

Memory back-reaction in black-hole scattering at fifth post-Minkowskian order

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Abstract

As two black holes swing past each other, their gravitational waves leave test masses permanently displaced. This memory is a leading-order effect, but its back-reaction on the orbit enters at fifth post-Minkowskian order and second order in the mass ratio, where radiation reaction has not yet been computed. There a spurious divergence must cancel, which fixes the back-reaction up to a constant set by the propagator prescription for the zero-frequency gravitons. The integrals involve periods of a K3 surface found by Klemm, Naga, Sauer, and Plefka. We rederive this surface from the maximal cut and show that the divergence occurs at the leading Landau singularity of the cut. For the smallest integral family we compute boundary constants on three of its four leading top sectors and continue the K3 and Calabi–Yau periods to bound orbits. These results follow from maximal cuts and boundary limits, without an integration-by-parts reduction of the full four-loop integrand, and supply analytic structure and boundary data that radiation reaction at this order requires.

Contents

1	Introduction	2
2	Gravitational memory	4
2.1	Displacement and memory	4
2.2	Linear and nonlinear memory	5
2.3	Memory in the loop expansion	6
2.4	The zero-mode language	6
3	Post-Minkowskian integrals at fifth order	6
3.1	Kinematics and integral families	6
3.2	The four geometries	7
3.3	The memory-region integrals	9
4	Periods from maximal cuts	10
4.1	The Apéry K3 surface	10
4.2	The other three geometries	12

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5	The dissipative function space	13
5.1	Master-integral counts and parity	14
5.2	Response matrices	14
5.3	Boundary constants of PX2	16
5.4	Relations among boundary constants	18
5.5	K3 and Calabi–Yau sectors	20
6	Continuation to bound orbits	21
6.1	Bound kinematics and singular points	21
6.2	Periods on the bound arc	22
6.3	Bound-orbit observables	23
7	Evaluation of the memory-region integrals	24
7.1	Frequency representations	25
7.2	The value of c_M	27
7.3	The singular point $\gamma = 3$	27
8	Conclusions and outlook	28
A	Auxiliary-mass flow on worldline families	30
A.1	Boundary data on worldline families	30
A.2	Tests at lower orders	30
A.3	The solver	30
B	Beyond the maximal cut at 4PM	31
B.1	Function space and non-holomorphic components	31
B.2	A fixed point of the flow	32
B.3	Master ratios at leading order in ϵ	33
C	Landau analysis with linear propagators	33
C.1	Landau equations and first-type letters	33
C.2	The topology-41 alphabet	34
D	Numerical checks	35
D.1	Maximal cuts and period identities	35
D.2	Boundary constants	36

1 Introduction

Gravitational-wave memory is the permanent displacement a burst of radiation leaves behind in an array of freely falling detectors [1, 2, 3]: after the wave has passed, the strain does not return to zero. It is a leading-order effect, known since the 1970s, and its nonlinear part is sourced by the energy flux of the radiation itself. The same zero-frequency field that builds up the memory also acts back on the system that emits it. In the post-Minkowskian (PM) approach to two-black-hole scattering, observables are expanded in Newton’s constant G with exact velocity dependence. Radiation reaction on the orbit begins at third order in G [4, 5, 6, 7]. In the four-loop integrals at fifth order in G and second order in the mass ratio, Driesse et al. identified a memory region, in which the stored zero-frequency field re-enters the orbital dynamics [8].

The PM expansion, revived as a program a decade ago [9], has been developed in three formalisms. Scattering amplitudes matched to classical potentials and observables [10, 11] produced the conservative dynamics at third order (3PM) [12, 13] and at 4PM [14, 15]. Worldline effective field theory [16] reproduced the 3PM result [17] and gave the 4PM dynamics with full velocity dependence [18]. With worldline quantum field theory (WQFT) [19] the conservative dynamics has been computed to fifth order in G , initially at first order in the mass ratio ν [20] and recently at second [8]. Bern et al. have also given the potential-graviton contribution at that order as a velocity expansion [21]. The expansion in the mass ratio is the self-force (SF) expansion [22], whose second-order results for bound orbits match numerical relativity even at comparable masses [23, 24]. We write 5PM-2SF for fifth order in G at second self-force order (2SF).

The dissipative observables, the radiated energy and the radiation-reacted impulse, are known at 3PM from amplitudes and reverse unitarity [6, 25], from linear response [4], from the eikonal [5], and from WQFT [26, 7]. At 4PM they are known [27], including spin [28], but at 5PM only at first self-force order (1SF) [29]. Both Refs. [8, 21] name radiation reaction, radiated energy, and recoil at 5PM-2SF as the next step, and to our knowledge no results for them have appeared. The nearest results are in the extreme-mass-ratio limit, where the bremsstrahlung waveform is known through 8PM and the radiation-reacted deflection through 6PM [30], and where the WQFT diagrammatic treatment of angular-momentum loss has been set up [31].

In the memory region, which appears at 5PM-2SF besides the potential and tail regions, two graviton momenta are radiative while the net frequency delivered to the worldlines vanishes [8]. The impulse then depends on the $i0^+$ prescription adopted at zero frequency, and in the static limit two boundary integrals survive in the memory region. Cancellation of the poles in the dimensional regulator ϵ , and of a spurious divergence at relative Lorentz factor $\gamma = 3$, fixes the two integrals up to a single constant c_M [8]. Driesse et al. fix c_M by adopting a routing prescription: retarded propagators directed toward the three-graviton vertex, averaged under time reversal. Porto and Riva, analyzing the corresponding sector at fifth post-Newtonian order, argue instead for the Feynman prescription and observe that the rerouting would be incompatible with the conjecture that all π^2 terms arise from the potential region [32]. Dlapa, Kälín, Liu, and Porto note that the 5PM-2SF results of Driesse et al. are fixed only up to the two memory-region boundary integrals [33].

Through 3PM the two-body observables are polylogarithms, and at 4PM complete elliptic integrals appear, the periods of a K3 surface [34, 35]. At 5PM Frellesvig, Morales, and Wilhelm identified the first Calabi–Yau threefold in a post-Minkowskian integral [36]. Klemm, Nega, Sauer, and Plefka showed that the periods of this threefold govern the 5PM-1SF dissipative integrals and gave its Picard–Fuchs operator [37]. These periods enter the 1SF radiated energy and recoil [29]. At 2SF Klemm et al. also found a second K3 surface, governed by the Apéry operator, and a second threefold [37]. Brammer, Frellesvig, Morales, and Wilhelm then completed the loop-by-loop Baikov classification of Ref. [38]: across the four-loop families, conservative and dissipative, at every self-force order, exactly these two threefolds and two K3 surfaces occur [39]. In the conservative 5PM-2SF result of Ref. [8] the Apéry K3 period remains in the function space while the threefold integrals drop out. By contrast, Bern et al. argue that neither Calabi–Yau nor complete elliptic integrals contribute to conservative observables at this order [40]. Brammer et al. call for a bootstrap that uses the known geometries and their special functions in place of the integration-by-parts (IBP) reduction, which is already the bottleneck at 1SF [39]. The dissipative function space at 2SF, the boundary data for its differential equations, and its continuation to bound orbits have not been determined.

In this paper we show that $\gamma = 3$, the point in the scattering region at which the spurious divergence occurs, is the leading Landau singularity of the maximal cut of the K3 sector of the planar family. By an elementary factorization of the Baikov polynomial of Brammer et al. [39] we

map the cut surface of this sector onto the Apéry–Fermi K3 [41] and obtain its torus period in closed form. The Picard–Fuchs operator of this period is the Apéry operator, which Klemm et al. found from differential equations [37] and Duhr et al. from the maximal cut in other variables [42]. At $\gamma = 3$ the cut surface acquires a node on the integration torus, so the divergence that cancels under the retarded routing lies at a Landau singularity of the cut, a fact that alone does not fix the routing. The same construction determines, for each of the four geometries, the constant relating the maximal cut to the holomorphic period. For the 2SF threefold it also gives the Frobenius coefficients of the holomorphic period in closed form.

We take the four dissipative families to be those of the conservative calculation [8], with two-sided worldline cuts and both $i0$ routings of the linearized propagators retained, and give their first-stage (fixed-seeding) master-integral counts. We work in detail on PX2, the smallest family, and there on three of its four leading top sectors. On these sectors the auxiliary-mass flow [43, 44] maps boundary data to master integrals through response matrices, whose left null spaces give ten linear relations among those integrals. All ten are exact reduction identities that the first-stage reduction did not generate. At one reference kinematic point we compute, with Feynman graviton propagators, twelve of the master integrals in these relations; these values are boundary constants for the differential equations of the family. They lie in $i\pi^5 \mathbb{Q}[\ln 2, \gamma_E]$ up to real π^4 terms, apart from a few finite parts known only numerically, and satisfy those of the ten relations that involve only these twelve. We continue the Frobenius bases of the four Picard–Fuchs operators to bound kinematics and find the expected monodromies. We also evaluate the memory-region integrals in the frequency representation of Driesse et al., with their retarded routing as input, and find $c_M = 1$, in agreement with Ref. [8] and with Porto and Riva’s fifth-post-Newtonian implementation of the same routing [32].

Our results follow from maximal cuts, boundary limits, and first-stage or single-sector reductions, without a reduction of the full four-loop integrand to master integrals. We do not compute a 5PM observable: assembling the scattering angle, the radiation-reacted impulse, or the radiated energy at $O(G^5\nu^2)$ requires the coefficient of every master integral in the reduced integrand, and at 2SF those coefficients are available only from the complete IBP reduction of the four-loop integrand and have not been published. Such a computation also needs input that we supply in part: the normalization of the canonical bases in the geometric sectors, boundary constants for the differential equations, and, for bound orbits, the continued periods. After reviewing gravitational memory in Section 2, we introduce the integral families, geometries, and memory-region integrals in Section 3, derive the periods from maximal cuts in Section 4, and analyze the dissipative function space in Section 5. We then continue the periods to bound orbits in Section 6 and evaluate the memory-region integrals in Section 7.

2 Gravitational memory

In this section we review gravitational memory [45]. We then locate it in the four-loop integrals of the two-body problem at fifth post-Minkowskian order [8].

2.1 Displacement and memory

We take two free masses a distance L apart, suitably oriented transverse to a gravitational-wave burst of strain $h(t)$; their separation is then $L[1 + h(t)/2]$. We define the memory as the difference of two readings of this detector,

$$\Delta h = h(t_2) - h(t_1), \quad (1)$$

with t_1 before the burst arrives and t_2 after it has passed [1, 2, 3]. Once the flux dies away, the difference Δh in Eq. (1) becomes independent of t_2 . Thus the masses end at relative rest with a changed separation: a permanent offset of the radiation field, which an interferometer holding lock through the burst reads from then on.

A displaced test body alone does not signal memory: a magnet slid across a table moves a paperclip permanently through near-field forces, with nothing radiated. For a free charge q of mass m in an electromagnetic pulse $E(t)$ we integrate $m\ddot{x} = qE$ twice,

$$\Delta v = \frac{q}{m} \int_{t_1}^{t_2} dt E(t), \quad \Delta x = \frac{q}{m} \int_{t_1}^{t_2} dt (t_2 - t) E(t), \quad (2)$$

and if the pulse comes from a bounded source that returns to its initial state of motion, the first integral vanishes. The second is generically nonzero, and the charge ends displaced and at rest. Bieri and Garfinkle showed that Δv is nonzero only if the asymptotic state of the source changes [46], as for charges escaping to infinity in a flyby (a plasma that sloshes and settles back delivers no net kick). The electromagnetic analog of gravitational memory is therefore the velocity kick Δv . After the pulse the electromagnetic field relaxes to the Coulomb field of the settled charges, whereas in gravity the strain ends offset and the permanent change is in the metric.

2.2 Linear and nonlinear memory

Memory sourced by escaping matter exists in both electromagnetism and gravity. Zel’dovich and Polnarev found it in 1974 for stars on unbound orbits [1], and Braginsky and Thorne called such signals “bursts with memory” [2]. This linear memory, the analog of the electromagnetic velocity kick, occurs whenever the asymptotic momenta of the sources change. Gravity has in addition a nonlinear memory sourced by the energy flux of the waves themselves, found by Christodoulou [3] (the post-Newtonian derivations followed soon after [47, 48, 49]). We write it schematically as

$$\Delta h \sim \frac{4G}{r} \int dt \int d\Omega \frac{dE}{dt d\Omega} \times (\text{angular kernel}), \quad (3)$$

with r the distance to the source, $dE/dt d\Omega$ the energy flux of the radiation, and the angular kernel the transverse-traceless projection of Ref. [45]. Because photons are uncharged, electromagnetism has no counterpart of Eq. (3). In the scattering problem we study (unbound black holes), both kinds of memory are present, and for bound systems settling to a final black hole the nonlinear memory dominates [45].

In the frequency domain the step Δh becomes a $1/\omega$ pole,

$$\tilde{h}(\omega) \longrightarrow \frac{\Delta h}{i\omega} \quad (\omega \rightarrow 0), \quad (4)$$

and for a binary merger this low-frequency rise extends into the band of ground-based detectors. Lasky et al. estimated that Advanced LIGO at design sensitivity could detect memory at signal-to-noise ratio 3 by combining roughly 35 events with masses and distances like GW150914’s, and at 5 with about 90 [50]. Searches in the first LIGO–Virgo catalog (ten binary black hole mergers) [51] and in GWTC-5.0 [52, 53] find no decisive evidence yet. Such evidence would still require around two thousand detections. Pulsar timing arrays also search for memory, most recently in the NANOGrav 15-year data set [54]. Next-generation ground detectors and LISA should detect the memory of single events [55, 56].

2.3 Memory in the loop expansion

At 5PM-2SF the two-body observables are computed from four-loop integrals over graviton momenta [8]. Besides the potential and tail regions these integrals develop a memory region, in which two of the four gravitons are radiative, with zero net frequency delivered to the worldlines [8]. There the stored field, the Δh of Eq. (1), acts back on the orbit. The memory of the emitted radiation is known in the post-Minkowskian expansion from scattering amplitudes [57] and, in a multipole expansion through $O(G^5)$, from the one-loop bremsstrahlung waveform [58]. Here we study instead the four-loop integrals that contain the back-reaction of the memory on the scattering impulse. The zero-frequency limit of the memory region can be taken with retarded, advanced, or Feynman propagators on the two memory gravitons. The impulse depends on this choice even in dimensional regularization, and the cancellation conditions of Section 3.3 reduce the dependence at this order to a single constant c_M [8].

