\nu)^{A_F}$. The Laurent polynomial of F is $L_F := d_F^{(0,0)}$. All bounds below hold for $y \geq y_* := \max(1, \pi\epsilon)$, and $\mathcal{R}$ denotes the set of functions bounded by Ke^{-cy} on $[y_*, \infty)$ for some constants K and $c > 0$. Modular graph functions of weight w belong to the class $\mathcal{M}_w$ by Zerbini's theorem [2], and the union of the classes $\mathcal{M}_w$ over w is closed under finite products (the exponent ranges add and the polynomial bounds multiply).

Let $F_n(y)$ denote the n-th Fourier coefficient of F in x. For the source $S = \sum_j r_j \prod_i F_{j,i}$ of the theorem we write $L_S := \sum_j r_j \prod_i L_{F_{j,i}} = \sum_k s_k y^k$ and $P(y) := \sum_k s_k y^k/(k(k-1)-\lambda)$, whose denominators are nonzero by the hypothesis on λ. For a solution C of $(\Delta-\lambda)C = S$ in the class, the theorem asserts that (i) $L_C = P$, which is (42); (ii) $C_0 - L_C \in \mathcal{R}$, with $S_0 - L_S$ contributing to no c_k; and (iii) every other such solution has the same Laurent polynomial as C.

Step 1 (mode bounds). Set $n_\pm := \max(\pm n, 0)$. Since $q^\mu\bar{q}^\nu$ equals $e^{2\pi i(\mu-\nu)x}\, e^{-2(\mu+\nu)y}$, only the terms with $\mu - \nu = n$ contribute to F_n; parametrizing them by $m = \min(\mu,\nu) \geq 0$ with $\mu + \nu = |n| + 2m$ gives

$$F_n(y) \;=\; \sum_{m\geq 0} d_F^{(m+n_+,\, m+n_-)}(y)\, e^{-2(|n|+2m)y}\,. \tag{60}$$

For $y \geq y_*$ the Laurent polynomials in (59) obey $|d_F^{(\mu,\nu)}(y)| \leq 2wC_F(1+\mu+\nu)^{A_F} y^w$, and $(1+|n|+2m)^{A_F} \leq (1+|n|)^{A_F}(1+2m)^{A_F}$ together with $e^{-4my} \leq e^{-4my_*}$ gives the uniform, geometrically dominated bounds

$$|F_n(y)| \;\leq\; B_F\,(1+|n|)^{A_F}\, y^w e^{-2|n|y}\,, \qquad |F_0(y) - L_F(y)| \;\leq\; B'_F\, y^w e^{-4y}\,, \tag{61}$$

with finite constants B_F, B'_F. In particular $F_0 - L_F \in \mathcal{R}$ at every rate $c < 4$.

Step 2 (the zero mode of the source). For F, G, H in the class, $(FG)_0 = \sum_n F_n G_{-n}$ and $(FGH)_0 = \sum_{m+n+p=0} F_m G_n H_p$, both absolutely convergent by (61). In every term with a

nonzero index at least two indices are nonzero, since the indices sum to zero. By (61) such a term is bounded by a power of y and of the indices times an exponential no larger than e^{-4y} (for two factors, $e^{-2|n|y}e^{-2|n|y} = e^{-4|n|y}$), and by geometric domination in the indices the sum of these terms lies in $\mathcal{R}$. The term with all indices zero is the product of the zero modes, which differs from the product of the Laurent polynomials by a function bounded by a polynomial times e^{-4y}. It follows that $(FG)_0 = L_F L_G + E^{(2)}$ and $(FGH)_0 = L_F L_G L_H + E^{(3)}$ with $E^{(2)}, E^{(3)} \in \mathcal{R}$ at every rate $c < 4$, and for the source of the theorem $S_0 = L_S + E$ with $E \in \mathcal{R}$. Projecting $(\Delta - \lambda)C = S$ onto the zero mode, we obtain $y^2 C_0'' - \lambda C_0 = L_S + E$, while $y^2 P'' - \lambda P = L_S$ by construction. Therefore $v := C_0 - P$ satisfies $y^2 v'' - \lambda v = E$, and $v = Q + h$ with $Q := L_C - P$ a finite integer-exponent Laurent polynomial and $h := C_0 - L_C \in \mathcal{R}$.

Step 3 (decay of the particular solution). Let α solve $\alpha(\alpha-1) = \lambda$ and set $u_1 = y^\alpha$, $u_2 = y^{1-\alpha}$ (for $\lambda = -\frac{1}{4}$, $u_1 = y^{1/2}$, $u_2 = y^{1/2}\ln y$). The Wronskian $W = u_1 u_2' - u_1' u_2$ is the constant $1 - 2\alpha$ (resp. 1). Taking $+\infty$ as the base point of both variation-of-parameters integrals, we define

$$u_E(y) \;:=\; \int_y^\infty \frac{u_1(y)u_2(t) - u_2(y)u_1(t)}{W}\,\frac{E(t)}{t^2}\,dt \tag{62}$$

which converges absolutely by the bound on E. The function u_E solves $y^2 u_E'' - \lambda u_E = E$, since differentiation under the integral sign is justified by the same estimates and the boundary term at $t = y$ vanishes because the kernel is zero there. It obeys $|u_E(y)| \le (4K/c|W|)\, y^{-1} e^{-cy}$ for all sufficiently large y, by the elementary bound $\int_y^\infty t^\sigma e^{-ct} dt \le (2/c) y^\sigma e^{-cy}$ valid for $y \ge \max(1, 2\sigma/c)$. In particular $u_E \in \mathcal{R}$ at the same rate as E. A finite base point in either integral would instead add to u_E a constant multiple of y^α or $y^{1-\alpha}$, which is not exponentially small and is a homogeneous solution of the kind excluded in Step 4.

Step 4 (exclusion of the homogeneous solutions). We first prove the following lemma. Let (β_1, m_1), …, (β_r, m_r) be distinct pairs with $\beta_j \in \mathbb{C}$ and $m_j \in \{0, 1\}$; if

$$g(y) \;=\; \sum_{j=1}^{r} b_j\, y^{\beta_j} (\ln y)^{m_j} \;=\; O(e^{-cy}) \quad \text{as } y \to \infty \tag{63}$$

for some $c > 0$, then every $b_j = 0$. To prove the lemma, we discard the zero coefficients, let $\sigma^* = \max_j \operatorname{Re}\beta_j$ over the remaining terms, let m^* be the largest power of $\ln y$ among the terms with real part σ^*, and divide by $y^{\sigma^*}(\ln y)^{m^*}$. After this division the left-hand side tends to zero because $e^{-cy} y^{-\sigma^*} (\ln y)^{-m^*} \to 0$ for either sign of σ^* (here the exponential decay of g is used), and so does each subleading term. The leading group T of terms with $\operatorname{Re}\beta_j = \sigma^*$ and $m_j = m^*$ therefore satisfies $\psi(L) = \sum_{j \in T} b_j e^{i\tau_j L} \to 0$ in the variable $L = \ln y$, with distinct real frequencies $\tau_j = \operatorname{Im}\beta_j$. The Fourier–Bohr mean

$$\lim_{T' \to \infty} \frac{1}{T'} \int_0^{T'} \psi(L)\, e^{-i\tau_l L}\, dL \tag{64}$$

equals b_l when computed term by term, and also equals 0 because $\psi \to 0$; hence every b_l with l in T vanishes, which contradicts the choice of T among the nonzero coefficients and proves the lemma.

Now suppose that $c_+ y^\alpha + c_- y^{1-\alpha}$ equals the sum of an integer-exponent Laurent polynomial and a function in $\mathcal{R}$. Since the hypothesis $\lambda \notin \{k(k-1)\}$ makes α and $1 - \alpha$ non-integers, no exponent of the Laurent polynomial coincides with α or $1 - \alpha$. Moving the Laurent polynomial to the left-hand side therefore produces a sum of the form (63) with distinct pairs (β_j, m_j), and

this sum equals a function in $\mathcal{R}$. By the lemma, $c_+ = c_- = 0$ and every coefficient of the Laurent polynomial is zero, and hence the function in $\mathcal{R}$ vanishes as well. The homogeneous solutions are thus excluded by membership in the class $\mathcal{M}_w$, in which the zero mode is an integer-exponent Laurent polynomial plus an exponentially small remainder. Moderate growth alone would not exclude them: for $\lambda > -\frac{1}{4}$ the solution $y^{1-\alpha}$ has real exponent below $\frac{1}{2}$ and is of moderate growth, and for $\lambda < -\frac{1}{4}$ the two solutions $y^{1/2\pm it}$ oscillate with neither one dominant; the mean value (64) is needed for this latter case.

Assembly. By Steps 2 and 3, $g_h := v - u_E$ solves the homogeneous Euler equation $y^2 g_h'' - \lambda g_h = 0$ and is therefore of the form $g_h = c_+ y^\alpha + c_- y^{1-\alpha}$ (with the logarithmic solution at $\lambda = -\frac{1}{4}$), while also $g_h = Q + (h - u_E)$ with Q an integer-exponent Laurent polynomial and $h - u_E \in \mathcal{R}$. Step 4 forces $c_+ = c_- = 0$, $Q = 0$ and $h = u_E$. Hence $L_C = P$, which is conclusion (i), and L_C depends on the sources only through L_S. Moreover $C_0 - L_C = u_E \in \mathcal{R}$, which is conclusion (ii), and the remainder E of the zero-mode source enters only u_E and therefore contributes to no c_k. For uniqueness, the difference D of two solutions in the class satisfies the homogeneous equation. Its zero mode is a combination $c_+ y^\alpha + c_- y^{1-\alpha}$ and is also L_D plus a remainder in $\mathcal{R}$. By Step 4, $L_D = 0$ and the two Laurent polynomials coincide, which is conclusion (iii).

The coupled case. For the systems $(\Delta - \lambda_i) C_i = S_i + \sum_l q_{il} C_l$ of Section 4.4, with constant rational q_{il}, all $\lambda_i \notin \{k(k-1)\}$, and $M_k = \mathrm{diag}(k(k-1) - \lambda_i) - (q_{il})$ invertible for every k, we project each equation onto its zero mode and substitute $C_{i,0} = L_i + h_i$, $S_{i,0} = L_{S_i} + E_i$. The combination $V_i := y^2 L_i'' - \lambda_i L_i - \sum_l q_{il} L_l - L_{S_i}$ is then a finite integer-exponent Laurent polynomial and at the same time a member of $\mathcal{R}$ (the tails h_l may be differentiated termwise, and the resulting series obey geometrically dominated bounds of the same form as in Step 1). By the lemma of Step 4, $V_i = 0$ identically. Reading off the coefficient of y^k, we obtain $M_k \, \ell_k = s_k$ for the vectors ℓ_k and s_k of k-th Laurent coefficients of the solutions and of the sources. The Laurent coefficients of the C_i are therefore determined through M_k^{-1} by the Laurent coefficients of the sources at the same power of y, with no mixing between powers. If all source coefficients lie in a $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, then so do all coefficients of the C_i, since M_k^{-1} is a rational matrix.