2.4 The zero-mode language

Since the memory Δh is the coefficient of the $1/\omega$ pole in Eq. (4), memory can equivalently be described through the zero-frequency mode of the field. The form of that pole is universal by Weinberg’s soft graviton theorem [59], and the flat metrics before and after the burst differ by a Bondi–Metzner–Sachs supertranslation [60, 61]. Memory, the soft theorem, and supertranslation symmetry are pairwise equivalent [62]. We use this language only to name the memory region and its zero-frequency gravitons, and the quantities we compute are defined at finite times.

3 Post-Minkowskian integrals at fifth order

We set up the kinematics and integral families at 5PM-2SF, recall the four geometries that occur through 5PM [37, 39], and introduce the two memory-region integrals of Driesse et al. [8] and the constant c_M .

3.1 Kinematics and integral families

The post-Minkowskian expansion describes the scattering of two black holes, modeled as point masses, in powers of G and to all orders in the velocity. An L -loop computation yields the $(L+1)$ PM correction to the impulse Δp^μ and to the scattering angle. The conservative dynamics is known through 5PM-2SF [18, 63, 20, 29, 8]. We use the single-scale kinematics of Refs. [18, 34, 39], in which the two bodies scatter with four-velocities u_1, u_2 ($u_i^2 = 1$) and momentum transfer q , with $u_i \cdot q = 0$. The masses factor out, and we set $q^2 = -1$, since the dependence on q^2 is restored by dimensional analysis. The integrals then depend on a single variable,

$$\gamma = u_1 \cdot u_2 = \frac{1 + x^2}{2x}, \quad (5)$$

the relative Lorentz factor, where $x = \gamma - \sqrt{\gamma^2 - 1}$ rationalizes $\sqrt{\gamma^2 - 1}$ and the scattering region is $0 < x < 1$.

Besides quadratic graviton propagators $1/k^2$ the integrals have linear worldline propagators $1/(2u_i \cdot k)$. In the worldline quantum field theory formalism [19, 7] L of them are replaced at L loops by delta functions $\delta(2k \cdot u_i)$. The delta functions also set the self-force (mass-ratio) order: an integral with n_i of them on worldline i is of order $\min(n_1, n_2)$ SF. Moreover, the integrals have

definite parity under $u_i \rightarrow -u_i$, which reverses all worldline frequencies, and the even and odd sectors decouple [37].

At 5PM-2SF the integrals form four families, P, PX1, PX2, and NP, called P, NP1, NP2, and NPM in Ref. [8]. All have four graviton loops between the two worldlines and differ in how the graviton lines cross. For the crossed family PX2, whose boundary constants we compute, the integrals are

$$I_{n_5 \dots n_{22}} = \int_{k_1 \dots k_4} \frac{\delta(D_1) \delta(D_2) \delta(D_3) \delta(D_4)}{D_5^{n_5} D_6^{n_6} \dots D_{22}^{n_{22}}},$$

$$\begin{aligned} D_1 &= 2k_1 \cdot u_1, & D_2 &= 2k_2 \cdot u_1, & D_3 &= 2k_3 \cdot u_2, & D_4 &= 2k_4 \cdot u_2, \\ D_5 &= 2k_1 \cdot u_2, & D_6 &= 2k_1 \cdot u_2 - 2k_2 \cdot u_2, & D_7 &= 2k_3 \cdot u_1, & D_8 &= 2k_3 \cdot u_1 - 2k_4 \cdot u_1, \\ D_9 &= (k_1 - k_2)^2, & D_{10} &= (k_1 - k_3)^2, & D_{11} &= (k_1 - k_4)^2, & D_{12} &= (k_2 - k_3)^2, \\ D_{13} &= (k_2 - k_4)^2, & D_{14} &= (k_3 - k_4)^2, & D_{14+j} &= (k_j + q)^2, & D_{18+j} &= k_j^2, \end{aligned} \quad (6)$$

with loop momenta $k_1, \dots, k_4$ and $j = 1, \dots, 4$. The four cut propagators $D_1, \dots, D_4$ appear as delta functions, two on each worldline (2SF). The uncut linearized propagators $D_5, \dots, D_8$ enter as $1/(D_i + i\sigma 0)$ with a routing sign $\sigma = \pm 1$, and $D_9, \dots, D_{22}$ are graviton propagators, with no $i0$ prescription specified in the family definition.

A top sector has thirteen of the D_i as denominators, including the four cut ones [8], and the remaining D_i as numerators. A family's leading top sectors are those included in its first-stage reduction (Section 5). PX2 has four leading top sectors, whose denominators are $D_1, \dots, D_9$ together with $\{D_{13}, D_{14}, D_{15}, D_{21}\}$ in the sector we call S_1 , $\{D_{11}, D_{14}, D_{16}, D_{21}\}$ in S_2 , $\{D_{11}, D_{14}, D_{17}, D_{20}\}$ in S_3 , and $\{D_{13}, D_{14}, D_{17}, D_{19}\}$ in the fourth. The families P and PX1 differ from PX2 only in uncut linearized propagators: both have $D_8 = 2k_4 \cdot u_1$, and P has in addition $D_6 = 2k_2 \cdot u_2$. NP has $D_8 = 2k_4 \cdot u_1$ as well but different graviton propagators. All four definitions are given in the ancillary files.

For the dissipative problem the cut delta functions are genuine two-sided cuts and both routings $\sigma = \pm 1$ are retained [7]. Since the integration-by-parts system is the same for either sign, σ enters only through the boundary constants. The boundary constants of Section 5 are computed with Feynman graviton propagators; retarded ones differ by a cut term on the negative-frequency mass shell [7, 42], whose contribution we do not compute here.

For the first-stage reductions we use KIRA [64, 65] with its automatic sector symmetries, which for PX2 are 48 maps between sectors and 4 self-maps. Each is a relabeling of the loop momenta, possibly combined with $q \rightarrow -q$ or the worldline exchange $u_1 \leftrightarrow u_2$, that maps the linearized propagators $D_1, \dots, D_8$ into one another without sign changes and is therefore valid for a common routing sign σ . Thirty of the 52 also reverse the momentum of a graviton line. The reversal leaves Feynman graviton propagators, which are even in the momentum, unchanged but turns a retarded propagator into an advanced one [42]. Symmetry relations that hold only for time-symmetric worldline propagators, such as those under $u_i \rightarrow -u_i$, reduce the conservative master counts [8]; they are not among the 52 and are not imposed.

3.2 The four geometries

Through 3PM the scattering observables are multiple polylogarithms with the alphabet $\{x, 1 \pm x\}$ (see Appendix C). On the maximal cut of a top sector in the loop-by-loop Baikov representation [38,

[39] a contour integral over the remaining n Baikov variables survives,

$$\varpi(x) = \frac{1}{(2\pi i)^n} \oint_{T^n} \frac{d^n t}{\sqrt{Q(t; x)}}, \quad (7)$$

which we call the maximal-cut leading singularity. The hypersurface $Q = 0$ is an algebraic variety whose holomorphic period is the torus integral (7). Its periods are the homogeneous solutions of the sector's differential equation in x , and ϖ normalizes the canonical basis [34, 37]. The periods are annihilated by a Picard–Fuchs operator, of order three for a one-parameter family of K3 surfaces and four for a family of Calabi–Yau threefolds [37]. For each of the four geometries of Table 1 the operator has a point of maximal unipotent monodromy (MUM point), located at $z = 0$ in a suitable variable z . There the Frobenius basis consists of one power series $\varpi_0(z) = 1 + O(z)$, up to an overall power of z , and companions $\varpi_1, \dots, \varpi_n$, with ϖ_k of degree k in $\ln z$. The torus integral (7) is proportional to ϖ_0 , while the logarithmic periods enter the master integrals through the boundary data.

At 4PM the variety $Q = 0$ is a K3 surface of Sym²-Legendre type [34, 35], one complex dimension above an elliptic curve, and its holomorphic period is the square of a complete elliptic integral. At 5PM three more non-polylogarithmic geometries, none of banana type, appear [36, 37], and by the classification of Brammer et al. [39] these four are all that occur through 5PM. With $\theta = z d/dz$ we write the four Picard–Fuchs operators, in the normalization of Klemm et al. [37], as

$$L_{K3} = (2\theta - 1)^3 + z^2(2\theta + 1)^3 - 4z\theta(4\theta^2 + 1), \quad z = x^2, \quad (8)$$

$$L_{CY_3} = \theta^4 - 2^8 z(\theta + \frac{1}{2})^4, \quad z = 2^{-8} x^4, \quad (9)$$

$$L_A = \theta^3 - z(2\theta + 1)(17\theta^2 + 17\theta + 5) + z^2(\theta + 1)^3, \quad z = x^2, \quad (10)$$

$$L_{CY'_3} = \theta^4 - 2^4 z(192\theta^4 + 128\theta^3 + 112\theta^2 + 48\theta + 7) + 2^{14} z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7) - 2^{30} z^3(\theta + \frac{1}{2})^4, \quad z = \frac{1 - \gamma^2}{2^{10}}, \quad (11)$$

each listed with its variable z .

The 4PM operator L_{K3} is the symmetric square of the Legendre operator $\theta^2 - z(\theta + \frac{1}{2})^2$, up to a factor $\sqrt{z}$ absorbed in the master integral. It annihilates $\sqrt{z} K(z)^2 \propto x K(x^2)^2$, with K the complete elliptic integral of the first kind. The 5PM-1SF operator L_{CY_3} (topology 3 in the labels of Ref. [39]), whose threefold we denote CY_3 , is the Hadamard square of the same Legendre operator in a rescaled variable. Its holomorphic solution is

$$\varpi_0 = {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; x^4), \quad (12)$$

the period of a threefold cut out by four quadrics in $\mathbb{P}^7$ [37]. The 5PM-2SF Apéry operator L_A (topology 40, whose surface we denote $K3'$) has holomorphic solution $\varpi_0 = \sum_{n \geq 0} A_n z^n$, generated by the Apéry numbers

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = 1, 5, 73, 1445, \dots, \quad (13)$$

of Apéry's proof that $\zeta(3)$ is irrational. The surface is the K3 of Beukers and Peters [66, 37]. In the 5PM-2SF operator $L_{CY'_3}$ (topology 37, whose threefold we denote CY'_3) the coefficient of θ^4 is $(1 - 2^{10}z)^3$, and the threefold, of Euler characteristic $\chi = 80$, is again related to a Hadamard construction [37]. The holomorphic solution $\varpi_0 = \sum_n a_n z^n$, $a_n = 1, 112, 47376, \dots$, has the closed form (26).

geometry	order, sector	operator	variable	finite singular points
K3 (Sym ² Legendre)	4PM	L_{K3} (8)	$z = x^2$	$z = 1$
K3' (Apéry)	5PM-2SF, top. 40	L_A (10)	$z = x^2$	roots of $z^2 - 34z + 1$
CY ₃ (${}_4F_3$)	5PM-1SF, top. 3	L_{CY_3} (9)	$z = 2^{-8}x^4$	$z = 2^{-8}$
CY' ₃ (Hadamard, $\chi = 80$)	5PM-2SF, top. 37	$L_{CY'_3}$ (11)	$z = \frac{1-\gamma^2}{2^{10}}$	$z = 2^{-10}$

geometry	cut surface, cycle	maximal cut
K3	Gram determinant, 1d Euler integral	LS = $8xK(x^2)^2 = 2\pi^2x\varpi_0(x^2)$
K3'	$Q_6 = P_+P_-$, T^2	$\varpi(x^2) = \varpi_0(x^2) = \sum_n A_n x^{2n}$
CY ₃	P_8 (odd parity), T^3	$-\frac{i}{2}(1-x^2)^{-1/2} {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; x^4)$
CY' ₃	Q_8 (even parity), T^3	$x\varpi_{\text{maxcut}}(x) = \frac{1}{8}\varpi_0(z)$

Table 1: The four non-polylogarithmic geometries of post-Minkowskian two-body scattering through 5PM [39, 37]. In each case the maximal-cut leading singularity of the top sector (lower block, last column) is proportional to the holomorphic period ϖ_0 of the sector's Picard–Fuchs operator; the constants and kinematic prefactors are derived in Section 4. Upper block: the PM and self-force order with the topology label of Ref. [39]; the operator of Eqs. (8)–(11); its variable z in terms of x and γ of Eq. (5); and its singular points at finite z other than the MUM point $z = 0$. Lower block: the polynomial whose zero set is the maximal-cut variety and, for the two threefolds, in parentheses the parity under $u_i \rightarrow -u_i$ of the sector in which the period appears, followed by the torus cycle T^n of Eq. (7), into which $(2\pi i)^{-n}$ is absorbed; and the leading singularity LS. Here ϖ_0 is the Frobenius power-series solution at $z = 0$ in the normalization of Ref. [37], with $\varpi_0(z) = (2/\pi)^2 K(z)^2$ for the 4PM K3 and K the complete elliptic integral of the first kind; A_n are the Apéry numbers (13); ${}_4F_3$ is the period (12); ϖ_{maxcut} is the torus integral (7) of the CY'₃ sector. The 4PM surface is derived from the Gram determinant, and the maximal cut reduces to the Euler integral (22), which is one-dimensional once its inner integral is written as a complete elliptic integral. The 5PM rows use the loop-by-loop surfaces Q_6 , P_8 , Q_8 of Ref. [39], and $Q_6 = P_+P_-$ is the factorization (17). Primes distinguish the two 2SF geometries from the 4PM K3 and the 1SF threefold.

The Apéry operator is singular at a physical velocity: the smaller root of $z^2 - 34z + 1$ is

$$z_+ = 17 - 12\sqrt{2} = x^2|_{\gamma=3}, \quad (14)$$

that is, $x = 3 - 2\sqrt{2}$ and $\gamma = 3$, or $v/c = \sqrt{8}/3$, inside the scattering region. There the potential-region contribution at 5PM-2SF has a spurious divergence [8].