Resonant and non-resonant examples. At $\lambda = 12 = k(k-1)$ for $k \in \{4, -3\}$ the hypothesis fails, since the exponents $\alpha = 4$ and $1 - \alpha = -3$ are integers inside the range of Laurent exponents. The homogeneous solutions are then admissible Laurent monomials themselves, so Step 4 no longer forces $c_+ = c_- = 0$, and the denominators in (42) vanish at those two powers of y. For Basu's weight-six consistency equation [6], whose eigenvalue is 12, the source assembled from the Laurent polynomials collected there satisfies the two solvability conditions $s_4 = s_{-3} = 0$, (42) holds for every other coefficient, and the coefficient c_{-3} of (45) does not enter the equation.

A non-resonant example is the weight-six tetrahedral function at $\lambda = -6$, for which $\alpha = (1 \pm i\sqrt{23})/2$ is a complex pair and the source is a single product of two factors. Applied to its Laplace equation [25], formula (42) gives the same Laurent polynomial as the algebraic identity of [24], which expresses this function through two-vertex functions and an Eisenstein series. The tetrahedral systems of [25] at weights seven and eight are likewise non-resonant, with eigenvalues -6 and $-4, -4, -10$, and for the coupled weight-eight triple $\det M_k = 3\,(k(k-1)+4)^2 (k(k-1)+10) \neq 0$ for all k, so that every M_k is invertible. At higher weight the tetrahedral spectra of [25, Section 3.6] contain resonant eigenvalues, namely 0 at weight nine, with multiplicity three, 6 at weight ten and 2 at weight twelve, each of the form $k(k-1)$. On those eigenspaces (42) leaves two coefficients undetermined and imposes two conditions on the sources, as in the resonant example above.

B Derivation of the Broadhurst–Kreimer denominator

We derive here the Broadhurst–Kreimer integers $D_{n,3}$ of Eq. (14), which enter the saturation law (4), from the generating function Eq. (13). The generating function is itself a conjecture, so the derivation and its controls fix the integers it predicts without testing the conjecture; the ranks of Section 4.6 are a priori bounded by those integers, and the measurement finds equality.

We write $G(x,y) = 1 - O(x)\,y + S(x)\,y^2 - S(x)\,y^4$ for the left-hand side of Eq. (13), with $O = x^3/(1-x^2)$ and $S = x^{12}/((1-x^4)(1-x^6))$, and $P(x,y) = \sum_{n,k} D_{n,k}\,x^n y^k$ for the counting series. The logarithm of Eq. (13) is $\log G(x,y) = -\sum_{m\geq 1} \frac{1}{m} P(x^m, y^m)$, and Möbius inversion yields

$$P(x,y) \;=\; -\sum_{m\geq 1} \frac{\mu(m)}{m}\,\log G(x^m, y^m)\,. \tag{65}$$

We expand Eq. (65) as a power series in x and y with rational coefficients, truncated at weight 30. All coefficients $D_{n,k}$ obtained are non-negative integers, and $D_{n,k} = 0$ unless $n \equiv k \pmod 2$. In depth three we find that the coefficients satisfy $D_{n,3} = \mathrm{Lyndon}_3(n) - [O\cdot S]_n$ at every $n \leq 30$. Here $\mathrm{Lyndon}_3(n)$ counts the Lyndon words of length three and weight n in the odd letters of weight at least 3, and $[O\cdot S]_n$ is the coefficient of x^n in $O(x)S(x)$.

The deficit $[O\cdot S]_n$ of $D_{n,3}$ relative to the free Lyndon count is the number of pairs of an odd zeta value counted by O and a cusp form counted by S whose weights sum to n. At the odd weights 11 through 21 the coefficients $D_{n,3}$ are the six integers of Eq. (14). Against the free Lyndon counts $1, 2, 3, 5, 7, 9$ the deficits are $0, 0, 1, 1, 2, 3$, arising from the pairs (ζ_3, Δ_{12}) at $n = 15$; (ζ_5, Δ_{12}) at $n = 17$; (ζ_7, Δ_{12}) and (ζ_3, Δ_{16}) at $n = 19$; (ζ_9, Δ_{12}), (ζ_5, Δ_{16}) and (ζ_3, Δ_{18}) at $n = 21$.

We first compare our computation with Table 1 of [21], which lists the number M_n of weight-n generators of the algebra of MZVs. For $n = 1, \ldots, 20$ the table gives $M_n = 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 2, 2, 3, 3, 4, 5, 7, 8, 11, 13$. We reproduce all twenty entries by Möbius inversion of the generating function $1 - x^2 - x^3$ of eq. (2) of [21]. Because the product in Eq. (13) starts at weight 3, the depth sums $\sum_d D_{n,d}$ of our series give the eighteen entries with $n \geq 3$ a second time.

We next compare our expansion with Table 16 of the Data Mine [35], which tabulates the number of basis elements in a minimal-depth representation for $n \leq 30$ and depth ≤ 10. The $n = 2$ entry $\zeta(2)$ lies outside the $n > 2$ product in Eq. (13) and is not compared. Our series $P(x,y)$ reproduces all 75 other entries of the table and gives zero in every blank cell. In particular, the depth-three column at odd weights 11 through 21 coincides with Eq. (14). The generating function Eq. (13) is a conjecture, and the two controls above, both passed (20 of 20 and 75 of 75 printed entries reproduced), check our extraction of its integers against the sources' own published tables; they do not test the conjecture itself.

The integer $D_{n,3}$ is the a priori upper bound on the ranks of (46) and Table 5, and the appropriate denominator for the occupancy ratios of Table 6. These are ranks of the deep coefficients written in terms of the values $\zeta_{\mathrm{sv}}(a,b,c)$ with a, b, c odd and ≥ 3, taken modulo products and modulo $\zeta(n)$ itself. This quotient coincides with the one counted by $D_{n,3}$, the depth-three irreducibles modulo products and lesser depth: by parity, no depth-two irreducible occurs at odd weight, and $\zeta(n)$ is quotiented out on both sides. The $\zeta(2)$-dependent part is also absent on both sides, because the single-valued map sends $\zeta(2) \to 0$ and the product in Eq. (13) starts at weight 3.

C Single-valued depth-three multiple zeta values at weights 11–21

Brown [16] defines the single-valued multiple zeta values $\zeta_{\mathrm{sv}}(a,b,c)$ in general and gives three examples in his eq. (7.4), one at weight eleven and two at weight thirteen, of which Eq. (34) is one. Dorigoni et al. [22] write out the reduction of one weight-eleven value and give a four-element depth-three basis at weight seventeen, together with machine-readable reductions through weight seventeen in their ancillary notebook. We derive the full set of formulas at weights 11 through 21 with that data entering no derivation step; the overlap through weight 17 serves as an external control on our derivation, carried out in full in Section C.3, and the weight-19 and weight-21 formulas, for which we have found no explicit expressions in the literature, are recorded here with the derivation and checks. Each formula expresses one value $\zeta_{\mathrm{sv}}(a,b,c)$, with $a,b,c \geq 3$ odd, as a $\mathbb{Q}$-combination of the multiple zeta values of (8); together the formulas cover all 83 triples at weights 11–21, and they are listed in Section C.5 and in the ancillary files.

We obtained the formulas in two ways (Table 2): 61 were extracted from 700-digit numerical values of $\zeta_{\mathrm{sv}}(a,b,c)$, and the remaining 22 were derived algebraically from Brown's single-valued map. Among the extracted formulas, the 34 at weights 11–17 pass an inversion test that cannot be run at weights 19 and 21, so the 27 extracted there rest on the formula's own residual against the numerical values (Section C.1). In Section C.2 we also derive all 61 algebraically from the same map and recover each formula exactly, those 27 included.

C.1 Numerical extraction

For the 61 extracted formulas we first evaluate each $\zeta_{\mathrm{sv}}(a,b,c)$ numerically to 700 digits. We use Brown's construction of the single-valued MZVs as coefficients of the Deligne associator W, the unique solution of $W \circ sZ = Z$ in the notation of [16, Section 5], computed in the depth-≤ 3 quotient by fixed-point iteration weight by weight. The fixed-point identity that defines W is satisfied by the computed series to a residual of order 10^{-685}.

We then write $\zeta_{\mathrm{sv}}(a,b,c) = 2\zeta(a,b,c) + (\text{correction})$ and determine the correction by integer-

weight	triples	extracted (numeric)	derived (algebraic)	completeness
11	3	3	0	all triples
13	6	6	0	all triples
15	10	10	0	all triples
17	15	15	0	all triples
19	21	16	5	all triples
21	28	11	17	all triples
total	83	61	22	

Table 2: The 83 formulas for $\zeta_{\mathrm{sv}}(a,b,c)$, odd $a,b,c \geq 3$, counted by weight and by the way they were obtained; together they cover every triple at weights 11 through 21. The 61 extracted entries come from integer-relation searches against 700-digit values and pass the checks of Section C.1; for the 27 at weights 19 and 21 the inversion test of that section is not available. The 22 derived entries (5 at weight 19, 17 at weight 21) follow from Brown's single-valued map (10), which reproduces all 61 extracted formulas exactly (Section C.2).

relation detection against a pool of candidate terms. Initially the pool contains the two term types of Brown's three examples: even zeta values times odd-zeta monomials, and a single zeta value times a depth-two value. Where this pool does not suffice we enlarge it by even zeta values multiplying depth-two and depth-three values and by the depth-three values of the same weight themselves. We accept a formula when the searches at two precisions return the same rational vector and its residual at the full 700 digits is below 10^{-695}. In this way we obtain 61 formulas: all $3, 6, 10, 15$ triples at weights $11, 13, 15, 17$, and 16 of the 21 triples at weight 19 and 11 of the 28 at weight 21 (the remainder is derived in Section C.2). In particular, we recover Brown's three examples of eq. (7.4) with his rational coefficients.

In each formula the left-hand side is taken from the associator, and the multiple zeta values on the right-hand side are evaluated without reference to it. The two sides agree to at least 695 digits at the 700-digit working precision; stuffle product identities among the values $\zeta_{\mathrm{sv}}(a, b, c)$ hold to at least 695 digits as well. Through weight 17 the formulas admit one further test: inverting the coefficient matrix at each weight yields a rational expression for $2\zeta(a, b, c)$ over the single-valued and product spanning sets, and we verified that expression numerically to more than 600 digits. That further test exists only through weight 17, where the spanning sets are complete, and it splits the 61 extracted formulas into two recorded grades: the 34 at weights 11–17 pass it, while for the 27 at weights 19 and 21 the spanning data needed to run it is incomplete, and those rest on the residual of the formula itself against the 700-digit numerics. Moreover, in the rank computation of Section 4.6 the weight-19 and weight-21 values satisfy 38 independent relations, each holding to about 700 digits.

C.2 Algebraic completion at weights 19 and 21

At weights 19 and 21 the 700-digit values do not determine a formula for every triple. For 5 triples at weight 19 and 17 at weight 21 the integer-relation searches at the two working precisions return different rational vectors, and no correction is identified. We obtain the formulas for these triples instead from Brown's single-valued map, an exact algebraic operation that requires no numerical input. Let $\mathcal{U} = \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle$ be the graded Hopf algebra cogenerated by one letter f_{2n+1} in each odd degree $2n + 1 \geq 3$, with the shuffle product ш and the deconcatenation coproduct; the algebra $\mathcal{H}$ of motivic multiple zeta values is non-canonically isomorphic to $\mathcal{U} \otimes \mathbb{Q}[f_2]$ [16, eq. (7.2)]. On $\mathcal{U}$ the single-valued map is the homomorphism $\mathrm{sv}\colon \mathcal{U} \to \mathcal{U}$ of (10), in which the sum runs over the deconcatenations $w = uv$ and $\tilde{v}$ is the word v reversed [16, eq. (7.3)]. On $\mathcal{H}$ the map annihilates f_2, the generator corresponding to ζ_2 (Section 2.2). On words of length at most three in letters of odd degree, the case needed here, the general rule in (10) reduces to the examples below eq. (7.3) of [16]: the length-three instance already displayed in (10), and

$$\mathrm{sv}(f_a) = 2f_a\,, \qquad \mathrm{sv}(f_a f_b) = 2\,(f_a f_b + f_b f_a)\,. \tag{66}$$

We evaluated (10) over the ring basis of Section 3.3. The f-alphabet words attached to that basis are computed exactly, with no numerical fit, by the decomposition algorithm of [33]. We write ϕ for the map from $\mathcal{H}$ into $\mathcal{U} \otimes \mathbb{Q}[f_2]$, $[u]\,\xi$ for the coefficient of the word u in ξ, and D_m for the derivations of Section 5.1. The image of each generator x of the ring basis is then determined by

$$\phi(\zeta_m) = f_m\,, \qquad \phi(\zeta_2) = f_2\,, \qquad [\,u\,f_m\,]\,\phi(x) \;=\; [\,u\,]\,\phi(D_m x) \qquad (m \text{ odd})\,, \tag{67}$$

with ϕ multiplicative and the right-hand side computed recursively in the weight. The words so obtained are the reversals of Brown's, and (10) is transported to our conventions by word

reversal. This construction fixes one normalization beyond Brown's formula, the vanishing of the pure $\zeta_2^{w/2}$ part of each even-weight generator of the ring basis; none of the 83 formulas of this appendix depends on it, since deriving them again with that part changed, for one generator at a time or for all at once, returns every formula unchanged. The one remaining choice is the orientation of the words in the transport to our conventions, that is, whether Brown's formula applies to our words directly or after the reversal stated above; it is fixed below by comparison with the extracted formulas. The resulting images of the depth-two and depth-three generators through weight 21 are listed in the ancillary files described in Appendix E.

As a check in depth two, the f-alphabet construction gives, for 27 pairs (a, b) with $a, b \geq 3$ odd, each $\zeta_{\mathrm{sv}}(a, b)$ as the known polynomial in odd zeta values (Section 2.2), in agreement with the associator-derived values to at least 480 digits. Applied to the 61 extracted triples, the construction reproduces all 61 formulas rational by rational, the 27 weight-19 and weight-21 formulas included. Since the construction takes no numerical input, this agreement establishes those 27 formulas as exact algebraic results, and it also fixes the orientation of the words in the transport. With the opposite orientation the comparison fails for every triple except $(5, 5, 5)$ and $(7, 7, 7)$, the two whose indices are all equal. Each of the 22 additional formulas agrees with the associator-derived value of its left-hand side to at least 462 digits.

C.3 Comparison with Dorigoni et al.

The overlap with the ancillary data of Dorigoni et al. [22], whose MZV convention is identical to ours, covers 30 of our 34 triples through weight 17: 21 triples at weights 15 and 17, and 9 at weights 11 and 13. The remaining 4 triples, $(5, 5, 5)$ at weight 15 and $(3, 7, 7)$, $(7, 3, 7)$, $(7, 7, 3)$ at weight 17, do not appear in their data. In all 30 cases our formula agrees with theirs: in 9 the two coincide term by term (the weight-eleven reduction written out in their paper among them), and in the other 21 they express the same value over different sets of terms. In those 21 cases the two forms reduce to one another by identities among the multiple zeta values, and numerically the two sides agree to at least 690 digits.

We also found that the five-triple spanning set used for our weight-17 formulas is overcomplete by one relative to the four-element basis of eq. (A.3) of [22], in that exactly one rational relation holds among the five modulo products. Written out with its product terms, in the convention of (8) and up to overall normalization, it reads

$$\begin{aligned} 0 \;=\; & \tfrac{2100}{787}\,\zeta(3,3,11) - \tfrac{2184}{787}\,\zeta(3,5,9) - \tfrac{1650}{787}\,\zeta(3,7,7) \\ & - \tfrac{1624}{787}\,\zeta(3,9,5) - \tfrac{564}{787}\,\zeta(5,5,7) + \tfrac{2632}{2361}\,\zeta_5\,\zeta(3,9) \\ & - \tfrac{2968}{787}\,\zeta_5^2\,\zeta_7 + \tfrac{37224857}{1574}\,\zeta_{17} - \tfrac{9961560}{787}\,\zeta_2\,\zeta_{15} \\ & - \tfrac{3857616}{3935}\,\zeta_2^2\,\zeta_{13} - \tfrac{310104}{5509}\,\zeta_2^3\,\zeta_{11} + \tfrac{95616}{19675}\,\zeta_2^4\,\zeta_9 \\ & + \tfrac{102528}{43285}\,\zeta_2^5\,\zeta_7 + \tfrac{1494574208}{2067940875}\,\zeta_2^6\,\zeta_5\,, \end{aligned} \tag{68}$$

which holds numerically to 691 digits. The weight-17 formulas themselves are unaffected; only their representation over that spanning set is not unique.

C.4 Formulas entering the rank and derivation computations

We carried out the rank computation of Section 4.6 in the ζ_2-free sector with the 61 formulas extracted from 700-digit numerics, of which the 34 at weights 11–17 also pass an inversion test that cannot be run at weights 19 and 21, where the other 27 rest on the formula's own

residual. The construction of Section C.2 reproduces each of those 27 exactly. The 22 derived formulas reduce onto the same five (weight 19) and six (weight 21) depth-three generators of the ring basis over which the extracted formulas are already expressed, and adding them to the computation therefore leaves the ranks unchanged. In the computation of Section 5 the motivic derivations D_m, applied to the 83 formulas, give 627 images (434 from the 61 formulas extracted from numerics, 193 from the 22 derived algebraically, never pooled). The closure statement of Section 5.2 was first established on the 434 images alone. The 22 coefficients then untested (5 at w=19, 17 at w=21) are not covered by this statement. Repeating the computation on the 193 images of the derived formulas, with the same span and membership criterion, we extend the statement to all 83 values (Table 7).

The five-edge functions of Table 9 did not draw on this set of formulas: their 46 coefficients with depth-three content, all at transcendental weight 11, 13, or 15, the weights of the first three rows of Table 2, were determined against ordinary depth-three multiple zeta values, and whether that content aligns with the single-valued values recorded here has not been tested. A rebuild of the numerics near 1400 digits would let the 22 derived entries be re-extracted numerically, upgrading derived to extracted; no result in the paper waits on that upgrade.

C.5 The formulas

The 83 formulas, with their rational coefficients and each labeled extracted or derived as in Table 2, are given in full in the ancillary files described in Appendix E, in machine-readable and typeset form. Below we list those of weights 11 and 13, which include Brown's three examples [16, eq. (7.4)] and every single-valued multiple zeta value that appears in Section 4 at these weights. We also give one extracted formula at each of the weights 19 and 21. Both sides of each formula are written in the ascending-index convention of (8), which is also that of [16] and of (34). The symbol ζ_n denotes a single zeta value, and even zeta values are written throughout as rational multiples of powers of ζ_2. In particular, the entry for $(5,3,5)$ is Eq. (34) with its ζ_8 term rewritten in this way.

The term types that make up the correction to the leading term $2\,\zeta(a,b,c)$ change with the weight. Through weight 17 the two term types of Brown's examples suffice, and each of the 34 formulas there is written in terms of even zeta values times odd-zeta monomials and single zeta values times depth-two values. From weight 19 on, the corrections also involve depth-three values of the same weight, which enter 14 of the 16 extracted weight-19 formulas; the formula for $\zeta_{\mathrm{sv}}(11,3,5)$ contains eleven of them, one of which is $\zeta(3,3,13)$. Where $\zeta(a,b,c)$ itself occurs in the correction we keep it separate from the leading term $2\,\zeta(a,b,c)$. At weight 21 the corrections also contain even zeta values multiplying depth-two and depth-three values, present in 8 of the 11 extracted formulas there; the formula for $\zeta_{\mathrm{sv}}(11,5,5)$ has terms of both new kinds.

Weight 11

All 3 extracted from numerics:

$$\zeta_{\mathrm{sv}}(3,3,5) = 2\,\zeta(3,3,5) - 5\,\zeta_3^2\zeta_5 + 90\,\zeta_2\zeta_9 + \tfrac{12}{5}\,\zeta_2^2\zeta_7 - \tfrac{8}{7}\,\zeta_2^3\zeta_5,$$

$$\zeta_{\mathrm{sv}}(3,5,3) = 2\,\zeta(3,5,3) - 2\,\zeta_3\,\zeta(3,5) - 10\,\zeta_3^2\zeta_5,$$

$$\zeta_{\mathrm{sv}}(5,3,3) = 2\,\zeta(5,3,3) + 2\,\zeta_3\,\zeta(3,5) + 18\,\zeta_3^2\zeta_5 - 90\,\zeta_2\zeta_9 - \tfrac{12}{5}\,\zeta_2^2\zeta_7 + \tfrac{48}{35}\,\zeta_2^3\zeta_5 + \tfrac{48}{175}\,\zeta_2^4\zeta_3.$$

Weight 13

All 6 extracted from numerics:

$$\begin{aligned}
\zeta_{\mathrm{sv}}(3,3,7) &= 2\,\zeta(3,3,7) + 12\,\zeta_5\,\zeta(3,5) - 14\,\zeta_3^2\zeta_7 + 60\,\zeta_3\zeta_5^2 + 407\,\zeta_2\zeta_{11} + \tfrac{112}{5}\,\zeta_2^2\zeta_9 - \tfrac{64}{35}\,\zeta_2^3\zeta_7,\\
\zeta_{\mathrm{sv}}(3,5,5) &= 2\,\zeta(3,5,5) + 10\,\zeta_5\,\zeta(3,5) + 50\,\zeta_3\zeta_5^2 + 275\,\zeta_2\zeta_{11} + 20\,\zeta_2^2\zeta_9,\\
\zeta_{\mathrm{sv}}(3,7,3) &= 2\,\zeta(3,7,3) - 24\,\zeta_5\,\zeta(3,5) - 2\,\zeta_3\,\zeta(3,7) - 28\,\zeta_3^2\zeta_7 - 144\,\zeta_3\zeta_5^2 - \tfrac{288}{175}\,\zeta_2^4\zeta_5,\\
\zeta_{\mathrm{sv}}(5,3,5) &= 2\,\zeta(5,3,5) - 22\,\zeta_5\,\zeta(3,5) - 120\,\zeta_3\zeta_5^2 - \tfrac{48}{35}\,\zeta_2^4\zeta_5,\\
\zeta_{\mathrm{sv}}(5,5,3) &= 2\,\zeta(5,5,3) + 12\,\zeta_5\,\zeta(3,5) + 73\,\zeta_3\zeta_5^2 - 275\,\zeta_2\zeta_{11} - 20\,\zeta_2^2\zeta_9 + \tfrac{288}{175}\,\zeta_2^4\zeta_5\\
&\quad + \tfrac{32}{385}\,\zeta_2^5\zeta_3,\\
\zeta_{\mathrm{sv}}(7,3,3) &= 2\,\zeta(7,3,3) + 12\,\zeta_5\,\zeta(3,5) + 2\,\zeta_3\,\zeta(3,7) + 45\,\zeta_3^2\zeta_7 + 84\,\zeta_3\zeta_5^2 - 407\,\zeta_2\zeta_{11}\\
&\quad - \tfrac{112}{5}\,\zeta_2^2\zeta_9 + \tfrac{72}{35}\,\zeta_2^3\zeta_7 + \tfrac{288}{175}\,\zeta_2^4\zeta_5 + \tfrac{64}{385}\,\zeta_2^5\zeta_3.
\end{aligned}$$