3.3 The memory-region integrals

The four-loop radiative integrals develop a distinct integration region, the memory region (see Section 2.3), which at 5PM-2SF gives rise to the memory back-reaction on the impulse [8]. In the conservative calculation of Driesse et al. [8] the contribution of the memory region reduces, at the static point $\gamma = 1$, to two boundary master integrals,

$$I_1^{(M)} = \int_{\ell_{1\dots 4}} \frac{\hat{\delta}(\ell_1 \cdot v_1) \hat{\delta}(\ell_2 \cdot v_1) \hat{\delta}(\ell_3 \cdot v_2) \hat{\delta}(\ell_4 \cdot v_2)}{D_{q3} D_{34} D_4 D_{31} D_{24}}, \quad (15)$$

and $I_2^{(M)}$, the same integral with D_{31} and D_{24} squared. In the notation of Ref. [8], $v_{1,2}$ are the worldline velocities, ℓ_i the loop momenta, $\int_\ell = \int d^{4-2\epsilon} \ell / (2\pi)^{4-2\epsilon}$, $\hat{\delta}(x) = 2\pi \delta(x)$, $D_{ij} = (\ell_i - \ell_j)^2$, $D_{q3} = (\ell_3 + q)^2$, and $D_4 = \ell_3 \cdot v_1$ is uncut. Both integrals depend on the $i0^+$ routing assigned to the two graviton propagators D_{31} and D_{24} .

In $4-2\epsilon$ dimensions, the ϵ poles of the memory-region contribution must cancel against those of the potential and tail regions. The spurious $(\gamma-3)$ divergence of the potential region must likewise drop out of the sum over regions. These two conditions fix the leading poles up to one constant [8],

$$\begin{aligned} I_1^{(M)} &= \frac{1}{15(8\pi)^4 \epsilon} + \mathcal{O}(\epsilon^0), \\ I_2^{(M)} &= -\frac{5 c_M}{6(8\pi)^4 \epsilon^2} + \mathcal{O}(\epsilon^{-1}). \end{aligned} \quad (16)$$

Ref. [8] fixes c_M by placing retarded propagators on D_{31} and D_{24} , oriented toward the three-graviton vertex, and averaging with the time-reversed routing. This is their γ -3 prescription, named for the divergence at $\gamma = 3$, Eq. (14), that it cancels. Once the routing is specified, $I_2^{(M)}$ is a convergent frequency integral order by order in ϵ , and c_M is its normalized ϵ^{-2} coefficient, which equals one under the γ -3 prescription [8]. The constant first enters the conservative scattering angle [8] at fifth post-Newtonian order, through $\frac{64}{5} c_M - \frac{11456}{135}$, and the fifth-post-Newtonian result of Porto and Riva, obtained with the same routing, agrees with $c_M = 1$ [32].

4 Periods from maximal cuts

For each of the four geometries in Table 1 the leading singularity of the top sector on its maximal cut, Eq. (7), is proportional to the holomorphic period ϖ_0 of its Picard–Fuchs operator [37, 39]. This construction fixes the normalization of the canonical basis in each geometric sector but does not evaluate the master integrals.

4.1 The Apéry K3 surface

The K3 sector of the planar 5PM-2SF family is topology 40 of Brammer et al. [39]. It is governed by the Apéry operator L_A (10), which Klemm et al. established from IBP relations and differential equations on its nine master integrals [37]. Duhr et al. obtained L_A from the maximal cut by expanding the torus period as a power series in x whose coefficients are the Apéry numbers [42]. Here we derive in closed form the torus period of the cut surface of Ref. [39], and from it L_A . From the same surface we obtain the leading Landau singularities of the cut, among them $\gamma = 3$.

In the loop-by-loop Baikov representation of Ref. [39], Eq. (3.26), the leading-singularity (LS) surface of the sector is $Q_6 = (x(1+t_1)(t_1+t_2^2) - (1+x^2)t_1t_2)^2 - 4x^2t_1^2t_2^2$ (in unfactored form). With $w := (1+t_1)(t_1+t_2^2)/(t_1t_2)$ and $2\gamma = x + 1/x$, the identity $(1+x^2) \pm 2x = (1 \pm x)^2$ factorizes this difference of squares, as Duhr et al. found for this sector in other variables [42]:

$$Q_6 = P_+ P_- , \quad P_\pm := x(1+t_1)(t_1+t_2^2) - (1 \pm x)^2 t_1 t_2 = x t_1 t_2 (w - 2\gamma \mp 2), \quad (17)$$

or equivalently $Q_6 = x^2 t_1^2 t_2^2 [(w - 2\gamma)^2 - 4]$, a spectral-curve form. The loop-by-loop reduction of Duhr et al. (Section 7.2.1 of Ref. [42]) leads instead to the fully factored surface $P_{4,4} = (1-t_1^2)(1-t_2^2)[(1-x)^2 + 4xt_1t_2][(1+x)^2 + 4xt_1t_2]$.

On the unit torus ($t_j = e^{i\phi_j}$) one has $w = 4 \cos \frac{\phi_1}{2} \cos \frac{\phi_1 - 2\phi_2}{2}$. With the unimodular change of angles $(\alpha, \beta) = (\frac{\phi_1}{2}, \frac{\phi_1 - 2\phi_2}{2})$ we then prove the constant-term lemma

$$\text{CT}_{t_1, t_2}[w^n] = \binom{n}{n/2}^2 [n \text{ even}] \quad \text{for all } n \geq 0. \quad (18)$$

In terms of $a := x/(1+x)^2$, $b := x/(1-x)^2$ and $z := x^2$, the holomorphic torus period is

$$\varpi(z) := \frac{1}{(2\pi i)^2} \oint_{T^2} \frac{dt_1 dt_2}{\sqrt{Q_6}} = \frac{1}{1-z} \sum_{m \geq 0} \frac{\binom{2m}{m}^2}{16^m} \sum_{j+k=2m} \binom{2j}{j} \binom{2k}{k} a^j b^k = \sum_{n \geq 0} A_n z^n, \quad (19)$$

where A_n are the Apéry numbers (13). The second equality follows from $(w - 2\gamma)^2 - 4 = (w - 1/a)(w - 1/b)$ on expanding each inverse square root in powers of w and taking constant terms with Eq. (18). The last we have verified as an identity of rational power series through order z^{50} , so that $L_A \varpi$ vanishes through that order; we have not constructed the creative-telescoping certificate that would prove $L_A \varpi = 0$ identically from Eq. (19). Granted the identification of this sector's K3 with the surface of Beukers and Peters [66, 37], the all-order statement follows from the Apéry recurrence $n^3 c_n - (2n-1)(17n^2 - 17n + 5)c_{n-1} + (n-1)^3 c_{n-2} = 0$ and the initial data $(1, 5, 73)$. The Picard–Fuchs operator of the torus period (19) is thus L_A , Eq. (14) of Klemm et al. [37]. The full leading singularity of Eq. (3.26) of Brammer et al. is $\text{LS} \propto -\frac{z}{1-z} \varpi(z)$, in the master-integral normalization used there; the prefactor agrees with that of Ref. [42].

We also evaluate the torus integral numerically at $x = 1/11$, $1/13$ and $3/20$, inside the disk $z < (3 - 2\sqrt{2})^2$ where the Frobenius series of L_A converges. The values agree with the Frobenius solution ϖ_0 , with no adjustable normalization since $\varpi(0) = 1$ on both sides (Table 8 in Appendix D). Conversely, fitting a third-order operator with cubic polynomial coefficients to these contour values and their z -derivatives yields L_A .

To identify the K3 surface of the sector, we lift the square root with a Poisson kernel in a third unit-circle variable s . The spectral-curve form then exhibits Q_6 as a Laurent polynomial in $w + s + 1/s$, which under the same unimodular change of angles (α, β) is the simple-cubic Fermi surface $\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = 2\gamma$. The torus period therefore satisfies

$$(1+z) \varpi(z) = \sum_{N \geq 0} B_N \left(\frac{z}{(1+z)^2} \right)^N, \quad (20)$$

an identity of integer series that we have verified through order z^{80} , in which B_N are the return numbers of the simple cubic lattice (OEIS A002896). Equation (20) is the relation between holomorphic periods implied by the Peters–Stienstra identification [41] of the Apéry pencil with the Fermi K3 (the “Apéry–Fermi” K3). Here it follows from the Feynman LS surface, with the moduli map $1/s^2 = z/(1+z)^2$.

We now show that the finite singular points of L_A , the roots of $z^2 - 34z + 1$, are leading Landau singularities of the maximal cut. The torus period is a two-fold integral over the surface $Q_6(t_1, t_2; x) = 0$, whose leading Landau singularities in x are the projection of $\{Q_6 = \partial_{t_1} Q_6 = \partial_{t_2} Q_6 = 0\}$. By an iterated resultant we find

$$\text{disc}_t Q_6 \propto x(x^2 - 1)(x^4 - 34x^2 + 1) = x(x^2 - 1)(x^2 - 6x + 1)(x^2 + 6x + 1), \quad (21)$$

and the unique root in the physical region $x \in (0, 1)$ is $x = 3 - 2\sqrt{2}$, i.e. $\gamma = (1 + x^2)/2x = 3$. The quartic $x^4 - 34x^2 + 1$ is the singular polynomial of L_A , and up to the sign of the radicand its square

root is the algebraic letter $r(x)$ of the canonical differential equation of Driesse et al. [8]. At $\gamma = 3$ the factor P_- of Eq. (17) has an ordinary node at $t_1 = t_2 = 1$, a point of the integration torus, so the torus period (19) is pinched there.

On the subtopologies of topology 40 for which our elimination of the Feynman parameters terminated (Appendix C), the first-type Landau equations [67] give only the letters $\{x, x \pm 1\}$. The elimination did not terminate on the ten-propagator topology or on some of the remaining subtopologies, and was not attempted on the others, so we have not determined the type of the $\gamma = 3$ singularity on the full topology. The polylogarithmic letters $\{1 + x^2, x^2 \pm x + 1\}$ of the 5PM-2SF alphabet of Driesse et al. do not arise on any subtopology we resolved, and presumably enter through other topologies or through mixing under IBP reduction.

New, to our knowledge, are the identities $Q_6 = P_+ P_-$ and $Q_6 = x^2 t_1^2 t_2^2 [(w - 2\gamma)^2 - 4]$ in the variables of Ref. [39], the identification of a post-Minkowskian Feynman LS surface with the Apéry–Fermi K3 by an explicit moduli map, and the Landau analysis of the sector’s cut surface and subtopologies. Other K3 surfaces also arise from Feynman integrals, among them the two-parameter families of Duhr and Maggio [68].

4.2 The other three geometries

At 4PM (three loops) we derive the cut surface of the K3 sector of Dlapa et al. [34, 35] from the Gram determinant. On its maximal cut the worldline propagators $\delta(2u_i \cdot k)$ are linear in the Baikov variables and fix scalar products inside the Gram determinant, so no quartic extension of the Baikov polynomial is needed. The remaining irreducible scalar products are then integrated out one at a time with Dodgson/Jacobi–Sylvester determinant identities. Each integration produces an Euler Beta function and lowers the order of the determinant by one. Four variables integrate out, and the $d = 4$ maximal cut reduces to the two-fold integral

$$\text{LS}_{\text{shape}}(x) = \int_{-\infty}^{-1/2} \frac{dc_2}{\sqrt{-2c_2 - 1}} \oint \frac{db_2}{\sqrt{(b_2^2 + c_2^2(\gamma^2 - 1))(b_2^2 + 2c_2 + 1)}}, \quad \gamma = \frac{1 + x^2}{2x}. \quad (22)$$

For $c_2 < -\frac{1}{2}$ the radicand has real branch points $b_2 = \pm\sqrt{-2c_2 - 1}$, and $\oint db_2$ denotes the integral taken once along the segment between them, with the radicand replaced by its modulus. This segment integral is one half of the integral around the closed cycle enclosing that segment, on which the constant 8 found below would read 16. The b_2 integral is a complete elliptic integral, and we evaluate the remaining c_2 integral by tanh–sinh quadrature.

The ratio $\text{LS}_{\text{shape}}(x)/(x K(x^2)^2)$ takes the same value at five rational points (see the supplementary material). An integer-relation search on this value over $\{1, \pi, 1/\pi, \pi^2, 1/\pi^2, 1/\pi^3, \ln 2\}$ returns the pure rational 8, and hence

$$\text{LS}_{\text{shape}}(x) = 8x K(x^2)^2 = 2\pi^2 x \varpi_0(x^2), \quad (23)$$

where $\varpi_0(z) = (2/\pi)^2 K(z)^2$ and $\sqrt{z} \varpi_0(z)$ is the holomorphic Frobenius solution of L_{K3} (8) [37].

Dlapa et al. [34, 35] treated the 4PM K3 sector with differential equations. In the notation of Ref. [35] it is $M_{000;010101110}^{(112;+++)}$, equal to $G_{1,1,1,0,0,0,0,1,1,1,0,0,1,1}$ of Ref. [34] under $k_2 \leftrightarrow k_3$, $u_1 \leftrightarrow u_2$. Reducing the sector by IBP on the maximal cut we find five master integrals, two of which are odd under the velocity reflection $(u_1, u_2) \rightarrow (-u_1, -u_2)$ and vanish. The three even ones (the scalar integral and its (k_1^2) and $(2k_1 \cdot u_2)^2$ numerator partners), together with the one even subsector master integral, constitute the “3 + 1” elliptic block of Ref. [35].

The first-type Landau locus $\{x, 1-x, 1+x\}$ of the one- and two-loop topologies coincides with the alphabet of the sector [35, 39]. Tensoring this alphabet with the period ring of L_{K3} gives, at order ϵ^0 where the polylogarithmic weight is zero, three functions per elliptic master integral,

$$\{x K(x^2)^2, x K(x^2) K'(x^2), x K'(x^2)^2\}, \quad (24)$$

which form the Picard–Fuchs basis of Ref. [35], Eq. (5.38). Here $K'(z) \equiv K(1-z)$ is the complementary elliptic integral. The leading singularity (23) is also the canonical normalizer $g_1 = f_1/(x \Psi_1^2)$ of the elliptic block, Eqs. (4.24)–(4.29) of Ref. [34].