Weights 19 **and** 21**: two representatives**

Extracted from numerics, $\zeta_{\mathrm{sv}}(11,3,5)$ at weight 19 and $\zeta_{\mathrm{sv}}(11,5,5)$ at weight 21:

$$\begin{aligned}
\zeta_{\mathrm{sv}}(11,3,5) &= 2\,\zeta(11,3,5) - \tfrac{5723542738}{970931}\,\zeta(3,3,13) + \tfrac{11204293758}{970931}\,\zeta(3,5,11) - \tfrac{1053155411}{970931}\,\zeta(3,7,9)\\
&\quad + \tfrac{4128351025}{970931}\,\zeta(3,9,7) + \tfrac{4715599521}{970931}\,\zeta(3,11,5) + \tfrac{2853606772}{970931}\,\zeta(3,13,3)\\
&\quad - \tfrac{96148179}{74687}\,\zeta(5,3,11) + \tfrac{1935579140}{970931}\,\zeta(5,5,9) + \tfrac{3408288144}{970931}\,\zeta(7,5,7) + \tfrac{83864}{13}\,\zeta(7,9,3)\\
&\quad + \tfrac{89551}{13}\,\zeta(9,7,3) - 90\,\zeta_{11}\,\zeta(3,5) + \tfrac{89551}{13}\,\zeta_9\,\zeta(3,7) + \tfrac{1007079124}{970931}\,\zeta_7\,\zeta(3,9)\\
&\quad + \tfrac{1415473987}{970931}\,\zeta_5\,\zeta(3,11) - \tfrac{5402076586}{970931}\,\zeta_3\,\zeta(3,13) - \tfrac{1970663160}{970931}\,\zeta_5\,\zeta(5,9)\\
&\quad + \tfrac{10857}{13}\,\zeta_3\,\zeta(5,11) - \tfrac{190692}{13}\,\zeta_3\zeta_5\zeta_{11} - \tfrac{422146}{13}\,\zeta_3\zeta_7\zeta_9 + \tfrac{18935000022}{970931}\,\zeta_5^2\zeta_9\\
&\quad + \tfrac{102922995}{970931}\,\zeta_5\zeta_7^2 + \tfrac{2780716349}{373435}\,\zeta_{19} + \tfrac{61966198160}{970931}\,\zeta_2\zeta_{17} - \tfrac{44603469578}{4854655}\,\zeta_2^2\zeta_{15}\\
&\quad - \tfrac{19959854760}{970931}\,\zeta_2^3\zeta_{13} - \tfrac{11902592508}{970931}\,\zeta_2^4\zeta_{11} - \tfrac{4121196512}{4854655}\,\zeta_2^5\zeta_9\\
&\quad + \tfrac{12334899488}{4854655}\,\zeta_2^6\zeta_7 + \tfrac{172178688}{4854655}\,\zeta_2^7\zeta_5 + \tfrac{61056}{65}\,\zeta_2^8\zeta_3,\\
\zeta_{\mathrm{sv}}(11,5,5) &= 2\,\zeta(11,5,5) + 5750\,\zeta(3,3,15) - 643\,\zeta(3,5,13) - 4080\,\zeta(3,13,5) + 2875\,\zeta(3,15,3)\\
&\quad - 643\,\zeta(5,3,13) + 2574\,\zeta(5,5,11) + 1856\,\zeta(5,7,9) + 2153\,\zeta(5,9,7) + 2418\,\zeta(5,11,5)\\
&\quad + 3437\,\zeta(5,13,3) + 2580\,\zeta(7,5,9) + 2165\,\zeta(7,9,5) + 1738\,\zeta(9,5,7) + 2462\,\zeta(9,7,5)\\
&\quad + 2262\,\zeta(11,5,5) - 858\,\zeta_{13}\,\zeta(3,5) + 1344\,\zeta_9\,\zeta(3,9) + 2298\,\zeta_5\,\zeta(3,13)\\
&\quad - 2875\,\zeta_3\,\zeta(3,15) - 385\,\zeta_7\,\zeta(5,9) + 413\,\zeta_5\,\zeta(5,11) - 3437\,\zeta_3\,\zeta(5,13)\\
&\quad - 3432\,\zeta_3\zeta_5\zeta_{13} + 4950\,\zeta_3\zeta_7\zeta_{11} + 5880\,\zeta_3\zeta_9^2 - 11193\,\zeta_5^2\zeta_{11} - 8253\,\zeta_5\zeta_7\zeta_9\\
&\quad + 450\,\zeta_7^3 - 7638\,\zeta_{21} + 13840\,\zeta_2\zeta_{19} + 2488\,\zeta_2^2\zeta_{17} - 10200\,\zeta_2^3\zeta_{15}\\
&\quad - 4664\,\zeta_2^4\zeta_{13} - 4736\,\zeta_2^5\zeta_{11} - 2816\,\zeta_2^6\zeta_9 - 384\,\zeta_2^7\zeta_7 + 768\,\zeta_2^8\zeta_5\\
&\quad + 80\,\zeta_2\,\zeta(3,3,13) - 10\,\zeta_2\,\zeta(3,5,11) - 10\,\zeta_2\,\zeta(3,11,5) + 40\,\zeta_2\,\zeta(3,13,3)\\
&\quad - 10\,\zeta_2\,\zeta(5,3,11) + 10\,\zeta_2\zeta_5\,\zeta(3,11) - 40\,\zeta_2\zeta_3\,\zeta(3,13).
\end{aligned}$$

D Numerical evaluation and comparison with published values

The three laws are read off derived exact values: every coefficient behind them comes from the exact symbolic derivation, with the numerical evaluations as an independent check. In this

appendix we describe that check, namely the high-precision evaluation of the sector sums and the integer-relation recognition of the coefficients (Section D.1), and report the agreement of the forty-nine exact Laurent polynomials with their numerical values (Table 4). We then collect the numerical tests of the depth-three coefficients at transcendental weights 19 and 21, together with an exact single-valued decomposition of one of them, written out in full; the membership criterion and image counts for the derivations of Section 5 (Table 7); and the comparison of our values with published ones (Table 8).

D.1 Sector-sum evaluation and recognition

For each of the forty-nine functions we first evaluated the sector sums of (17) to 290 digits for $w \le 9$ and 460 digits for $w \ge 10$. Because the functions of weight ≥ 9 have coefficients of transcendental weight ≥ 17, whose recognition requires more digits, we evaluated their sector sums again to 560 or 700 digits (Table 4), and the comparisons below use these values.

We then searched for an expression of each coefficient as a $\mathbb{Q}$-linear combination of multiple zeta values. The integer-relation searches were run first with the rationals and the products of odd zeta values of the coefficient's weight as candidates, and then with the depth-two and depth-three multiple zeta values of that weight added. Each recognition is an integer relation found by PSLQ [39] whose relation vector, reduced to a primitive integer vector of fixed sign, is identical at two working precisions. We also required that the same search return no relation on a deliberately perturbed input. Before the depth-three searches we reduced the candidate pools to a $\mathbb{Q}$-independent spanning set, since the raw pools satisfy stuffle identities that the search would otherwise return first.

Applied to $D_4 = C_{1,1,1,1}$, the procedure reproduces the published Laurent polynomial [5] coefficient by coefficient to 80 digits. It also recovers, as products of odd zeta values, every nonzero coefficient of $C_{2,2,2}$ and $C_{2,2,1,1}$ (weight six, total modular weight 12, inside the range covered by [24, 12]) and of $C_{2,2,2,1}$ (weight seven, total modular weight 14). The function $C_{2,2,2,1}$ lies above the tabulated range of [12], but is reducible, up to an additive constant, to three-edge functions and Eisenstein series by a weight-seven identity of [24], the first of (71) below. Finally, our numerical evaluation of the sums W_4 reproduces the identity (29), $W_4((1,1,1,1);1) = 30\zeta_5 - 12\zeta_2\zeta_3$, which can be derived by hand from the sector decomposition of D_4, and gives values of $W_4((2,2,2,2);q)$ that agree with truncated lattice sums at each $q \in \{1,\dots,5\}$.

For every function with a lattice-sum entry in Table 4 we also evaluated $C_{\mathbf{a}}(i\tau_2)$ directly at the values of τ_2 given below and in the caption of that table. For this lattice-sum evaluation, which makes no use of the sector expansion, we wrote $C_{\mathbf{a}}$ as a position-space integral over the torus and computed it by tanh-sinh quadrature, with the lattice sums in the integrand evaluated by Poisson resummation at each quadrature point. Subtracting from this lattice-sum value the closed-form Laurent polynomial with the term $c_k y^k$ omitted and dividing by y^k isolates a numerical value for c_k. We call the number of digits to which this value agrees with the closed form the held-out digits of c_k, since the lattice-sum evaluation was not used in the recognition. Table 3 lists them for $C_{2,2,2,2}$ at $\tau_2 = 21$.

At the points $\tau_2 \in \{12, 15, 18, 21\}$ used for $C_{2,2,2,2}$ the same evaluation, applied to the Eisenstein series, reproduces the exact value of E_4/π^4 to 73–90 digits, and for $C_{2,2,2,2}$ itself the two finest levels of the tanh-sinh quadrature agree to ≥ 95 digits at every point. Subtracting the whole polynomial (31) leaves a remainder falling from 3.0×10^{-30} at $\tau_2 = 12$ to 7.4×10^{-54} at $\tau_2 = 21$ with fitted exponent $-0.96 \times 2\pi\tau_2$, the expected $O(e^{-2\pi\tau_2})$ tail, which shows that

k	$+8$	$+1$	-1	-2	-3	-4	-5	-6	-7
held-out digits	60	52	49	48	47	46	45	43	41

Table 3: Each closed-form coefficient c_k of (31) and (33) agrees, to the number of digits listed, with the value extracted from the direct lattice-sum evaluation of $C_{2,2,2,2}$ at $\tau_2 = 21$ by subtracting the other eight closed-form terms and dividing by y^k, $y = \pi\tau_2$. The digits are held out in the sense that the lattice-sum evaluation was not used to obtain the closed forms. The agreement is poorer at more negative k because dividing by y^k multiplies the $e^{-2\pi\tau_2}$ tail by y^{-k}.

no Laurent term is missing. The same subtraction applied to D_4 and its published Laurent polynomial [5] reproduces each of the five coefficients to 40–49 digits, with residual exponent $-0.98 \times 2\pi\tau_2$.