The hypergeometric threefold CY_3 of topology 3 (5PM-1SF) appears only in the odd-parity sector of that topology, the sector with a dotted linear propagator [37, 39]. Its loop-by-loop cut surface is P_8 , Eq. (3.5) of Ref. [39]. Integrating $d^3t/\sqrt{P_8}$ over the vanishing torus T^3 at the MUM point, with the branch of $\sqrt{P_8}$ followed by homotopy in x from $x=0$, we find

$$\frac{1}{(2\pi i)^3} \oint_{T^3} \frac{d^3t}{\sqrt{P_8}} = -\frac{i}{2\sqrt{1-x^2}} {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; x^4\right), \quad (25)$$

where the ${}_4F_3$ is the holomorphic period ϖ_0 (12) of L_{CY_3} . The even-parity scalar integral of the same topology is polylogarithmic (Table 2 of Ref. [39]), and its maximal cut is not proportional to ϖ_0 for any kinematic prefactor.

The Hadamard threefold CY'_3 of topology 37 (5PM-2SF) has the even-parity cut surface Q_8 , Eq. (3.8) of Ref. [39]. We evaluate its maximal-cut integral over T^3 by a residue series and find $x \varpi_{\text{maxcut}}(x) = \frac{1}{8} \varpi_0(z)$, with z as in Eq. (11). From the (t_1, t_2) constant-term reduction of Q_8 we also obtain a closed formula for the Frobenius coefficients $a_n = 1, 112, 47376, \dots$ of ϖ_0 ,

$$a_n = 16^n \binom{2n}{n}^2 J_n, \quad J_n = \sum_{j,k=0}^n \binom{2n}{n-j} \binom{2n}{n-k} \frac{(-1)^{j+k} (n + \frac{1}{2})_{j+k}}{j! k!}, \quad (26)$$

which holds for all n . The inner sum telescopes by Gosper’s algorithm, and the remaining single sum satisfies an order-two recurrence produced by Zeilberger’s algorithm. In the algebra of shift operators the coefficient recurrence of $L_{CY'_3}$ is a left multiple of this order-two recurrence (with zero remainder). Induction from three initial terms completes the proof. Equation (26) reflects the Hadamard structure of $L_{CY'_3}$ identified in Ref. [37].

At four loops the nested Beta-function integration of the Gram determinant removes eight of the twelve irreducible scalar products for topology 3 and six of the eleven for topology 37, without reaching the three-dimensional leading-singularity surfaces. For the 5PM rows of Table 1 we therefore take the loop-by-loop surfaces Q_6 , P_8 and Q_8 of Brammer et al. [39] as input. Since these surfaces were derived without reference to the Picard–Fuchs operators of Ref. [37], the comparison is between independent constructions. In Appendix D we compare each contour integral with its Frobenius series at rational points inside the disk of convergence and fix the constants of the two threefold rows by an integer-relation search.

5 The dissipative function space

On three of the four leading top sectors of PX2, the smallest family, which contain 35 of its 37 master integrals, we determine the ten linear relations among master integrals that the response matrices of the auxiliary-mass flow [43, 44] imply. These relations are exact reduction identities of the PX2 family (Section 5.4). We compute twelve distinct master integrals of these sectors at one kinematic point, where their values are boundary constants for the differential equations in x , and check them against the seven relations that involve only these twelve. The four families themselves are not evaluated here.

Table 2: First-stage master-integral counts of the four dissipative 5PM-2SF families against the conservative counts of Ref. [8] (where PX1, PX2 and NP are called NP1, NP2 and NPM). The first-stage count is the number of master integrals onto which the integrals with unit propagator powers of the listed top sectors and their subsectors reduce. The integration-by-parts identities are seeded to total propagator power 14, one unit above the top sectors, with at most one numerator power (for P and PX1 no numerator and at most one dot), and KIRA’s automatic sector symmetries are included. With these symmetries switched off, the integrals of the top sector S_1 of PX2 and its subsectors alone reduce at the same seeding to 30 master integrals, and those of S_2 to 28. The integrals of the other two top sectors are the images of these under $k_i \rightarrow k_i + q$, $q \rightarrow -q$, so that the four top sectors reduced separately give $2 \times (30 + 28) = 116$ master integrals, against 37 with the symmetries (Section 3.1). These counts are not converged basis dimensions and have not been tested for stability under deeper seeding. The two columns are obtained with different relations, targets and sector coverage and do not correspond entry by entry. “Top sectors” counts the leading top sectors of each family; for PX2 the 36 sectors in parentheses are the further top sectors that contain the K3 and Calabi–Yau subtopologies (Section 5.5). The parity split, determined for PX2 only, is the number of even plus the number of odd master integrals under $u_i \rightarrow -u_i$, which reverses all worldline frequencies. The reduction setup for PX2 and the four master-integral lists are given in the ancillary files, and the PX2 list also in the supplementary material.

Family	Top sectors	Conservative	First stage	Parity split
P	56	321	195	—
PX1	44	144	256	—
PX2	4 (+36)	46	37	12 + 25
NP	9	220	243	—

5.1 Master-integral counts and parity

We reduce the four dissipative families with KIRA [64, 65], using its automatic sector symmetries (Section 3.1) and the targets and seeding stated in Table 2. There we list the resulting first-stage master-integral counts with the parity split for the crossed family PX2 and the conservative counts of Ref. [8]. The four dissipative counts sum to exactly 731, numerically equal to the conservative total. Because the dissipative counts are taken at a fixed seeding depth and the conservative counts after imposing relations that hold only under time symmetry, we regard the equality of the totals as a coincidence at this seeding depth, not as a correspondence between the bases.

Under $u_i \rightarrow -u_i$ the 37 master integrals of PX2 split into 12 even and 25 odd ones. Even master integrals contribute to the scattering angle and odd ones to the radiated energy and momentum [42]. The odd block contains the radiation-reaction integrals, which change sign when the worldline time direction is reversed [7, 27].

5.2 Response matrices

In the auxiliary-mass flow [43, 44] a subset of the propagators of a top sector is deformed, $D_i \rightarrow D_i - \eta$, and the master integrals are evolved at fixed kinematics from $\eta \rightarrow \infty$, where they reduce to simpler boundary integrals, to the physical point $\eta \rightarrow 0$. On cut worldline families, however, the boundary integrals are not vacuum integrals (see Appendix A). The evolution is linear in the boundary data. We define the response matrix $A(\epsilon)$ by

$$M(\epsilon) = A(\epsilon) b(\epsilon), \quad (27)$$

Table 3: Response matrices of the three analyzed top sectors of PX2 and of the first NP top sector. The rank deficit of $A(\epsilon)$ implies 1, 4 and 5 linear relations among the master integrals of S_1, S_2, S_3 , supported on 3, 12 and 16 rows (31 rows, at most 25 distinct master integrals), and two relations on six master integrals for the NP sector, which is solved on the double cover of the flow variable. Columns: the propagators D_j of Eq. (6) that define the sector ($D_1, \dots, D_4$ cut); the number n_J of master integrals of the η -deformed differential system; the number n_b of boundary integrals at $\eta \rightarrow \infty$; the rank of $A(\epsilon)$ of Eq. (27), constant over the twelve sampled values of ϵ ; the number $n_v = n_J - \text{rank } A$ of flow-free directions (left null vectors of A); and the number n_{supp} of master integrals on which they are supported. In the ancillary files, which include the three response matrices at the sampled values of ϵ and their ten left null vectors, the sectors S_1, S_2, S_3 are labeled 1077759, 1091071 and 599551 (the integer $\sum_j 2^{j-1}$ over the propagators D_j of the sector).

Sector	Propagators	n_J	n_b	rank $A(\epsilon)$	n_v	n_{supp}
S_1	$D_1, \dots, D_9, D_{13}, D_{14}, D_{15}, D_{21}$	14	27	13	1	3
S_2	$D_1, \dots, D_9, D_{11}, D_{14}, D_{16}, D_{21}$	27	68	23	4	12
S_3	$D_1, \dots, D_9, D_{11}, D_{14}, D_{17}, D_{20}$	44	101	39	5	16
NP, first top sector	(double cover)	31	98	29	2	6

where M is the vector of master integrals of the η -deformed system of the sector at $\eta \rightarrow 0$ and b the vector of boundary integrals at $\eta \rightarrow \infty$. Column j of A is the value of M for b equal to the j -th unit vector. We call a left null vector v of A , $v^T A = 0$, a flow-free direction. Each such vector implies a linear relation $v^T M = 0$ among the master integrals (Section 5.4).

In Table 3 we give the propagators of the PX2 top sectors S_1, S_2 and S_3 and the rank of $A(\epsilon)$ for each sector at twelve values of ϵ near 3×10^{-4} . We determine the ranks numerically, from a gap of more than forty orders of magnitude in the singular-value spectrum of A that is stable across the twelve values of ϵ . The ranks 13, 23 and 39 fall short of the numbers 14, 27 and 44 of master integrals of the η -deformed systems by 1, 4 and 5, which implies ten linear relations. The corresponding left null vectors are supported on 3, 12 and 16 rows of the three response matrices. By the relabeling identities of Section 5.3 these 31 rows involve at most 25 distinct master integrals. The three master integrals supporting the null vector of S_1 , which we call the corner integrals, have the four cut worldline propagators and five graviton propagators, and none of the sector's uncut linearized propagators $D_5, \dots, D_8$ (Eq. (31) below). Two of the ten flow-free directions have rational, ϵ -independent weights,

$$v^{(S_1)} = (1, \frac{16}{3}, -6) \text{ on } (I_0, I_2, I_3), \quad v_0^{(S_2)} = (-\frac{1}{6}, 1, -\frac{8}{9}) \text{ on } (I_{r0}, I_{r2}, I_{r3}), \quad (28)$$

in the notation of Table 4. The other eight depend on ϵ (Section 5.4).

For the dissipative NP family the η -evolution requires one modification, since at the $\eta \rightarrow \infty$ boundary one coupled block of the differential system has local exponents

$$\{-\frac{1}{2}, -\frac{3}{2}, -1, +\frac{1}{2}\} + 4\epsilon, \quad (29)$$

integers and half-integers together. A fundamental system of power-series solutions then exists only in the double-cover variable t (with t^2 the flow variable), in which the NP systems become solvable by the same method [43, 44] as PX2. For the first of the nine NP top sectors the 31×98 response matrix has rank 29, with a two-dimensional left null space supported on six master integrals; we have not analyzed the other eight NP top sectors.

5.3 Boundary constants of PX2

At one reference kinematic point ($\gamma = 5/4$, $q^2 = -1$) we compute twelve of the at most 25 distinct master integrals on which the ten left null vectors of Section 5.2 are supported; through the relabeling and reflection identities below they account for 19 of the 31 rows. Integrals are normalized as $\prod_i d^D k_i / (i\pi^{D/2})$ with $D = 4 - 2\epsilon$, each cut worldline propagator is replaced by $2\pi \delta(2k \cdot u)$, and no $e^{\gamma_E \epsilon}$ prefactor is included (the γ_E terms below are artifacts of this normalization). Throughout this section the graviton propagators, to which Eq. (6) assigns no $i0$, are Feynman propagators. The reference point is

$$\gamma \equiv u_1 \cdot u_2 = \frac{5}{4}, \quad q^2 = -1, \quad x = \gamma - \sqrt{\gamma^2 - 1} = \frac{1}{2}. \quad (30)$$

Except for the finite parts of the radiative corner masters, which we know only numerically, these constants are exact and lie in the ring $i\pi^5 \mathbb{Q}[\ln 2, \gamma_E]$ joined by real π^4 terms where an ultraviolet pole survives.

The three corner integrals supporting $v^{(S_1)}$ are I_0 and its two numerator partners,

$$I_0 = \int \prod_{i=1}^4 \frac{d^D k_i}{i\pi^{D/2}} \frac{(2\pi)^4 \delta(2k_1 \cdot u_1) \delta(2k_2 \cdot u_1) \delta(2k_3 \cdot u_2) \delta(2k_4 \cdot u_2)}{(k_1 - k_2)^2 (k_2 - k_4)^2 (k_3 - k_4)^2 (k_1 + q)^2 k_3^2}, \quad (31)$$

$$I_2 = I_0[(k_1 - k_3)^2], \quad I_3 = I_0[(k_1 - k_4)^2],$$

where $I_0[N]$ denotes I_0 with the numerator N inserted. In the labeling of Eq. (6) the denominators are $D_9 D_{13} D_{14} D_{15} D_{21}$ and the numerators D_{10} and D_{11} . The four delta functions confine each loop momentum to a Euclidean hyperplane orthogonal to u_1 or u_2 . The k_1 and k_3 integrations are then massless bubbles in Γ -function form, and a single graviton propagator, $(k_2 - k_4)^2$, connects the two hyperplanes. In the two surviving worldline frequencies a and b this propagator contains the indefinite form $a^2 + b^2 - 2ab\gamma$, which vanishes on two rays for $\gamma > 1$ (the radiation cone). The three corner integrals are therefore radiative, two-frequency Bessel- K integrals of the same type as the memory-region integrals of Ref. [8] evaluated in Section 7, with an ultraviolet pole from a factor $\Gamma(4\epsilon - 1)$. The pole coefficients are

$$I_0|_{\epsilon^{-1}} = \pi^4 \frac{80 \ln 2 - 48}{27} + \frac{i\pi^5}{27},$$

$$I_2|_{\epsilon^{-1}} = \pi^4 \frac{150 - 220 \ln 2}{27} - \frac{7i\pi^5}{864}, \quad (32)$$

$$I_3|_{\epsilon^{-1}} = \pi^4 \frac{1128 - 1640 \ln 2}{243} - \frac{i\pi^5}{972},$$

while we know the finite parts only numerically and have not identified them in closed form. Loop-momentum relabelings and reflections of the integrands identify the corner integrals with master integrals of the other two sectors: I_0, I_2, I_3 coincide with I_{r0}, I_{r3}, I_{r2} of S_2 and with $\tilde{I}_0, \tilde{I}_3, \tilde{I}_2$ of S_3 , respectively. The reflection $k_i \rightarrow -k_i - q$ maps S_2 onto S_3 and also gives $\tilde{I}_4 = I_{r4}$ (Eq. (35) below).

The odd-parity master integrals I_{r5}, I_{r6}, I_{r7} of S_2 and their counterparts $\tilde{I}_5, \tilde{I}_6, \tilde{I}_7$ in S_3 contain one uncut linearized worldline propagator $1/(2k \cdot u + i\sigma 0)$, with $\sigma = \pm 1$ labeling its orientation [7, 27]. We split it into a principal value and a delta function,

$$\frac{1}{b + i\sigma 0} = \text{PV} \frac{1}{b} - i\pi\sigma \delta(b). \quad (33)$$

Table 4: Boundary constants of the crossed family PX2 at the reference point (30), $\gamma = 5/4$, $q^2 = -1$, in the normalization $\prod_i d^D k_i / (i\pi^{D/2})$, $D = 4 - 2\epsilon$, with $2\pi \delta(2k \cdot u)$ for each cut worldline propagator and no $e^{\gamma_E \epsilon}$ factor. All entries lie in $i\pi^5 \mathbb{Q}[\ln 2, \gamma_E]$ joined by real terms in $\pi^4 \mathbb{Q}[\ln 2]$, apart from the finite parts of the radiative corner integrals I_0, I_2, I_3 of S_1 , Eq. (31). These three have an ultraviolet pole with coefficient in the same ring, while their finite parts are known numerically and are quoted to ten figures, truncated (Appendix D). All values are computed with Feynman propagators on the graviton lines. I_{rk} and $\tilde{I}_k$ denote the master integrals in row k of the response matrices of S_2 and S_3 . The static-slice integrals have no pole and are quoted at $\sigma = +1$, with σ the orientation of their uncut worldline propagator in Eq. (33); their S_3 counterparts $\tilde{I}_{5,6,7}$ take identical values through $O(\epsilon)$. The dotted master integral I_{r4} is given in Eq. (35). The closed forms were identified from numerical values by an integer-relation search (Appendix D). The supplementary material gives the propagator exponents of all 31 supporting master integrals, the reductions of I_0, I_2, I_3 and $I_{r4}, \dots, I_{r7}$ to frequency integrals and the complete numerical values; the exponents and values are also provided in the ancillary files.

Master	ϵ^{-1}	ϵ^0
<i>Radiative corner (S_1; also realized as I_{r0}, I_{r2}, I_{r3} in S_2 and $\tilde{I}_0, \tilde{I}_2, \tilde{I}_3$ in S_3)</i>		
I_0	$\pi^4(80 \ln 2 - 48)/27 + i\pi^5/27$	$388.8416383 \dots + 161.9427314 \dots i$
I_2	$\pi^4(150 - 220 \ln 2)/27 - 7i\pi^5/864$	$-131.6922906 \dots - 35.15238209 \dots i$
I_3	$\pi^4(1128 - 1640 \ln 2)/243 - i\pi^5/972$	$-52.25287419 \dots - 4.256106616 \dots i$
<i>Static slice (S_2), no pole; an entry $c_0 [c_1]$ denotes $(c_0 + c_1 \epsilon) i\pi^5 + O(\epsilon^2)$</i>		
I_{r5}	0	$-\frac{16}{3} \left[-128 + \frac{160}{3} \ln 2 + \frac{64}{3} \gamma_E \right]$
I_{r6}	0	$+\frac{128}{81} \left[\frac{1024}{27} - \frac{1280}{81} \ln 2 - \frac{512}{81} \gamma_E \right]$
I_{r7}	0	$+\frac{32}{81} \left[\frac{736}{81} - \frac{320}{81} \ln 2 - \frac{128}{81} \gamma_E \right]$
<i>Vanishing master integrals (all orders, all prescriptions)</i>		
I_{r1}	0	0
$\tilde{I}_1$	0	0

A reflection of both in-plane frequency axes preserves the quadratic propagators and the cut deltas and reverses the sign of the linear forms. The principal-value part therefore integrates to zero and only the $\delta(b)$ slice survives. On that slice, which we call static, the worldline frequency vanishes, the radiation cone is not crossed, and the integrals are purely imaginary, linear in σ and independent of the graviton $i0$ prescription.

On the $\delta(b)$ slice one frequency integral remains. It reduces to a radial Weber–Schafheitlin integral and a Mellin product of two Bessel- K functions, both in closed Γ -function form to all orders in ϵ . The $\Gamma(4\epsilon - 1)$ pole of the corner integrals is absent on this slice. In Table 4 we list the values through order ϵ at $\sigma = +1$. Two further master integrals, I_{r1} of S_2 and $\tilde{I}_1$ of S_3 , have a single odd worldline frequency as numerator and no linear denominator to pair it with. They therefore vanish under the same reflection of the frequency axes, to all orders in ϵ and for any $i0$ prescription,

$$I_{r1} = \tilde{I}_1 = 0. \quad (34)$$

On S_1 all three master integrals supporting the flow-free direction, I_0, I_2 and I_3 , are known. On

S_2 eight of the twelve are known, the block I_{r0} through I_{r7} , while the principal-value pair I_{r9}, I_{r10} and the double-static pair I_{r18}, I_{r19} are not yet computed. On S_3 eight of sixteen are known, the counterparts $\tilde{I}_0, \dots, \tilde{I}_7$ of the known S_2 integrals, the open eight being the eikonal-dressed and doubly dressed masters. The master integrals still to be computed contain the uncut propagator D_7 or D_8 , alone or together with D_5 , and fall into two reduction classes: a one-loop eikonal bubble dressing the static slice with a ${}_2F_1$, and the doubly dressed combination with two uncut propagators.

The dotted integral $I_{r4} = \tilde{I}_4$, a Bessel- K integral of the corner type, is the one computed master integral with a purely real ultraviolet pole,

$$I_{r4} = \frac{\pi^4}{\epsilon} + \pi^4 \left(4 + \frac{20}{3} \ln 2 - 4\gamma_E \right) - \frac{2}{3} i\pi^5 + O(\epsilon). \quad (35)$$

Its pole arises at a static endpoint of the radial integration, off the radiation cone, which first contributes at the finite order through the $-\frac{2}{3}i\pi^5$ term. We have not yet computed its order- ϵ coefficient. We leave open which sign of the worldline $i0$ label σ corresponds to retarded and which to advanced boundary conditions. The values in Table 4 and Eq. (35) are known for both signs of σ . Fixing the retarded sign requires a convention for the worldline frequency frame [7]. Retarded graviton propagators are Feynman propagators plus cut terms on the negative-frequency mass shell [7]; we have set up this expansion but not evaluated the cut integrals, and give no in-in boundary constants here.

5.4 Relations among boundary constants

The three corner constants satisfy

$$I_0 + \frac{16}{3} I_2 - 6 I_3 = 0 \quad (36)$$

order by order in ϵ . At the pole the π^4 , $\pi^4 \ln 2$ and $i\pi^5$ components of Eq. (32) cancel separately, and at the finite order we find the relation satisfied to the accuracy of the numerically known constants of Table 4. The weights are the components of $v^{(S_1)}$ in Eq. (28), fixed by the differential system independently of the values. Equation (36) is one instance of a general relation, valid under three assumptions.

- (i) Closure. The η -deformed master integrals $J(\eta, \epsilon)$ satisfy a first-order homogeneous linear system $\partial_\eta J = C(\eta, \epsilon) J$ with rational coefficients, and the physical values are $M = \lim_{\eta \rightarrow 0^+} J$ along the flow contour [43, 44].
- (ii) Boundary completeness. As $\eta \rightarrow \infty$ along the contour the physical solution matches onto the enumerated boundary integrals, $J(\eta) = \Phi(\eta, \epsilon) b(\epsilon) + o(1)$ with Φ the power-by-power matching matrix. The large- η expansion of J contains nothing outside the matched sectors: no powers of η absent from the matching and no recessive exponentials on the cut cycles. For the present families the former excludes odd or half-integer-shifted powers, which the linearized worldline propagators can generate.
- (iii) Genericity. The rank of $A(\epsilon)$ is constant near the evaluation points; nothing is claimed across a locus where the rank jumps.

Under (i)–(iii) the evolution from a matching point η_* to $\eta = 0$ is a single linear map $U(0, \eta_*)$, the same for all solutions, and the response matrix is the composition

$$A = \lim_{\eta_* \rightarrow \infty} U(0, \eta_*) \Phi(\eta_*) \quad (37)$$

column by column. Linearity in the boundary data then gives Eq. (27) for the physical solution and, for every flow-free direction v ,

$$v^T A = 0 \implies v^T M = 0 \quad (38)$$

identically in ϵ on the genericity domain. We call Eq. (38) the null law; its physical input is assumption (ii). If the enumeration is incomplete, let $N \equiv M - A b$ denote the part of the physical solution that the enumerated boundary integrals do not generate. The sharper form of the relation is then

$$v^T M = v^T N, \quad (39)$$

so that a nonvanishing contraction $v^T M$ equals the projection of the missed boundary terms onto the direction of v . The law does not extend to the self-similar family of Appendix B, where the boundary vector coincides with M and a single flow gives only the fixed-point condition $M = AM$.

The ten relations $v^T M = 0$ on S_1 , S_2 and S_3 are reduction identities of the undeformed PX2 family, obtained from its integration-by-parts and Lorentz-invariance identities together with the sector symmetries of Section 3.1. Each involves integrals with a numerator or a dot, which were not among the targets of the first-stage reduction of Table 2. Reducing the same sectors with these integrals as targets, we obtain all ten relations with coefficients rational in d and γ , and Eq. (36) with the weights $(1, \frac{16}{3}, -6)$ for all d and γ . The tests below therefore check our direct evaluations of the boundary constants against these identities and do not probe assumption (ii).

With $v(\epsilon) = \sum_k v^{(k)} \epsilon^k$ and $M(\epsilon) = \sum_{k \geq -1} M^{(k)} \epsilon^k$ the relation (38) reads

$$v^{(0)T} M^{(-1)} = 0, \quad v^{(0)T} M^{(0)} + v^{(1)T} M^{(-1)} = 0, \quad \dots \quad (40)$$

An ϵ -independent null vector such as $v^{(S_1)}$ therefore annihilates each $M^{(k)}$ separately, whereas an ϵ -dependent one constrains the $M^{(k)}$ only through the combinations (40). We use these constraints order by order where $M^{(-1)}$ vanishes or is known on the support (Eqs. (41) and (42)), and otherwise at fixed ϵ . Through the relabeling identities of the corner integrals, the left null spaces of S_2 and S_3 contain the same relation (36), listed as rows 2 and 5 of Table 5. At $\epsilon = 0$ all the left null vectors we extracted have rational weights, the ϵ -dependent ones within the uncertainty of their extrapolation in ϵ . These weights are the coefficients of the reduction identities at $d = 4$ and $\gamma = 5/4$.

On the PX2 top sectors S_1 , S_2 , S_3 we test the relation (38) in the seven of their ten flow-free directions that the computed constants reach (Table 5). The relation applies separately to each assignment of graviton propagators that is even in the graviton momenta (Feynman, anti-Feynman or their average), because the master integrals with these assignments satisfy the same differential system and differ only in their boundary data. On S_2 the two directions of the parity-odd block are supported on the vanishing integral I_{r1} and on the pairs $\{I_{r5}, I_{r6}\}$ and $\{I_{r5}, I_{r7}\}$. Given I_{r5} , the relation (38) fixes the other two,

$$\begin{aligned} I_{r6}|_{\epsilon_0} &= -\frac{8}{27} I_{r5}|_{\epsilon_0} = \frac{128}{81} i\pi^5, \\ I_{r7}|_{\epsilon_0} &= -\frac{2}{27} I_{r5}|_{\epsilon_0} = \frac{32}{81} i\pi^5, \end{aligned} \quad (41)$$

and, applied at fixed ϵ , their order- ϵ terms as well. Neither integral can have a pole, because the other integrals in the support of these two directions are I_{r5} , which has no pole, and I_{r1} , which vanishes. The static-slice reduction gives I_{r6} and I_{r7} directly (Table 4), in agreement with Eq. (41) and with the predicted order- ϵ terms.

The two parity-odd directions of S_2 reappear on S_3 with identical weights. Because the reflection $k_i \rightarrow -k_i - q$ maps the uncut propagator $1/(2k \cdot u + i\sigma 0)$ to $-1/(2k \cdot u - i\sigma 0)$ and the static-slice

Table 5: The seven flow-free directions (left null vectors v of the response matrix $A(\epsilon)$ of Eq. (27)) of the PX2 top sectors S_1, S_2, S_3 that are supported on computed master integrals, with their weights at $\epsilon = 0$. All ten flow-free directions correspond to reduction identities of the undeformed family (Section 5.4), which are given in the ancillary files. The relation (38), $v^T M = 0$ for all ϵ , is verified in seven of the ten flow-free directions of the three sectors, at the reference point (30) and with Feynman propagators on the graviton lines; the other three directions are supported on master integrals not yet computed. Rows 1, 2 and 5 are the relation (36) as it appears in the response matrices of the three sectors (the relabeling identities of Section 5.3 equate their supports); the relations in rows 3, 4, 6 and 7 express I_{r6}, I_{r7} in terms of I_{r5} , and their S_3 counterparts $\tilde{I}_6, \tilde{I}_7$ in terms of $\tilde{I}_5$, as in Eq. (41). Master labels as in Table 4.