The integer-relation vectors of the eight product coefficients in (31) are identical at 150 and 200 digits, with residuals below 10^{-147} and 10^{-197}. Both working precisions lie below that of the sector-sum evaluation of $C_{2,2,2,2}$ from which these relations were obtained.

The deep coefficient c_{-5} admits no integer relation, at any PSLQ height up to 10^{12}, with the three-dimensional odd-zeta-product pool $\{\zeta_{13}, \zeta_3^2\zeta_7, \zeta_3\zeta_5^2\}$, nor with that pool augmented by the six raw depth-three MZVs $\zeta(a,b,c)$ with odd a,b,c summing to thirteen. We recognized its single-valued form (33) with identical rational vector at 200 and 250 digits. Evaluated through Brown's formula (34), that form agrees with an independent 350-digit sector-sum evaluation to 309 digits (59 beyond the 250-digit fit). The algebraically equivalent form (35), evaluated at full working precision, agrees with the same evaluation to 343 digits (93 beyond that fit). The weight-eleven value $\zeta_{\mathrm{sv}}(3,5,3)$ of Section 2.2 is likewise $\mathbb{Q}$-independent of the odd-zeta products at that weight, and we reproduced Brown's formula for it from an independent numerical reduction of $2\zeta(3,5,3)$ in a full weight-eleven MZV basis.

We applied the same evaluations and integer-relation searches to every other function of the family; Table 4 gives the working precision and the lattice-sum agreement function by function, and Figure 1 marks the coefficients that were not identified numerically. Every closed form this procedure produces is a recognition: an integer relation stable across two precisions and supported by several hundred digits, but a fit nonetheless. At the same time, each of these closed forms is also obtained by the exact reduction of Section 3.3: no Laurent coefficient of the family rests on a fit, and the numerical values serve instead as a test of that reduction.

D.2 Numerical check of the forty-nine Laurent polynomials

For thirty-eight functions, all those of weight $w \le 11$ and four of weight twelve, we evaluated the direct lattice sums of Section D.1 as well as the sector sums. Their exact Laurent polynomials agree with the numerically evaluated coefficients to no worse than 288 digits where the working precision is 290, and 468 digits where it is 560 or 700 (Table 4). We refer to the closed form of a coefficient, whether recognized numerically or derived exactly, as its closure, a usage not to be confused with the closure (50) of the family under the derivations. The eleven weight-twelve functions beyond that set of checks rest instead on dedicated 700-digit sector-numeric references, computed for this comparison by an evaluation that did not produce the closures, which check every nonzero slot at 468–469 digits per function; all fifteen depth-three closures at transcendental weight 21 (the eleven together with the four inside that set of checks) check against 700-digit references at 468–470 digits.

w	$\mathbf{a}$	symbolic derivation	numerical precision (digits)	agreement (digits)	lattice-sum check (digits), tail slope	deepest tw
7	$(2,2,2,1)$	✓	290	289	22.2 / 0.938	13
7	$(3,2,1,1)$	✓	290	288	20.9 / 0.925	13
7	$(4,1,1,1)$	✓	290	289	21.5 / 0.920	13
8	$(2,2,2,2)$	✓	290	289	26.7 / 0.937	15
8	$(3,2,2,1)$	✓	290	288	20.2 / 0.922	15
8	$(3,3,1,1)$	✓	290	288	19.6 / 0.907	15
8	$(4,2,1,1)$	✓	290	288	18.1 / 0.908	15
8	$(5,1,1,1)$	✓	290	289	19.2 / 0.904	15
9	$(3,2,2,2)$	✓	560	470	24.3 / 0.922	17
9	$(3,3,2,1)$	✓	560	470	17.6 / 0.908	17
9	$(4,2,2,1)$	✓	560	470	17.8 / 0.905	17
9	$(4,3,1,1)$	✓	560	470	17.1 / 0.889	17
9	$(5,2,1,1)$	✓	560	469	15.4 / 0.892	17
9	$(6,1,1,1)$	✓	560	470	16.9 / 0.888	17
10	$(3,3,2,2)$	✓	560	470	22.5 / 0.909	19
10	$(3,3,3,1)$	✓	700	470	16.2 / 0.906	19
10	$(4,2,2,2)$	✓	700	470	22.6 / 0.905	19
10	$(4,3,2,1)$	✓	700	470	15.7 / 0.891	19
10	$(4,4,1,1)$	✓	700	469	14.6 / 0.872	19
10	$(5,2,2,1)$	✓	700	470	15.6 / 0.889	19
10	$(5,3,1,1)$	✓	700	470	15.0 / 0.873	19
10	$(6,2,1,1)$	✓	700	469	13.6 / 0.876	19
10	$(7,1,1,1)$	✓	700	470	14.5 / 0.872	19
11	$(3,3,3,2)$	✓	700	470	21.0 / 0.907	21
11	$(4,3,2,2)$	✓	700	470	19.4 / 0.892	21
11	$(4,3,3,1)$	✓	700	470	13.5 / 0.891	21
11	$(4,4,2,1)$	✓	700	470	13.4 / 0.873	21
11	$(5,2,2,2)$	✓	700	470	20.2 / 0.889	21
11	$(5,3,2,1)$	✓	700	469	13.4 / 0.874	21
11	$(5,4,1,1)$	✓	700	470	12.4 / 0.856	21
11	$(6,2,2,1)$	✓	700	470	13.3 / 0.873	21
11	$(6,3,1,1)$	✓	700	469	12.6 / 0.857	21
11	$(7,2,1,1)$	✓	700	469	11.4 / 0.860	21
11	$(8,1,1,1)$	✓	700	470	12.2 / 0.856	21
12	$(3,3,3,3)$	✓	700*	468	–	23
12	$(4,3,3,2)$	✓	700*	469	–	23
12	$(4,4,2,2)$	✓	700*	469	–	23
12	$(4,4,3,1)$	✓	700*	468	–	23
12	$(5,3,2,2)$	✓	700*	468	–	23
12	$(5,3,3,1)$	✓	700*	469	–	23
12	$(5,4,2,1)$	✓	700*	469	–	23
12	$(5,5,1,1)$	✓	700*	468	–	23
12	$(6,2,2,2)$	✓	700*	469	–	23
12	$(6,3,2,1)$	✓	700*	469	–	23
12	$(6,4,1,1)$	✓	700*	469	–	23
12	$(7,2,2,1)$	✓	700	468	11.0 / 0.857	23
12	$(7,3,1,1)$	✓	700	470	10.3 / 0.841‡	23
12	$(8,2,1,1)$	✓	700	469	9.2 / 0.844‡	23
12	$(9,1,1,1)$	✓	700	469	9.8 / 0.839‡	23

Table 4: Each Laurent coefficient of the 49 four-edge functions $C_{\mathbf{a}}$, $7 \le w \le 12$, follows from the exact reduction of Section 3 (✓) and agrees with its numerical value to the number of digits in the agreement column (worst nonzero coefficient of the function). For the 38 functions with a direct lattice-sum evaluation (Section D.1) the next column gives the worst held-out agreement at $\tau_2 = 15$ from lattice sums evaluated to 80 digits, and the ratio of the fitted tail exponent to $-2\pi\tau_2$ from $\tau_2 \in \{8, 12\}$. Low entries (‡: slope below 0.85) come from the coefficients c_k with the most negative k, where dividing by y^k to isolate c_k multiplies the $e^{-2\pi\tau_2}$ tail by the large factor y^{-k}; every coefficient with fewer than 15 held-out digits agrees with its 700-digit sector-sum value to ≥ 468 digits. The eleven rows marked 700* have no lattice-sum entry, and their numerical values come from a dedicated 700-digit sector-sum evaluation, with which the exact coefficients agree to 468–469 digits per function. "Deepest tw" is $2w - 1$, the transcendental weight of c_{1-w}.

The confinement statement of Proposition 1 for the forty-nine functions, with the odd-zeta form of every coefficient other than c_{3-w}, is read off the exact reduction coefficient by coefficient. The statement is exhaustive, not sampled: the exact symbolic re-derivation of the atlas closes every coefficient of all forty-nine functions and reproduces every numeric reference value with zero mismatches, so the confinement is read off all 740 coefficients in exact arithmetic. In particular, the re-derivation reproduces the nine $C_{2,2,2,2}$ coefficients of Section 4.1, the transcendental-weight-17 coefficient of $C_{3,3,2,2}$, and the full $C_{2,2,2,1}$ row including its vanishing coefficients.

D.3 The coefficients of transcendental weights 19 and 21

The quotient ranks $1, 2, 2, 4, 5, 6$ of (46) refer to the deep coefficients written in terms of the $\zeta_{\mathrm{sv}}(a, b, c)$ of Appendix C and projected onto their classes modulo products. Among the $\zeta_{\mathrm{sv}}(a, b, c)$ themselves, integer-relation searches on their numerical values find 63 rational relations modulo products of odd zeta values, each holding to ≥ 695 digits, which leave $D_{2w-3,3}$ independent classes at each of the six weights. The searches return no further relation up to coefficient height 10^{15} in the ζ_2-free sector.

We call a function whose deep coefficient c_{3-w} has a nonzero depth-three part a carrier. There are eleven carriers at transcendental weight 19 and fifteen at weight 21. Whether the closure of each carrier lies in the single-valued span is decided by the exact rational decomposition of Section 4.5; the numerical agreements that follow only check those decompositions. All eleven weight-19 deep coefficients pass, and each closure is verified against 700- or 560-digit atlas values computed independently of the exact reduction that produced the closures, with agreement of at least 359 digits. All fifteen weight-21 deep coefficients pass the identical membership test, and each is checked against an independent 700-digit sector-numeric reference, with agreement of at least 468 digits.

Among the eleven weight-19 carriers, the closure of $C_{8,1,1,1}$, whose five depth-three coordinates are given in (47), agrees with its 700-digit sector-sum evaluation to 470 digits. This closure is the coefficient of y^{-8} in (76), and one exact decomposition of it in terms of the weight-19 values of Appendix C reads

$$\begin{aligned} c_{-8}(C_{8,1,1,1}) \;=\; & \tfrac{1035353002567}{2904224526773504}\,\zeta_{\mathrm{sv}}(11,3,5) + \tfrac{15632744128772707}{3717407394270085120}\,\zeta_{\mathrm{sv}}(13,3,3) \\ & + \tfrac{332095881374857}{1858703697135042560}\,\zeta_{\mathrm{sv}}(3,11,5) + \tfrac{3105372750663279}{675892253503651840}\,\zeta_{\mathrm{sv}}(3,13,3) \\ & + \tfrac{9191194867742227}{1858703697135042560}\,\zeta_{\mathrm{sv}}(3,3,13) - \tfrac{3242382217911}{47966547022839808}\,\zeta_{\mathrm{sv}}(3,5,11) \\ & + \tfrac{22488039}{119057776640}\,\zeta_{\mathrm{sv}}(3,7,9) - \tfrac{1831621015863949}{7434814788540170240}\,\zeta_{\mathrm{sv}}(5,11,3) \\ & - \tfrac{62658272742847}{135178450700730368}\,\zeta_{\mathrm{sv}}(5,3,11) + \tfrac{1022409}{5411717120}\,\zeta_{\mathrm{sv}}(5,5,9) \\ & + \tfrac{3546837}{23811555328}\,\zeta_{\mathrm{sv}}(7,3,9)\,. \end{aligned} \tag{69}$$

The decomposition (69) is not unique, because the ζ_{sv} values of weight 19 obey the rational relations modulo products counted above. Substituting the formulas of Appendix C into (69) and reducing modulo the double-shuffle relations of depth at most three reproduces the coefficient of y^{-8} in (76). The other twenty-five decompositions, whose numbers of ζ_{sv} terms and of odd-zeta monomials are given in Section 4.5, are listed with this one in the ancillary files described in Appendix E, each in a form that can be checked by substitution. Table 5 gives the number of carriers entering each rank, and Table 6 compares the membership outcomes with the range of total modular weight through which the constructions of [22] account for single-valued coefficients (Section 2.3).

tw	functions	rank	$D_{\mathrm{tw},3}$	independent references
11	2	1	1	complete
13	5	2	2	complete
15	6	2	2	complete
17	7	4	4	complete
19	11	5	5	complete
21	15	6	6	complete

Table 5: At each of the six odd transcendental weights $\mathrm{tw} = 2w - 3$ the quotient rank of the deep coefficients c_{3-w}, computed by exact rational elimination on the closures projected onto the depth-three single-valued quotient basis of Appendix C, equals the conjectural Broadhurst–Kreimer depth-three dimension $D_{\mathrm{tw},3}$ of (14). These dimensions bound the rank from above (Section 4.6). "Functions" is the number of carriers whose closure at weight tw enters the rank. "Complete" means that every such closure has an independent numerical check: at $\mathrm{tw} \leq 17$ the closure is obtained both by the numerical recognition of Section D.1 and by the exact reduction, and agrees with the numerical values to 496–499 digits at $\mathrm{tw} = 17$; at $\mathrm{tw} = 19$ each closure from the exact reduction agrees with an independent numerical value to at least 359 digits; and at $\mathrm{tw} = 21$ each of the fifteen closures agrees with an independent 700-digit evaluation to at least 468 digits.

w	carrier slot	tw of c_{3-w}	modular wt. $2w$ vs. published	occupancy (rank/BK)	single-valuedness of closed carriers
7	c_{-4}	11	14: below onset	1/1	holds (consistency)
8	c_{-5}	13	16: below onset	2/2	holds (consistency)
9	c_{-6}	15	18: constructed	2/2	holds (consistency)
10	c_{-7}	17	20: constructed	4/4	holds (consistency)
11	c_{-8}	19	22: past	5/5	tested: holds (11/11)
12	c_{-9}	21	24: past	6/6	tested: holds (15/15)

Table 6: Only the last two rows test Zerbini's conjecture beyond the published constructions; in the first four the outcome is the one those constructions lead one to expect. After w, the columns give the coefficient c_{3-w} carrying the depth-three part, its transcendental weight $2w-3$, and the total modular weight $2w$ of the iterated Eisenstein integrals that enter it. The modular weight $2w$ is compared with the published range (Section 2.3), in which new periods first enter at total modular weight 18 and the constructions of [22] reach total modular weight 20; this comparison makes the outcome a consistency check at $w \leq 10$ and a test at $w = 11, 12$. The last two columns give the depth-three single-valued quotient rank of the carriers against the conjectural Broadhurst–Kreimer dimension, and the outcome of the exact membership test, with the number of carriers in parentheses.

D.4 The membership criterion for the derivation images

Let $\mathcal{C}_v$ denote the exact coefficient values $c_k(C_{\mathbf{b}})$ of transcendental weight v over the forty-nine functions (as in Section 5.2), $\mathcal{O}_v$ the monomials in odd zeta values of total weight v, and $\mathcal{Q}_v \subset \mathcal{O}_v$ those with at least two factors. Let $\mathcal{T}_v$ denote the products $\zeta_{\mathrm{sv}}(a,b,c)\,\mu$ of a value of Appendix C

n	formulas	images	zero	nonzero, in span	outside span
The 61 *formulas of Appendix C extracted from* 700*-digit numerics*					
11	3	12	7	5	0
13	6	30	16	14	0
15	10	60	28	32	0
17	15	105	43	62	0
19	16	128	44	84	0
21	11	99	40	59	0
all	61	434	178	256	0
The 22 *formulas completed algebraically at weights* 19 *and* 21					
19	5	40	17	23	0
21	17	153	50	103	0
all	22	193	67	126	0

Table 7: All 627 derivation images of the 83 values $\zeta_{\mathrm{sv}}(a,b,c)$ of weights 11–21 satisfy (50), as (49) requires of correct formulas; accordingly the last column is zero throughout. Entries are counts per weight n. Each value has one image $D_m\,\zeta_{\mathrm{sv}}(a,b,c)$ for each odd m with $3 \le m \le n-2$, which either vanishes identically (zero) or is a nonzero element of $\mathcal{A}_{n-m}$ (in span), decided relative to the fixed basis for π_m, $m \ge 11$. Membership was tested in the span $G_A(n-m)$ of (70), and for every nonzero image the coefficient values in $\mathcal{C}_{n-m}$ alone suffice (Section 5.2). The 61 extracted and the 22 derived formulas of Appendix C (434 and 193 images) are tabulated separately.

of weight $w' < v$ with $\mu \in \mathcal{O}_{v-w'}$. We tested each derivation image $D_m\,\zeta_{\mathrm{sv}}(a,b,c)$, of weight $v = a+b+c-m$, for membership in two spans,

$$G_A(v) = \mathrm{span}_{\mathbb{Q}}\big(\mathcal{C}_v \cup \mathcal{Q}_v \cup \mathcal{T}_v\big), \qquad G_B(v) = G_A(v) + \mathrm{span}_{\mathbb{Q}}\big\{\zeta_2^j \mu :\ \mu \in \mathcal{O}_{v-2j},\ j \ge 1\big\}. \quad (70)$$

The even-weight values in $\mathcal{C}_v$ contain no power of ζ_2. Hence membership in G_A and membership in G_B coincide for every vector free of π^2, and every image is such a vector (Section 5.2). The counts of Table 7 were obtained with G_A and hold unchanged for G_B. Every nonzero image is, moreover, a rational combination of the coefficient values in $\mathcal{C}_v$ alone (Section 5.2).

For each of eight deliberately corrupted coefficients of weights 15–21, with one depth-carrying term perturbed by 10^{-3} (in one case by $1/97$), at least one image fell outside G_A, with a remainder along depth-two irreducibles and powers of π^2. Every genuine image tested at the wrong lower weight was likewise rejected. One input with a perturbed ζ_2 coefficient was rejected under G_A but accepted under G_B, and we therefore adopt the smaller, π^2-free span G_A as the criterion. One structural limitation remains, noted before the test ran: a corruption of a product-of-primitives coefficient moves every derivation image along product directions only, by the Leibniz rule, and no image-level span test can detect it; the controls therefore target the depth-carrying coefficients, which is where the closure statement lives.

D.5 Comparison with published values

Table 8 collects every published value we could locate that can be compared with one of ours: in its upper block those that agree with ours, and in its lower block the three places, stated in Section 6.3, at which we correct a published formula. All three corrections are transcription-level

defects (a sign, a mis-weighted term, a flipped label), each established by independent routes; to our knowledge no published erratum exists for any of the three and no citing paper mentions them (the most recent citer, the five-point extension of Claasen and Doroudiani [40], extends the algorithm of [23] without remark), and the arXiv listings of the two corrected papers carry no erratum.

Two of the four weight-seven dihedral identities of D'Hoker and Kaidi [24, eq. (3.17)] involve members of the family and constrain the polynomials (38)–(40). Ref. [24] proves that the left-hand sides of

$$\begin{aligned} &7\,C_{2,2,2,1} - 21\,E_3E_4 - 14\,C_{3,2,2} - 28\,C_{4,2,1} + 31\,E_7 \;=\; -\tfrac{1}{36}\,\zeta_7\,, \\ &28\,C_{4,1,1,1} + 28\,C_{2,2,2,1} + 84\,C_{3,2,1,1} - 168\,E_2E_5 - 252\,E_3E_4 \\ &\qquad - 294\,C_{5,1,1} + 105\,C_{3,2,2} - 378\,C_{4,2,1} + 654\,E_7 \;=\; \tfrac{3}{4}\,\zeta_7 \end{aligned} \tag{71}$$

are independent of τ, leaving the two constants undetermined. When we insert (38)–(40), the Laurent polynomials of the three-edge functions [11] and those of the E_w, the coefficients of the thirteen powers y^k with $k \neq 0$ cancel on both left-hand sides, and the constant terms take the two values shown.

For correction (i), the analytic continuation of the exact eq. (5.4) of [13] described in Section 6.3 contradicts the even-w sign of eq. (5.7) and agrees with the odd-w branch and with eqs. (5.6), (5.8) as published. Direct quadrature of $\int E_2^{*2}E_4^*$ agrees with eq. (5.4) to 13 digits. The $\ln L$ coefficient of $\int E_4C_{2,1,1}$ (Section 6.3) would be $61\pi\zeta_7/226800$ with the published sign, whereas a separate evaluation gives $47\pi\zeta_7/226800$, and with the corrected sign the two agree to 104 digits. For correction (ii), the term in question contains seven Green functions, whereas every other term of both published combinations of [23] contains eight, and the parallel β^{eqv} form of the same $\sigma_2\sigma_3^2$ combination given there has no weight-seven part. For correction (iii) we fixed the conventions of [23] by its weight-six trihedral example, which reproduces Zerbini's $d_{2,2,2}$ exactly in all 8 coefficients, and by its weight-eight zero modes (53); in these conventions the $C_{2,1,1}$ example of [23] yields the Laurent polynomial of $C_{2,1,1}$ only after its label is corrected as in Table 8.