Sector	Support	Weights at $\epsilon = 0$	Consequence
S_1	(I_0, I_2, I_3)	$(1, 16/3, -6)$	Eq. (36), all orders
S_2	(I_{r0}, I_{r2}, I_{r3})	$(-1/6, 1, -8/9)$	Eq. (36)
S_2	(I_{r1}, I_{r5}, I_{r6})	$(-7/54, 8/27, 1)$	$I_{r6} = -\frac{8}{27}I_{r5} + O(\epsilon)$
S_2	(I_{r1}, I_{r5}, I_{r7})	$(-2/27, 2/27, 1)$	$I_{r7} = -\frac{2}{27}I_{r5} + O(\epsilon)$
S_3	$(\tilde{I}_0, \tilde{I}_2, \tilde{I}_3)$	$(1, -6, 16/3)$	Eq. (36)
S_3	$(\tilde{I}_1, \tilde{I}_5, \tilde{I}_6)$	$(-7/54, 8/27, 1)$	$\tilde{I}_6 = -\frac{8}{27}\tilde{I}_5 + O(\epsilon)$
S_3	$(\tilde{I}_1, \tilde{I}_5, \tilde{I}_7)$	$(-2/27, 2/27, 1)$	$\tilde{I}_7 = -\frac{2}{27}\tilde{I}_5 + O(\epsilon)$

integrals are odd in σ , the equalities $\tilde{I}_{5,6,7} = I_{r5,6,7}$ hold at equal σ , consistent with the identical weights. Direct evaluation through order ϵ confirms the three equalities. The contractions of these two directions with M change only by an overall sign when σ is flipped and vanish term by term for a principal-value worldline propagator. Thus these two relations do not distinguish the retarded from the advanced assignment of σ .

The remaining ϵ -dependent direction of S_2 fixes, at the pole, a combination of four principal-value-class integrals that we have not computed,

$$\begin{aligned}
& \frac{1}{2}I_{r9}^{(-1)} + \frac{5}{12}I_{r10}^{(-1)} + \frac{1}{8}I_{r18}^{(-1)} + I_{r19}^{(-1)} \\
&= \frac{5}{12}I_{r0}^{(-1)} = \pi^4 \frac{100 \ln 2 - 60}{81} + \frac{5}{324} i\pi^5.
\end{aligned} \tag{42}$$

The two remaining directions of S_3 , both in its even block, likewise fix combinations of that sector's eight uncomputed master integrals at the pole and at the finite order. In the seven directions we can test, the relation holds exactly at the pole, in rational arithmetic, on S_1 and S_2 , and at the finite order to the accuracy of the numerically known corner constants on all three sectors.

5.5 K3 and Calabi–Yau sectors

The K3 and Calabi–Yau sectors at 5PM-2SF [36, 37, 38, 39] are subtopologies with six or seven graviton propagators, whereas the leading top sectors of dissipative PX2 have five (Table 3). A period can enter a master integral only through a sector contained in its top sector. Hence no K3 or Calabi–Yau period appears in any of the 37 master integrals of the leading top sectors, consistent with the reductions to Γ , Bessel- K and ${}_2F_1$ functions found above. Any K3 or Calabi–Yau period of dissipative PX2 must instead enter through one of the 36 further top sectors of PX2 that contain these subtopologies (Table 2). We have enumerated these sectors but do not reduce them in this paper, so which of them couple to the dissipative masters remains undetermined.

The planar family P contains the Apéry K3 sector (topology 40 [39]), whose periods, singular at the point z_+ of Eq. (14), we continue to bound-orbit kinematics in Section 6. For the conservative problem Klemm et al. [37] report nine master integrals in this sector, and Duhr et al. [42] list nine master integrals on its maximal cut, obtained with Feynman propagators and all symmetry relations, in their ancillary files. Reducing the even-parity block of the sector on the maximal cut with integration-by-parts and Lorentz-invariance identities, seeded with four dots and four numerator powers and without relabeling symmetries, we find eleven master integrals, unchanged at a second kinematic point; they are listed in the supplementary material. When the sector's worldline-exchange symmetry ($u_1 \leftrightarrow u_2$ with a loop-momentum relabeling) is also imposed, the eleven reduce to nine, the number found in Ref. [42]. We have not mapped the nine integrals of Ref. [42] onto our nine, nor completed the symbolic reduction that would give their differential equation.

6 Continuation to bound orbits

A bound binary has $\gamma < 1$, outside the scattering region $\gamma > 1$. The K3 and Calabi–Yau periods of the 4PM and 5PM integrals [39, 29, 37], as solutions of the Picard–Fuchs equations (8)–(11), nevertheless extend analytically to $\gamma < 1$ regardless of any scattering-to-bound map. We continue their 3×3 and 4×4 Frobenius bases onto the bound domains, evaluate them at twelve bound energies, and compute the monodromy matrices relating continuations along inequivalent paths. However, the complete master sectors, whose bound values need the ϵ -expanded sector transport of the scattering computation, are not continued here. To our knowledge the full Frobenius bases of the Apéry K3' and ${}_4F_3$ geometries have not been evaluated at bound kinematics before.

6.1 Bound kinematics and singular points

For bound motion we write

$$\gamma = \cos \theta \in (0, 1), \quad x = e^{\mp i\theta}, \quad (43)$$

and bound orbits map to the unit circle $|x| = 1$. We fix the branch by the Feynman prescription $s + i0$ on the Mandelstam invariant s of the two-body system, a prescription distinct from the $i0^+$ routing of the memory gravitons. Since $s + i0$ implies $\gamma + i0$, and $d\gamma/dx < 0$ on the scattering segment $x \in (0, 1)$, we continue along the lower arc $x = e^{-i\theta}$, $\theta \in (0, \pi/2]$, which we call the physical branch. The four Picard–Fuchs operators have real coefficients, and on the upper arc the periods take the Schwarz-conjugate values. Each geometry has its own variable on the arc,

$$z_{\text{K3}} = z_{\text{K3}'} = x^2, \quad z_{\text{CY}_3} = \frac{x^4}{28}, \quad z_{\text{CY}_3'} = \frac{1 - \gamma^2}{2^{10}}, \quad (44)$$

the last from $\gamma = (x + 1/x)/2$.

The finite singular points of each operator are the zeros of its leading symbol $s_r(z)$, where $p_r(z) = z^r s_r(z)$ is the leading coefficient of the order- r operator. For the four operators we find

$$s_3^{\text{K3}}(z) = (1 - z)^2, \quad s_3^{\text{K3}'}(z) = 1 - 34z + z^2, \quad s_4^{\text{CY}_3}(z) = 1 - 256z, \quad s_4^{\text{CY}_3'}(z) = (1 - 1024z)^3, \quad (45)$$

with the MUM point at $z = 0$, and we write z_+ for the zero of each symbol nearest to it, as in Eq. (14) for L_A . We can then decide algebraically whether a singular point lies on the bound arc (see Fig. 1). The Apéry symbol has roots $z_{\pm} = 17 \mp 12\sqrt{2}$, with $z_+ \approx 0.0294$, the image of $\gamma = 3$, inside the unit circle and z_- outside. On $|z| = 1$ we have $\bar{z} = 1/z$, hence

$$z^2 - 34z + 1 = z \left(z + \frac{1}{z} - 34 \right) = z (2 \operatorname{Re} z - 34), \quad (46)$$

and since $|2 \operatorname{Re} z| \leq 2 < 34$ the symbol has no zero on the arc. Both endpoints are ordinary as well, $s_3(1) = -32$ and $s_3(-1) = 36$.

For the 4PM K3 of Refs. [34, 35] the arc $z = e^{-2i\theta}$ is again the unit circle $|z| = 1$. It meets the singular point $z = 1$ only at the weak-binding endpoint $\gamma \rightarrow 1$, with the classical threshold behavior

$$\varpi_0 = \left(\frac{2}{\pi}\right)^2 K(z)^2 \sim \frac{1}{\pi^2} \ln^2 \frac{16}{1-z} \quad (\gamma \rightarrow 1^-). \quad (47)$$

The open arc and the $\gamma = 0$ endpoint $z = -1$ are ordinary. The ${}_4F_3$ arc $z = 2^{-8}e^{-4i\theta}$ is instead the circle $|z| = 2^{-8}$ through the singular point z_+ , traversed once clockwise as θ runs over $(0, \pi/2]$. Only its endpoints are singular, both at $z_+ = 2^{-8}$ but on different windings: $\gamma = 1$ at winding zero and $\gamma = 0$ after one full clockwise wind, on the sheet shifted by the inverse MUM monodromy M_0^{-1} . Finally, the bound domain $z \in (0, 2^{-10})$ of the Hadamard threefold is a real segment of ordinary points on which the Frobenius series at the MUM point converge and are real. Its endpoints are the MUM point ($\gamma = 1$) and the triple root $z = 2^{-10}$ of the symbol ($\gamma = 0$). The latter has local exponents $\{0, \frac{1}{2}, \frac{3}{2}, 2\}$ [37] in place of the conifold pattern $\{0, 1, 1, 2\}$, and the monodromy about it squares to the identity. Of the four period bases only that of the Apéry K3' is therefore analytic at the threshold $\gamma = 1$ and along the whole arc, although the complete master sector is singular there (see Section 6.3).

6.2 Periods on the bound arc

We evaluate the full Frobenius basis of each operator at the bound energies

$$\gamma \in \left\{ \frac{199}{200}, \frac{99}{100}, \frac{49}{50}, \frac{97}{100}, \sqrt{\frac{8}{9}}, \frac{23}{25}, \frac{9}{10}, \frac{\sqrt{3}}{2}, \frac{4}{5}, \frac{\sqrt{2}}{2}, \frac{3}{5}, \frac{1}{2} \right\}. \quad (48)$$

The grid is densest at the weak-binding end $\gamma = 199/200$ through $9/10$, which contains the inspiral regime, and extends to $\gamma = 1/2$. The point $\gamma = \sqrt{8/9} \approx 0.9428$, the specific energy of the Schwarzschild innermost stable circular orbit, is included to mark the transition from inspiral to plunge. We integrate each Picard–Fuchs equation from the MUM point along the physical path, and list the resulting holomorphic periods ϖ_0 in Table 6.

The K3 column of Table 6 equals $(2/\pi)^2 K(z)^2$ on the principal arithmetic-geometric-mean branch of $K(z)^2$, and the physical continuation $x = e^{-i\theta}$ therefore stays on that branch along the whole arc. The CY₃ values at γ and $\sqrt{1-\gamma^2}$, mirror points on the singular circle one wind apart, are complex conjugates because ϖ_0 is invariant under M_0 . The Apéry K3' values, by contrast, are nowhere real on the arc and form no complex-conjugate pairs.

The physical K3' path runs from the scattering segment $z \in (z_+, 1)$ through the lower half plane ($z - i0$) onto the arc and encloses no singular point. Any other path inside $|z| < z_-$ differs from it by a word in the unipotent MUM monodromy M_0 and the monodromy around z_+ ,

$$M(z_+) = \begin{pmatrix} 0 & 0 & \pi^2/3 \\ 0 & 1 & 0 \\ 3/\pi^2 & 0 & 0 \end{pmatrix}, \quad M(z_+)^2 = \mathbb{1}. \quad (49)$$

We continue the K3' basis to $z = e^{-i\pi/3}$ ($\gamma = \sqrt{3}/2$) once through each half plane. The path dependence of the continuation is measured and found equal to the expected monodromy: the quotient of the two bases is the involutive Picard–Lefschetz matrix (49), entrywise. On the ${}_4F_3$ arc we integrate to $\gamma = 1/2$ once with the physical clockwise winding and once along the short

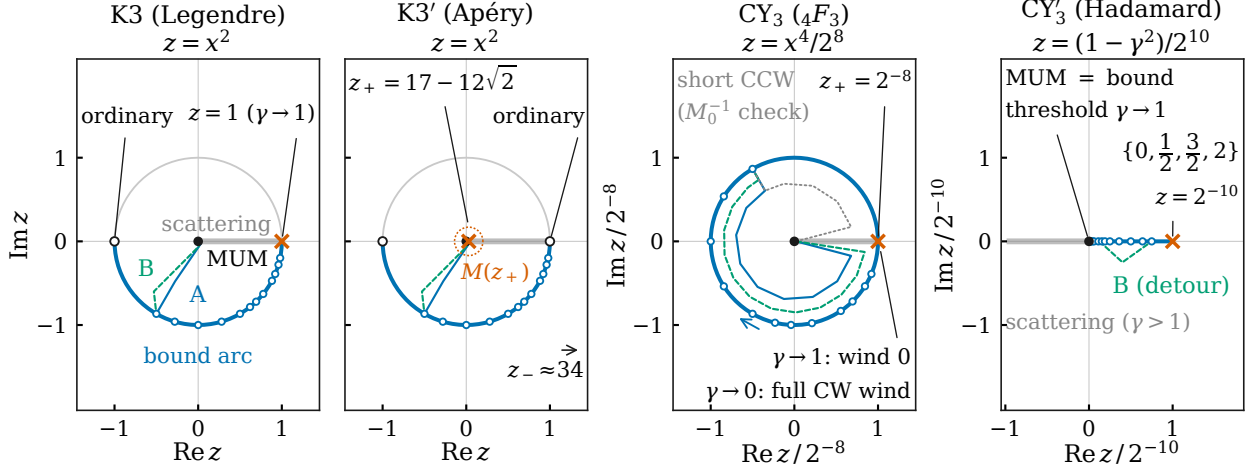


Figure 1: The four bound domains differ, and only the Apéry $K3'$ arc has no singular point on it or at its endpoints. Each panel shows one of the four geometries of Table 1 in its own variable z of Eq. (44): singular points (crosses), the MUM point $z = 0$ (dot), the scattering image $\gamma > 1$ (real segment), the bound arc on the physical branch $x = e^{-i\theta}$ with the twelve grid points of Eq. (48) (open circles), arc endpoints that are regular points of the operator (marked “ordinary”), and the two homotopically equivalent continuation paths from the MUM point along which each bound value is computed (labeled A, solid, and B, dashed; in the CY'_3 panel B is drawn as a detour through the complex plane). The short counterclockwise path in the CY_3 panel, labeled “ M_0^{-1} check,” is the second winding used for Eq. (50) in Section 6.2. $K3$ (4PM): the unit circle, through the singular point $z = 1$ at $\gamma \rightarrow 1$. $K3'$ (Apéry): the unit circle, with $z_+ = 17 - 12\sqrt{2}$ inside (the dotted loop around it has monodromy $M(z_+)$ of Eq. (49)) and $z_- = 17 + 12\sqrt{2}$ outside the frame. CY_3 (${}_4F_3$): the singular circle $|z| = 2^{-8}$, both endpoints at $z_+ = 2^{-8}$ on different windings. CY'_3 (Hadamard): the real segment $(0, 2^{-10})$, ending at the singular point $z = 2^{-10}$ with exponents $\{0, \frac{1}{2}, \frac{3}{2}, 2\}$.

counterclockwise path (see Fig. 1). We find that the quotient of the two bases is the inverse MUM monodromy, with Frobenius-frame entries

$$(M_0^{-1})_{jk} = \frac{(-2\pi i)^{k-j}}{(k-j)!}, \quad k \geq j, \quad (50)$$

and zero below the diagonal, again entrywise.