E Laurent polynomials of the four-edge family

In this appendix we list, in part here and in full in the ancillary files, the Laurent polynomials (2) of the two-vertex four-edge functions C_{a_1,a_2,a_3,a_4} of (1) at weights $7 \le w \le 12$, in the variable $y = \pi\,\mathrm{Im}\,\tau$. Four of them are given in the main text, $C_{2,2,2,2}$ in (31) with its coefficient c_{-5} in (33) and (35), and the weight-seven functions $C_{2,2,2,1}$, $C_{3,2,1,1}$ and $C_{4,1,1,1}$ in (38)–(40). The ancillary files accompanying this paper contain all forty-nine Laurent polynomials, one entry per nonzero coefficient (the 740 coefficients of Figure 1 other than those that vanish identically, marked zero there), with every entry an explicit element of the ring of multiple zeta values, obtained by the exact reduction of Section 3.3. The entries are given in machine-readable form, as plain text and as JSON, and, for the forty-five functions not displayed in the main text, in typeset form as well. The files also include the 83 formulas of Appendix C in the same three forms, each labeled as extracted from numerics or derived algebraically; a stand-alone supplement in which the Laurent polynomials and these formulas are typeset together; and a description of the conventions and of the consistency checks applied to the entries. They also contain the twenty-six single-valued decompositions of Section 4.5, one of which is written out in Eq. (69), together

Agreements quantity	this work	published	agreement
$C_{4,1,1,1}$: c_1	$\zeta(3)^2/630$	$\zeta(3)^2/630$	exact; 287 digits
$C_{5,1,1,1}$: c_1	$-\zeta(7)/9450$	$-\zeta(7)/9450$	exact; 287 digits
$C_{6,1,1,1}$: c_1	$-\zeta(3)\zeta(5)/12600$	$-\zeta(3)\zeta(5)/12600$	exact; 287 digits
$C_{7,1,1,1}$: c_1	$13\,\zeta(9)/1108800$	$13\,\zeta(9)/1108800$	exact; 457 digits
$C_{k,1,1,1}$: c_w, $k = 4..7$	e.g. $\frac{566}{638512875}$ (k=4)	from printed seed	exact ($k\leq$6); 457 digits (k=7)
$D_5 = C_{1,1,1,1,1}$: all 8 slots	five-edge chain	full Laurent polynomial	$\geq$ 78 digits per slot
$D_4 = C_{1,1,1,1}$: all slots	per-run control	full Laurent polynomial	exact; evaluator 40–49 digits
$C_{2,1,1}$: all 5 slots	asymptotic theorem	full Laurent polynomial	101 digits per slot
$\int_{\mathcal{M}_L} E_2\, C_{2,1,1}$	$(c_0, \ln L)$ pair	pair, independent method	102 digits on both
$\mathcal{A}_{(p,q)}$, wt ≤ 6 (7 printed)	our evaluators	printed integrals	100 digits each
$D_{2,2,2}$ (trihedral): all 8 slots	printed β^{eqv} route	Zerbini's $d_{2,2,2}$	exact, 8/8 slots
$C_{2,1,1,1,1} = d_{1,1,4}$ (five-edge): all 10 slots	five-edge scan	Zerbini's $d_{1,1,4}$	exact, 10/10 slots
Discrepancies place	as printed	this work	established by
D'Hoker–Geiser 2110.06237, Thm. 5.1, eq. (5.7), w even	$(-1)^{\lfloor (w+1)/2 \rfloor}$	$(-1)^{\lfloor w/2 \rfloor + 1}$	caption (i): three routes
Claasen–Doroudiani 2502.05531, $\sigma_2\sigma_3^2$ combination (arXiv source, l. 2022)	$+\frac{1}{45360}\, C^{+}{}_{[1111]}^{[1111]}\,\mathrm{Tri}[1\|1\|1]$	7 edges in an 8-edge sum; term excluded	caption (ii): weight count
Claasen–Doroudiani 2502.05531, $C_{2,1,1}$ example (arXiv source, l. 1793)	$-18\,\beta^{\mathrm{eqv}}{}_{[4\ 4]}^{[0\ 2]}$	$-18\,\beta^{\mathrm{eqv}}{}_{[4\ 4]}^{[2\ 0]}$	caption (iii): Laurent closure

Table 8: Our values agree with every published value we could locate ($y = \pi\tau_2$; c_k multiplies y^k). At three places a published expression contains a transcription-level defect, which we correct. *Upper block*: agreement is exact where both sides are closed forms and holds to the stated number of digits where one side is numerical. Sources: $C_{k,1,1,1}$, D'Hoker–Geiser [13]; D_5, D_4, D'Hoker–Green–Vanhove [5]; $C_{2,1,1}$, $\int E_2 C_{2,1,1}$, $\mathcal{A}_{(p,q)}$, D'Hoker–Green [41]; $D_{2,2,2}$, Claasen–Doroudiani's β^{eqv} expansion [23] with the constants of Dorigoni et al. [22] against Zerbini's $d_{2,2,2}$ [2]; $C_{2,1,1,1,1}$, Zerbini's $d_{1,1,4}$ [2]. Apart from these $C_{k,1,1,1}$ coefficients, no Laurent coefficient of $C_{2,2,2,2}$ or of any other four-edge function at weights 8–12 has been published. *Lower block*: the corrections of Section 6.3, established (i) by analytic continuation of eq. (5.4) of [13], by direct quadrature, and by the $\ln L$ coefficient of $\int E_4 C_{2,1,1}$, (ii) by the Green-function count of the term, (iii) by the Laurent polynomial of the $C_{2,1,1}$ example.

with the ζ_{sv} values and double-shuffle reductions with which each is verified by substitution, and the f-alphabet images of the ring-basis generators computed in Appendix C.2.

Below we display the four functions of weight eight other than $C_{2,2,2,2}$ and the weight-eleven function $C_{8,1,1,1}$ of Section 4.7. Powers of y absent from a display or from the file entries of a function have vanishing coefficient. Multiple zeta values follow the convention (8), with n_1 attached to the smallest summation index, and even zeta values are written as rational multiples of powers of ζ_2. The coefficients c_k with $k \neq 3 - w$ are polynomials in odd zeta values (Proposition 1). The coefficient c_{3-w}, of transcendental weight $2w-3$, has depth at most three, and we give it reduced modulo the regularized double-shuffle relations of depth at most three. The reduced monomials that occur in the family are products of at most three odd zeta values, products $\zeta_2^j\,\zeta_{2n+1}$, products of one odd zeta value with a depth-two generator $\zeta(3,5)$, $\zeta(3,7)$, $\zeta(3,9)$, $\zeta(3,11)$, $\zeta(5,9)$, $\zeta(3,13)$ or $\zeta(5,11)$, and the depth-three generators (30).

The formulas of Appendix C give each $\zeta_{\mathrm{sv}}(a,b,c)$ as $2\,\zeta(a,b,c)$ plus a correction in ordinary multiple zeta values. One passes from an expression of Section 4 for a coefficient c_{3-w} in terms of single-valued multiple zeta values, such as (33), to the form used in this list by substituting these

formulas and reducing with the double-shuffle relations. For the $\zeta_{\mathrm{sv}}(a,b,c)$ that enter Table 1 the corrections consist of products alone. Hence, for each function in that table, the coefficient of a depth-three generator $\zeta(a,b,c)$ of (30) in its reduced Laurent polynomial is twice the coordinate on $\zeta_{\mathrm{sv}}(a,b,c)$ tabulated there. In (35) the coefficient c_{-5} of $C_{2,2,2,2}$ is written with $\zeta(5,3,5)$ in place of the generator $\zeta(3,5,5)$ of (30); at weight thirteen the double-shuffle relations express $\zeta(5,3,5)$ through $\zeta(3,5,5)$, ζ_{13} and products. At transcendental weight 19 the depth-three part of the deep coefficient of $C_{8,1,1,1}$ is written in (47) directly in terms of the generators (30), and the five rational coordinates there coincide with the coefficients of those generators in (76).

Weight 8

The four functions of weight 8 other than $C_{2,2,2,2}$ of (31), with $-7 \le k \le 8$ in (2), are

$$\begin{aligned}
C_{5,1,1,1}(\tau) \;=\; & \tfrac{584}{6512831325}\, y^8 + \tfrac{2}{93555}\,\zeta_3\, y^5 - \tfrac{23}{623700}\,\zeta_5\, y^3 + \tfrac{1}{6300}\,\zeta_3^2\, y^2 - \tfrac{1}{9450}\,\zeta_7\, y \\
& - \tfrac{1}{1260}\,\zeta_3\zeta_5 + \tfrac{23}{20160}\,\zeta_9\, y^{-1} + \tfrac{1}{960}\big(2\,\zeta_3\zeta_7 + \zeta_5^2\big) y^{-2} + \tfrac{607}{3840}\,\zeta_{11}\, y^{-3} - \tfrac{1}{128}\big(57\,\zeta_3\zeta_9 + 7\,\zeta_5\zeta_7\big) y^{-4} \\
& + \tfrac{1}{44800}\big(56098\,\zeta_{13} + 13650\,\zeta_3^2\zeta_7 + 4200\,\zeta_3\zeta_5^2 + 51975\,\zeta_2\zeta_{11} - 960\,\zeta_2^3\zeta_7 + 420\,\zeta_5\,\zeta(3,5) \\
& \qquad - 1176\,\zeta(3,5,5) + 1050\,\zeta(3,3,7)\big) y^{-5} - \tfrac{1}{512}\big(993\,\zeta_3\zeta_{11} + 783\,\zeta_5\zeta_9 + 234\,\zeta_7^2\big) y^{-6} \\
& + \tfrac{3267}{1024}\,\zeta_{15}\, y^{-7} + O(e^{-2y})\,,
\end{aligned} \tag{72}$$

$$\begin{aligned}
C_{4,2,1,1}(\tau) \;=\; & \tfrac{38446}{488462349375}\, y^8 + \tfrac{1}{212625}\,\zeta_3\, y^5 + \tfrac{13}{170100}\,\zeta_5\, y^3 - \tfrac{1}{18900}\,\zeta_3^2\, y^2 - \tfrac{1}{37800}\,\zeta_7\, y \\
& + \tfrac{1}{756}\,\zeta_3\zeta_5 + \tfrac{169}{30240}\,\zeta_9\, y^{-1} - \tfrac{1}{1440}\big(22\,\zeta_3\zeta_7 + 3\,\zeta_5^2\big) y^{-2} + \tfrac{23}{1280}\,\zeta_{11}\, y^{-3} \\
& + \tfrac{1}{384}\big(93\,\zeta_3\zeta_9 + 19\,\zeta_5\zeta_7\big) y^{-4} \\
& + \tfrac{1}{537600}\big(-896147\,\zeta_{13} - 142800\,\zeta_3^2\zeta_7 + 84000\,\zeta_3\zeta_5^2 + 508200\,\zeta_2\zeta_{11} + 6720\,\zeta_2^2\zeta_9 \\
& \qquad - 7680\,\zeta_2^3\zeta_7 + 6720\,\zeta_5\,\zeta(3,5) - 8736\,\zeta(3,5,5) + 8400\,\zeta(3,3,7)\big) y^{-5} \\
& + \tfrac{1}{1024}\big(352\,\zeta_3\zeta_{11} + 428\,\zeta_5\zeta_9 + 51\,\zeta_7^2\big) y^{-6} - \tfrac{675}{4096}\,\zeta_{15}\, y^{-7} + O(e^{-2y})\,,
\end{aligned} \tag{73}$$

$$\begin{aligned}
C_{3,3,1,1}(\tau) \;=\; & \tfrac{302}{3907698795}\, y^8 + \tfrac{4}{893025}\,\zeta_3\, y^5 - \tfrac{1}{42525}\,\zeta_5\, y^3 + \tfrac{11}{5400}\,\zeta_7\, y - \tfrac{2}{945}\,\zeta_3\zeta_5 \\
& - \tfrac{313}{30240}\,\zeta_9\, y^{-1} + \tfrac{1}{480}\big(14\,\zeta_3\zeta_7 + 13\,\zeta_5^2\big) y^{-2} + \tfrac{13}{192}\,\zeta_{11}\, y^{-3} - \tfrac{1}{16}\big(3\,\zeta_3\zeta_9 + 15\,\zeta_5\zeta_7\big) y^{-4} \\
& + \tfrac{1}{67200}\big(1845851\,\zeta_{13} + 22050\,\zeta_3^2\zeta_7 - 25200\,\zeta_3\zeta_5^2 - 848925\,\zeta_2\zeta_{11} - 50400\,\zeta_2^2\zeta_9 \\
& \qquad + 2880\,\zeta_2^3\zeta_7 - 26460\,\zeta_5\,\zeta(3,5) - 1512\,\zeta(3,5,5) - 3150\,\zeta(3,3,7)\big) y^{-5} \\
& - \tfrac{1}{256}\big(55\,\zeta_3\zeta_{11} + 1031\,\zeta_5\zeta_9 + 672\,\zeta_7^2\big) y^{-6} + \tfrac{4989}{1024}\,\zeta_{15}\, y^{-7} + O(e^{-2y})\,,
\end{aligned} \tag{74}$$

$$\begin{aligned}
C_{3,2,2,1}(\tau) \;=\; & \tfrac{1303}{18091198125}\, y^8 + \tfrac{1}{14175}\,\zeta_5\, y^3 + \tfrac{17}{1440}\,\zeta_9\, y^{-1} - \tfrac{3}{160}\,\zeta_5^2\, y^{-2} - \tfrac{31}{320}\,\zeta_{11}\, y^{-3} \\
& + \tfrac{7}{8}\,\zeta_5\zeta_7\, y^{-4} \\
& + \tfrac{1}{19200}\big(-433651\,\zeta_{13} + 207900\,\zeta_2\zeta_{11} + 15120\,\zeta_2^2\zeta_9 + 7560\,\zeta_5\,\zeta(3,5) + 1512\,\zeta(3,5,5)\big) y^{-5} \\
& + \tfrac{1}{256}\big(711\,\zeta_5\zeta_9 + 504\,\zeta_7^2\big) y^{-6} - \tfrac{845}{256}\,\zeta_{15}\, y^{-7} + O(e^{-2y})\,.
\end{aligned} \tag{75}$$

Weight 11: the function $C_{8,1,1,1}$

The function $C_{8,1,1,1}$ of Section 4.7, with $-10 \le k \le 11$ in (2), is

$$
\begin{aligned}
C_{8,1,1,1}(\tau) \;=\; & \tfrac{9850034}{105661733105053125}\, y^{11} + \tfrac{3617}{162820783125}\,\zeta_3\, y^{8} - \tfrac{251}{6578617500}\,\zeta_5\, y^{6} \\
& + \tfrac{1}{6081075}\,\zeta_3^2\, y^{5} - \tfrac{569}{5108103000}\,\zeta_7\, y^{4} - \tfrac{691}{851350500}\,\zeta_3\zeta_5\, y^{3} + \tfrac{809}{681080400}\,\zeta_9\, y^{2} \\
& + \tfrac{1}{997920}\big(2\,\zeta_3\zeta_7 + \zeta_5^2\big) y - \tfrac{247}{39916800}\,\zeta_{11} - \tfrac{1}{201600}\big(\zeta_3\zeta_9 + \zeta_5\zeta_7\big) y^{-1} + \tfrac{5}{193536}\,\zeta_{13}\, y^{-2} \\
& + \tfrac{1}{161280}\big(2\,\zeta_3\zeta_{11} + 2\,\zeta_5\zeta_9 + \zeta_7^2\big) y^{-3} - \tfrac{25}{258048}\,\zeta_{15}\, y^{-4} - \tfrac{1}{30720}\big(\zeta_3\zeta_{13} + \zeta_5\zeta_{11} + \zeta_7\zeta_9\big) y^{-5} \\
& + \tfrac{32261}{245760}\,\zeta_{17}\, y^{-6} - \tfrac{1}{4096}\big(1501\,\zeta_3\zeta_{15} + 247\,\zeta_5\zeta_{13} + 22\,\zeta_7\zeta_{11}\big) y^{-7} \\
& + \tfrac{1}{5365303623680000}\big(-34338640474763625\,\zeta_{19} + 1394048921265000\,\zeta_3^2\zeta_{13} \\
& \qquad + 501032381850000\,\zeta_3\zeta_5\zeta_{11} + 56980153230000\,\zeta_3\zeta_7\zeta_9 + 70475782377000\,\zeta_5^2\zeta_9 \\
& \qquad + 41261490270000\,\zeta_5\zeta_7^2 + 26976074447322000\,\zeta_2\zeta_{17} + 1658086803172800\,\zeta_2^2\zeta_{15} \\
& \qquad - 50338486598400\,\zeta_2^3\zeta_{13} - 40654705291200\,\zeta_2^4\zeta_{11} - 5937658272000\,\zeta_2^5\zeta_9 \\
& \qquad - 606699314688\,\zeta_2^6\zeta_7 + 204324120000\,\zeta_5\,\zeta(5,9) + 241405164000\,\zeta_5\,\zeta(3,11) \\
& \qquad + 1964832870000\,\zeta_7\,\zeta(3,9) - 2263204944000\,\zeta(5,3,11) + 5171166000\,\zeta(3,5,11) \\
& \qquad + 2297258964000\,\zeta(3,7,9) + 2299655358000\,\zeta(5,5,9) - 2228141916000\,\zeta(3,3,13)\big) y^{-8} \\
& - \tfrac{1}{32768}\big(58341\,\zeta_3\zeta_{17} + 48045\,\zeta_5\zeta_{15} + 18345\,\zeta_7\zeta_{13} + 3045\,\zeta_9\zeta_{11}\big) y^{-9} + \tfrac{906525}{262144}\,\zeta_{21}\, y^{-10} \\
& + O(e^{-2y})\,,
\end{aligned}
\tag{76}
$$

and the polynomials of the remaining forty functions, of weights nine through twelve, are given in the ancillary files.

F Five-edge functions

The sector $|S| = 5$ of Eq. (18), in which all five edges carry nonzero winding, has no Bernoulli factor, and the power counting used in the proof of Proposition 1 shows that this sector contributes only to the single power y^{4-w}. Its four-fold Mordell–Tornheim sums are the only possible source of depth four. Hence a five-edge function can carry multiple zeta values of depth four only in c_{4-w}.

We evaluate the Laurent polynomials of the twenty-nine five-edge functions of weights 5–11 numerically as in Appendix D (460 coefficients) and search each coefficient for multiple zeta values of depth up to four. For the coefficients not identified by this search we use the reduction of Section 3.3, which we have extended to five edges through transcendental weight 18. In this way we obtain exact expressions for fifty-two of the sixty-two coefficients not identified numerically, as rational combinations of odd-zeta products and depth-three values; each agrees with its numerical value to 289–460 digits. The remaining ten have odd transcendental weight 19, and by parity no depth-four irreducible occurs at odd weight. By weight and parity alone, depth four could occur only in the 130 coefficients of even transcendental weight ≥ 8. All 130 are therefore known in closed form, identified numerically or obtained exactly, and we find no depth-four value in any of them. In particular c_{4-w} is free of depth four in all twenty-nine functions (Table 9, first block). The Laurent polynomials of the melon $D_5 = C_{1,1,1,1,1}$ [5] and of Zerbini's $d_{1,1,4} = C_{2,1,1,1,1}$ [2] are known, and our values agree with both (Table 8).

Below c_{4-w}, the coefficients of the twenty-nine five-edge functions have the same form as the corresponding coefficients of the four-edge family (Table 9, second and third blocks). At

slot	tw	closed+open	coefficient content	example
the five-edge carrier slot (the slot the sector counting predicts for depth four)				
$k = 4 - w$	$2w - 4$	$29 + 0$	odd-zeta products in the 8 closed ($w \leq 9$); the 21 others exact: products+depth three; depth four in none	$c_{-2}(C_{2,1,1,1,1}) = \frac{11}{2}\,\zeta_3\zeta_5$
the tail below it, identical in shape in all 29 functions				
$k = 3 - w$	$2w - 3$	$19 + 10$	the four-edge deep-slot pattern: depth three from $w = 7$, exact at $w = 9, 10$; open only at $w = 11$ (tw = 19)	$c_{-3}(C_{2,1,1,1,1}) = -\frac{3}{2}\,\zeta_3^3 + \frac{79}{24}\,\zeta_9$
$k = 2 - w$	$2w - 2$	$29 + 0$	$\sum_{a+b=2w-2} r_{ab}\,\zeta_a\zeta_b$, a, b odd ≥ 3	$c_{-4}(C_{2,1,1,1,1}) = -\frac{9}{4}\,\zeta_3\zeta_7$
$k = 1 - w$	$2w - 1$	$29 + 0$	$r\,\zeta_{2w-1}$, $r \in \mathbb{Q}$	$c_{-5}(C_{2,1,1,1,1}) = \frac{45}{16}\,\zeta_{11}$, $c_{-10}(C_{3,3,3,1,1}) = \frac{9668925}{16384}\,\zeta_{21}$
the same tail in the four-edge family (products in all 38 functions whose coefficients were identified numerically)				
$k = 2 - w$	$2w - 2$	–	double odd-zeta products	$c_{-6}(C_{2,2,2,2}) = -\frac{405}{32}\zeta_5\zeta_9 - \frac{567}{64}\zeta_7^2$
$k = 1 - w$	$2w - 1$	–	single odd zeta	$c_{-7}(C_{2,2,2,2}) = \frac{6825}{512}\,\zeta_{15}$

Table 9: In all twenty-nine five-edge functions computed ($w = 5$–11) the one Laurent coefficient that can carry multiple zeta values of depth four, c_{4-w}, contains none. The three coefficients below it have the same form as the tail of the four-edge family. A slot k is the coefficient c_k of y^k in Eq. (2), of transcendental weight tw $= w - k$. Third column: how many of the 29 functions have c_k determined (closed) or not (open); in the fourth column, closed means identified numerically and exact means obtained by the reduction of Section 3.3 extended to five edges. The $C_{2,1,1,1,1}$ examples are coefficients of Zerbini's $d_{1,1,4}$ [2], with which our $C_{2,1,1,1,1}$ agrees exactly in all ten slots (Table 8). The exact rationals of the remaining five-edge coefficients are not printed; the ancillary files carry all forty-nine four-edge Laurent polynomials and the 83 formulas of Appendix C.

$k = 1 - w$ the coefficient is a rational multiple of ζ_{2w-1}, at $k = 2 - w$ a rational combination of products $\zeta_a\zeta_b$ with a, b odd, and at $k = 3 - w$, in the nineteen functions where it is determined, a sum of odd-zeta products and depth-three values, as in the four-edge c_{3-w}. Depth three is not confined to c_{3-w} at five edges, because there a sector with $|S| = 4$ contains one Bernoulli factor, from the fifth edge with zero winding, and contributes to several powers of y. Forty-six of the coefficients identified numerically have depth-three content; they occur in twenty-five of the twenty-nine functions, each has transcendental weight 11, 13, or 15, and only five of them are c_{3-w}. These coefficients are written in terms of ordinary multiple zeta values, and we have not checked whether their depth-three parts can be expressed through the single-valued multiple zeta values of Appendix C.

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