All entries of Table 6, and the full Frobenius matrices containing them, were computed along two homotopically equivalent paths in ball arithmetic at 600-bit working precision. The two evaluations agree within their enclosures at all 48 points, each value carries an enclosure radius below 10^{-167} , and the computed monodromy quotients contain the matrices (49) and (50) within radii of the same size. Moreover, the CY'_3 column agrees with the direct series and with evaluation through the Hadamard factorization of the operator [37].

6.3 Bound-orbit observables

The continued periods of the previous subsections are properties of the function space and do not determine any bound-orbit observable, which requires in addition a map from scattering data to

Table 6: Holomorphic period $\varpi_0(z)$ of the four geometries of Table 1 continued to the twelve bound energies γ of Eq. (48), on the physical branch $x = e^{-i\theta}$ and in the variables z of Eq. (44), rounded to ten decimal places. For K3', CY₃ and CY₃' the entry is the power-series solution $\varpi_0 = 1 + O(z)$ of the operator (10), (9) or (11) at the MUM point, continued along the physical path of Fig. 1; the K3 entry is the holomorphic period $(2/\pi)^2 K(z)^2$ of the 4PM operator (8), up to its overall power of z , continued along the same path, where it coincides with the principal arithmetic-geometric-mean branch of $K(z)^2$. The CY₃' values are real, as is the CY₃ value at $\gamma = \sqrt{2}/2$ ($z = -2^{-8}$), and the CY₃ values at γ and $\sqrt{1-\gamma^2}$, mirror points on the singular circle, are complex conjugates. The ancillary files, described in the supplementary material, contain in machine-readable form the full 3×3 and 4×4 Frobenius matrices $\varpi_k^{(i)}(z) = d^i \varpi_k / dz^i$ at the twelve energies with their enclosure radii, the continuation paths, and the four operators. The matrices are expressed in the Frobenius frame in which M_0^{-1} takes the form (50), namely $\varpi_k = \sum_{j=0}^k \frac{1}{j!} \ln^j z h_{k-j}(z)$ with $h_0 = \varpi_0$ and $h_1, \dots, h_k$ power series that vanish at $z = 0$.

γ	K3 (Legendre)	K3' (Apéry)	CY ₃ (${}_4F_3$)	CY ₃ ' (Hadamard)
199/200	1.8281307768 - 1.2199046645 <i>i</i>	0.3417124937 - 0.4225037111 <i>i</i>	1.0674161632 - 0.0559110288 <i>i</i>	1.0010955371
99/100	1.5700666433 - 1.0766284225 <i>i</i>	0.3493756077 - 0.4092605183 <i>i</i>	1.0508390588 - 0.0623022668 <i>i</i>	1.0021946619
49/50	1.3445888963 - 0.9303527036 <i>i</i>	0.3595283678 - 0.3905885532 <i>i</i>	1.0306152319 - 0.0658575680 <i>i</i>	1.0044037557
97/100	1.2280652917 - 0.8436346471 <i>i</i>	0.3668018392 - 0.3763102131 <i>i</i>	1.0172410822 - 0.0657218772 <i>i</i>	1.0066274461
$\sqrt{8/9}$	1.0661859232 - 0.7042192556 <i>i</i>	0.3805554073 - 0.3468311375 <i>i</i>	0.9943996701 - 0.0604035025 <i>i</i>	1.0127490157
23/25	0.9933589135 - 0.6310495834 <i>i</i>	0.3887433309 - 0.3274035760 <i>i</i>	0.9823803268 - 0.0543571087 <i>i</i>	1.0179707910
9/10	0.9492069920 - 0.5821243119 <i>i</i>	0.3945362395 - 0.3126148091 <i>i</i>	0.9746670311 - 0.0487989426 <i>i</i>	1.0226163710
$\sqrt{3}/2$	0.8964407888 - 0.5175603307 <i>i</i>	0.4024777460 - 0.2906339333 <i>i</i>	0.9653284283 - 0.0393789390 <i>i</i>	1.0306556121
4/5	0.8334376869 - 0.4279077907 <i>i</i>	0.4137693504 - 0.2549503465 <i>i</i>	0.9552449729 - 0.0219886529 <i>i</i>	1.0468367771
$\sqrt{2}/2$	0.7833203059 - 0.3402898785 <i>i</i>	0.4245374650 - 0.2137327660 <i>i</i>	0.9511579157 + 0.0000000000 <i>i</i>	1.0709536506
3/5	0.7495896393 - 0.2657485597 <i>i</i>	0.4328919173 - 0.1735454010 <i>i</i>	0.9552449729 + 0.0219886529 <i>i</i>	1.1009592492
1/2	0.7296147024 - 0.2096209167 <i>i</i>	0.4383293082 - 0.1402998262 <i>i</i>	0.9653284283 + 0.0393789390 <i>i</i>	1.1314087463

bound dynamics. For the local-in-time conservative sector such a map exists, the boundary-to-bound (B2B) dictionary of K  lin and Porto [69, 70]. Its revised 4PM form agrees with effective-one-body dynamics [71]. For the nonlocal-in-time (tail) sector the dictionary fails for generic bound orbits. Tail terms can be mapped only at large eccentricity [72], they do not continue smoothly between hyperbolic and elliptic motion [63, 73], and beyond perturbation theory the B2B relations fail in numerical relativity [74]. Accordingly, we make no claim in this section about bound-orbit observables obtained by B2B continuation, at this or any PM order.

The dissipative fluxes, on the other hand, are tail-free, and Jakobsen et al. note that a direct map to bound orbits therefore remains possible [28]. The flux is also the natural object for generic bound orbits, since the per-period losses given by the adiabatic map [72] degenerate at even PM orders [35]. The γ -dependent coefficient functions of a flux pass through the adiabatic map unchanged and need only be evaluated at $\gamma < 1$, with the maximal-cut period blocks given here on the branch fixed by $s + i0$. The regularity of the Ap  ry periods at the threshold $\gamma \rightarrow 1$ is, however, a property of the period block only. The differential system of the full K3' master sector has further rows singular at $x = 1$ through the lower-sector letters $x \mp 1$. The exponents of those rows at $x = 1$ require the differential equation of the complete sector, the nine-master system of Refs. [37, 42] or that of the even block of Section 5.5, and have not been determined.

7 Evaluation of the memory-region integrals

In this section we evaluate the two memory-region boundary integrals $I_1^{(M)}$ and $I_2^{(M)}$, Eq. (15), at the static point and extract the constant c_M , which the cancellation conditions leading to Eq. (16)

do not determine. We route the two memory gravitons as in the γ -3 prescription of Ref. [8] and take that routing as input, together with the frequency representations and the integration-by-parts identity of their supplement. We carry out the expansion in ϵ and the numerical evaluation anew, and find $c_M = 1$, the value obtained in Ref. [8]. With Feynman propagators on the two memory gravitons the ϵ^{-4} pole of $I_2^{(M)}$ does not cancel, as Driesse et al. report, and we compute its coefficient. In Section 7.3 we return to the singular point $\gamma = 3$ of Section 4.1, whose Landau locus fixes the position of the divergence but not c_M .

7.1 Frequency representations

Like the master integrals I_0 , I_2 , I_3 of the PX2 top sectors (Section 5.3), $I_1^{(M)}$ and $I_2^{(M)}$ are zero-frequency radiative boundary integrals, and at the static point $x = 1$ they admit two-fold frequency representations [8]. Since c_M is defined at $x = 1$ and the Apéry K3 periods enter only the x dependence of the master integrals, c_M contains no K3 period. These boundary integrals, and the coefficients of the identity (53) below, depend on γ only through overall powers of $\sqrt{\gamma^2 - 1}$ [8], which we strip by quoting all values at $\gamma = \sqrt{2}$, where these powers equal one. We write $\omega_3 = \omega_1 + \omega_2$ and $\int_\omega = \int_{-\infty}^{\infty} d\omega/(2\pi)$. With K_ϵ the modified Bessel function of the second kind, the first integral reads¹

$$I_1^{(M)} = N_1(\epsilon) \int_{\omega_1 \omega_2} \prod_{j=1,2} (0^+ - i\omega_j)^{1-2\epsilon} \prod_{i=1}^3 |\omega_i|^{-\epsilon} K_\epsilon(|\omega_i|),$$

$$N_1(\epsilon) = -2^{7\epsilon-1} \frac{\Gamma^2(\frac{2\epsilon-1}{2}) \Gamma(4\epsilon-1)}{\Gamma(2-5\epsilon) (4\pi)^{5-4\epsilon}}, \quad (51)$$

and a third integral $I_3^{(M)}$, obtained from $I_2^{(M)}$ by inserting the numerator $-D_4^3 D_5^3$ with $D_4 = \ell_3 \cdot v_1$ and $D_5 = \ell_4 \cdot v_1$ [8], is given by

$$I_3^{(M)} = N_3(\epsilon) \int_{\omega_1 \omega_2} \omega_1^3 \omega_2^3 \prod_{j=1,2} (0^+ - i\omega_j)^{-1-2\epsilon} \prod_{i=1}^3 |\omega_i|^{-\epsilon} K_\epsilon(|\omega_i|),$$

$$N_3(\epsilon) = 2^{7\epsilon-3} \frac{\Gamma^2(\frac{2\epsilon+1}{2}) \Gamma(4\epsilon-2)}{\Gamma(3-5\epsilon) (4\pi)^{5-4\epsilon}}. \quad (52)$$

An integration-by-parts identity [8], whose derivation uses propagator symmetries respected by both the γ -3 routing and Feynman propagators, expresses $I_2^{(M)}$ in terms of $I_1^{(M)}$ and $I_3^{(M)}$,

$$I_2^{(M)} = r_1(\epsilon) I_3^{(M)} + r_2(\epsilon) I_1^{(M)}, \quad (53)$$

with

$$r_1(\epsilon) = \frac{28(5\epsilon-2)(14\epsilon-9)(1088\epsilon^4 - 898\epsilon^3 + 275\epsilon^2 - 38\epsilon + 2)}{3\epsilon^3(96\epsilon^3 - 116\epsilon^2 + 43\epsilon - 5)},$$

$$r_2(\epsilon) = \frac{(1-2\epsilon)^2(121024\epsilon^6 - 237340\epsilon^5 + 181552\epsilon^4 - 70363\epsilon^3 + 14767\epsilon^2 - 1612\epsilon + 72)}{6\epsilon^3(96\epsilon^3 - 116\epsilon^2 + 43\epsilon - 5)}. \quad (54)$$

¹We find the opposite sign for the exponent of $|\omega_i|$ relative to Eq. (27) of the supplement of Ref. [8], where the Bessel factor is printed as $|\omega_i|^{+\epsilon} K_\epsilon(|\omega_i|)$. The exponent $-\epsilon$ follows from their Eq. (26) at unit impact parameter, and only with it does the ϵ^{-3} pole of $I_2^{(M)}$ cancel. With that sign our value of c_M agrees with theirs.

Both coefficients grow as ϵ^{-3} , while $I_1^{(M)}$ and $I_3^{(M)}$ start at order ϵ^{-1} . The ϵ^{-4} and ϵ^{-3} poles of the right-hand side of Eq. (53) must then cancel to leave the ϵ^{-2} pole of Eq. (16). Extracting c_M therefore requires $I_1^{(M)}$ and $I_3^{(M)}$ through order ϵ^1 , two orders beyond their leading poles.

In both $I_1^{(M)}$ and $I_3^{(M)}$ the integrand is a product of frequency factors $(0^+ - is_i\omega_i)^{a_i - q_i\epsilon}$, with $s_i = \pm 1$ the routing signs, and Bessel lines $|\omega_i|^{-\epsilon} K_\epsilon(|\omega_i|)$. Expanding the integrand in ϵ gives

$$\text{integrand} = \text{base} \times \exp\left[\epsilon(A + iB) + \epsilon^2 R + \mathcal{O}(\epsilon^3)\right]. \quad (55)$$

Here A collects the logarithms $\ln|\omega_i|$ with integer weights, B the phases of the frequency factors, and R the second derivatives of the Bessel functions in their index ν ,

$$\kappa_2(x) = \frac{1}{2} \partial_\nu^2 K_\nu(x)|_{\nu=0} = \frac{1}{2} \int_0^\infty t^2 e^{-x \cosh t} dt. \quad (56)$$

The $i0^+$ prescription enters through each frequency factor $(0^+ - is\omega)^{a - q\epsilon}$ as a whole, the integer power included. For a retarded ($s = +1$) or advanced ($s = -1$) propagator

$$\ln(0^+ - is\omega) = \ln|\omega| - \frac{i\pi}{2} s \text{sign}(\omega), \quad (57)$$

whereas a Feynman propagator gives $(0^+ - i|\omega|)^{a - q\epsilon}$, with logarithm $\ln|\omega| - i\pi/2$ for either sign of ω . Equations (51) and (52) have retarded propagators on both memory gravitons, $s_1 = s_2 = +1$, and the γ -3 prescription averages this routing with the all-advanced one [8]. The substitution $(\omega_1, \omega_2) \rightarrow (-\omega_1, -\omega_2)$ maps each retarded factor to the advanced one and leaves the rest of the integrand unchanged, so the two routings, and hence their average, give the same values.

Integrating the expanded integrand order by order in ϵ leaves frequency integrals of the base against polynomials in A , B and R , which we call cores. Cores that involve a single frequency reduce to Γ functions through the Mellin transform

$$\int_0^\infty \omega^{s-1} K_\epsilon(\omega) d\omega = 2^{s-2} \Gamma\left(\frac{s+\epsilon}{2}\right) \Gamma\left(\frac{s-\epsilon}{2}\right). \quad (58)$$

The two-frequency cores, in which the three Bessel lines are coupled through $\omega_3 = \omega_1 + \omega_2$, do not reduce to Γ functions. We evaluate them by two-dimensional numerical quadrature, taking the ν -derivative of K_ν in Eq. (56) by finite differences.² Their leading moments are rational multiples of π^2 ,

$$\begin{aligned} \int_{\mathbb{R}^2} \omega_1 \omega_2 \prod_{i=1}^3 K_0(|\omega_i|) d\omega_1 d\omega_2 &= -\frac{2\pi^2}{15}, \\ \int_{\mathbb{R}^2} \omega_1^2 \omega_2^2 \prod_{i=1}^3 K_0(|\omega_i|) d\omega_1 d\omega_2 &= \frac{32\pi^2}{105}. \end{aligned} \quad (59)$$

The first moment is equivalent, through N_1 , to the ϵ^{-1} pole of $I_1^{(M)}$ in Eq. (16). The second then follows from the cancellation of the ϵ^{-4} pole in the identity (53), and direct quadrature of both moments gives the same values.

²This numerical differentiation can be avoided by using the integral representation (56) throughout, which we have done only for part of the two-frequency cores.

7.2 The value of c_M

We expand the integrands of $I_1^{(M)}$ and $I_3^{(M)}$ as in Eq. (55) and integrate the cores order by order in ϵ . The identity (53) then yields $I_2^{(M)}$. The ϵ^{-1} pole of $I_1^{(M)}$ agrees with Eq. (16), the ϵ^{-4} and ϵ^{-3} poles of $I_2^{(M)}$ cancel, and the surviving ϵ^{-2} coefficient gives, under the retarded routing,

$$c_M = 1, \quad (60)$$

reproducing the value of Ref. [8] to the numerical accuracy stated in Table 7.

The subleading Laurent coefficients of $I_1^{(M)}$ and $I_3^{(M)}$ that enter the assembly are not given explicitly in Ref. [8]; we list them here. The ϵ^0 coefficients lie in $\mathbb{Q}[\ln 2\pi - \gamma_E]/\pi^4$, where the combination $\ln 2\pi - \gamma_E$ comes from the $(4\pi)^{-4\epsilon}$ and Γ -function prefactors of the normalizations $N_{1,3}$, with no $e^{\gamma_E \epsilon}$ factor:

$$\begin{aligned} I_1^{(M)}|_{\epsilon^0} &= \frac{1}{2048 \pi^4} \left[\frac{116}{225} + \frac{2}{15} (\ln 2\pi - \gamma_E) \right], \\ I_3^{(M)}|_{\epsilon^0} &= -\frac{1}{131072 \pi^4} \left[\frac{1456}{1575} + \frac{32}{105} (\ln 2\pi - \gamma_E) \right]. \end{aligned} \quad (61)$$

With these closed forms and the leading moments (59), the ϵ^{-3} pole in Eq. (53) cancels analytically, including the $\ln 2\pi - \gamma_E$ terms of $r_1 I_3^{(M)}$ and $r_2 I_1^{(M)}$. The ϵ^1 coefficients, which c_M also requires, we have only numerically; the constants entering c_M are collected in Table 7. An integer-relation search, at the roughly thirty significant figures the quadrature delivers, finds no admixture of $\{\pi^2, \ln 2, \zeta(3), \pi^2 \ln 2\}$ in c_M , consistent with a rational value.

With Feynman propagators on D_{31} and D_{24} the frequency factors of Eqs. (51) and (52) become $(0^+ - i|\omega_j|)^{1-2\epsilon}$ and $(0^+ - i|\omega_j|)^{-1-2\epsilon}$, so the integrand changes at order ϵ^0 : the leading factor $-\omega_1\omega_2$ of $I_1^{(M)}$ becomes $-|\omega_1\omega_2|$. The ϵ^{-1} coefficient of $I_1^{(M)}$ then changes by the factor $-1.3465\dots$ and no longer satisfies Eq. (16). Because the ϵ^{-1} coefficient of $I_3^{(M)}$ is multiplied by a different factor, $-0.9243\dots$, the ϵ^{-4} pole of $I_2^{(M)}$ in Eq. (53) survives. Its coefficient is $1/(6144 \pi^6)$ to the seventeen figures of our evaluation (exactly, if the moments (59) over single quadrants have the closed forms identified in the supplementary material), and no constant c_M in the sense of Eq. (16) is defined. Reference [8] reports the same uncanceled pole for Feynman $i0^+$ on the gravitons, without its coefficient, together with a $(\gamma - 3)^{-15/2}$ divergence of the resulting observable. Porto and Riva infer from their fifth-post-Newtonian computation that their prescription applied to Feynman boundary conditions would imply values of $I_1^{(M)}$ and $I_2^{(M)}$ equal to minus those of Eq. (16) at $c_M = 1$ [32]. These inferred values are not direct evaluations of Eqs. (51)–(52) with Feynman propagators, and our evaluation, which agrees with Ref. [8] on the uncanceled pole, does not bear on them.

7.3 The singular point $\gamma = 3$

Driesse et al. [8] find that the potential region diverges at $\gamma = 3$ through the integrals of the K3 sector, and that the memory region cancels this divergence once its boundary integrals have the pole structure of Eq. (16), for any value of c_M . In Section 4.1 we showed that $\gamma = 3$ is the leading Landau singularity [67] of the maximal cut of the K3 sector [37] (topology 40 [39]), where the cut surface acquires a node on the integration torus. The ten on-shell conditions of topology 40 have no simultaneous solution at real loop momenta, because a real null momentum orthogonal to a timelike velocity vanishes. The Coleman–Norton theorem [75] therefore associates no real classical process with this singularity. The divergence arises in the potential-region integrals, whose x dependence is insensitive to the graviton $i0^+$ [8], whereas c_M is fixed at $x = 1$ by the routing of D_{31} and

Table 7: Constants entering the assembly of $c_M = 1$, Eq. (60), through the identity (53); the ϵ^{-4} and ϵ^{-3} poles of $I_2^{(M)}$ cancel (last row). The entries are frequency-space moments and Laurent coefficients in ϵ of the memory-region boundary integrals $I_1^{(M)}$, $I_2^{(M)}$ of Eq. (15) and $I_3^{(M)}$ of Eq. (52) at the static point, in the normalization of Eqs. (51)–(52) with the overall powers of $(\gamma^2 - 1)$ stripped. The first two rows are the two-frequency moments of Eq. (59), with $\prod K_0 = \prod_{i=1}^3 K_0(|\omega_i|)$, $\omega_3 = \omega_1 + \omega_2$, integrated over $\mathbb{R}^2$; the next four rows are the leading poles of $I_1^{(M)}$ and $I_3^{(M)}$ and the ϵ^0 coefficients of Eq. (61), with γ_E Euler’s constant. The closed forms and the ϵ^{-1} coefficients are exact; the ϵ^1 coefficients of $I_1^{(M)}$ and $I_3^{(M)}$ are numerical, rounded to ten figures, and two implementations of the expansion and quadrature agree on them to at least seventeen figures. The pole cancellation of the last row holds for retarded and for advanced propagators on the two memory gravitons, Eq. (57), and fails for Feynman propagators. From these entries and the Laurent expansions of the coefficients (54) through order ϵ^{-1} , $I_2^{(M)}$ can be reassembled through order ϵ^{-2} ; with the entries as printed, the result is $c_M = 1$ to eight figures. The entries to the full working precision are given in the ancillary files and in the supplementary material.

Quantity	Value
$\int \omega_1 \omega_2 \prod K_0$	$-2\pi^2/15$
$\int \omega_1^2 \omega_2^2 \prod K_0$	$32\pi^2/105$
$I_1^{(M)} _{\epsilon^{-1}}$	$1/(15(8\pi)^4)$
$I_3^{(M)} _{\epsilon^{-1}}$	$-1/(1720320\pi^4)$
$I_1^{(M)} _{\epsilon^0}$	$[\frac{116}{225} + \frac{2}{15}(\ln 2\pi - \gamma_E)]/(2048\pi^4)$
$I_3^{(M)} _{\epsilon^0}$	$-[\frac{1456}{1575} + \frac{32}{105}(\ln 2\pi - \gamma_E)]/(131072\pi^4)$
$I_1^{(M)} _{\epsilon^1}$	$4.144449270 \times 10^{-5}$
$I_3^{(M)} _{\epsilon^1}$	$-1.061146240 \times 10^{-6}$
$I_2^{(M)} _{\epsilon^{-4}, \epsilon^{-3}}$	0

D_{24} in boundary integrals regular at $\gamma = 3$. The Landau locus (21) gives the position of the singularity, $\gamma = 3$, but the locus does not fix the value of c_M , which requires the explicit evaluation in Section 7.2. We adopt the routing of Ref. [8] as input; deriving it would require the conservative projection of the in-in integrand, whose graviton propagators are retarded [7].

8 Conclusions and outlook

We studied the four-loop integrals of fifth-order post-Minkowskian scattering at second self-force order through their maximal cuts and boundary data. We showed that in the scattering region the leading Landau singularity [67] of the maximal cut of the planar K3 sector lies at $\gamma = 3$, where the potential-region contribution diverges [8]. From the same cut we derived the Apéry K3 surface and its Picard–Fuchs operator, in agreement with Refs. [37, 42], and identified the surface with the Apéry–Fermi K3 [41] by a moduli map. For all four non-polylogarithmic geometries we fixed the constant relating the maximal-cut leading singularity to the holomorphic period, and for the Hadamard threefold CY'_3 we gave the Frobenius coefficients of that period in closed form, Eq. (26). We obtained first-stage master-integral counts for the four families. With the worldline-exchange symmetry imposed, the even block of the Apéry sector has nine master integrals on the maximal cut, the number found in Refs. [37, 42]. On three of the four leading top sectors of PX2 we computed twelve master integrals at the reference point, with Feynman graviton propagators. The response

matrices of the auxiliary-mass flow [43, 44] imply ten linear relations among these sectors’ master integrals, each an exact reduction identity of the family, and the computed integrals satisfy the seven that involve only them. We continued the Frobenius bases of the Picard–Fuchs operators to the bound arc and recovered the expected monodromies. None of these results is a 5PM observable, because the scattering angle, radiated energy, and recoil at $O(G^5\nu^2)$ require, beyond the functions and boundary data obtained here, the coefficient of every master integral in the reduced integrand.

The maximal-cut and boundary-data methods used here, like the leading-singularity analysis of Duhr et al. [42], avoid the IBP reduction of the full integrand, a major bottleneck of the post-Minkowskian program [39]. Neither approach yet supplies numerical values of the master integrals themselves, which the 5PM-2SF radiative observables require in addition to the functions constructed here. A dissipative observable at this order needs further input:

- The differential equations in x that would carry the boundary constants from the reference point to general kinematics have not been solved for any 2SF sector. Solving them also needs the remaining master integrals of PX2 (Section 5.3), the retarded or advanced assignment of the worldline label σ , and the conversion of the boundary data to retarded graviton propagators [7].
- The fourth leading top sector of PX2, its thirty-six further top sectors that contain a geometric subtopology, and all but one of the 109 top sectors of the other three families have not been analyzed.
- Which of those additional PX2 sectors couple K3 or Calabi–Yau periods into the dissipative master integrals is open.
- Converged dissipative master bases require reductions at deeper seeding than those behind Table 2.
- The continuation is carried out for the period functions only: the complete master sectors are not continued, no bound-orbit observable is claimed, and our statements about bound-orbit fluxes are limited to the adiabatic map. A bound-orbit flux would follow only through that map [72, 73] or through a direct map for tail-free fluxes, left open in Ref. [28].

With the retarded routing of Driesse et al. [8] as input, the memory-region integrals give $c_M = 1$, the value obtained there. Ref. [8] reports that Feynman propagators on the memory gravitons leave an ϵ^{-4} pole in $I_2^{(M)}$, as we also find in the boundary representation (Section 7.2). Refs. [32, 33] state that their Feynman-based prescription is free of ϵ poles, and Ref. [32] gives the boundary values it would imply. Our result fixes the retarded-side value and does not by itself decide between these statements. The routing is an input here, as in Ref. [8]; its derivation from the in-in integrand [7] and the type of the Landau singularity at $\gamma = 3$ remain open (Section 7.3).

The normalizations of the canonical bases and the boundary constants obtained here enter the differential equations from which radiation reaction at this order would follow. Given the second-self-force integrand coefficients and the inputs listed above, the ϵ -expanded solutions of the three PX2 top sectors treated here yield these sectors’ contribution to the dissipative observables. With those coefficients and numerical values of the master integrals, one could determine whether the K3 and Calabi–Yau periods survive in the radiated energy, as at first self-force order [29].

A Auxiliary-mass flow on worldline families

On a cut worldline family the auxiliary-mass flow [43, 44] requires boundary integrals that are not vacuum integrals, and we do not evaluate them for the 2SF sectors. The boundary constants reported in Section 5 are instead master integrals computed directly at the reference point (30), and we check them against the linear relations (38) that the flow implies.

A.1 Boundary data on worldline families

The auxiliary-mass flow method [43, 44] determines master integrals by deforming a subset of propagators, $D_i \rightarrow D_i - \eta$. The differential equation in η is then integrated from large auxiliary mass, where the integrals simplify, down to the physical point $\eta = 0$. For an ordinary Feynman family the boundary integrals at $\eta \rightarrow \infty$ are vacuum integrals, and the method is self-contained [44]. However, on dissipative worldline families the cut propagators $2\pi\delta(2k \cdot u)$ cannot be deformed, and the large- η regions of a cut sector contain boundary integrals with cut worldline propagators instead. The master integrals at $\eta = 0$ are related to these boundary integrals by the response matrix $A(\epsilon)$ defined in Eq. (27), which we compute at each value of ϵ separately.

A left null vector of $A(\epsilon)$ annihilates the response to any boundary data, so a rank deficit of $A(\epsilon)$ implies linear relations among the master integrals at $\eta = 0$, exact when the boundary matching is complete (Section 5.4). By contrast, at a fixed point of the boundary recursion (Appendix B) the response relation becomes Eq. (63) and a single flow leaves the combinations in $\ker(A - I)$ undetermined. A second η insertion or externally supplied values then fix them. In Section 5.2 we determine the ranks and left null spaces of $A(\epsilon)$ for S_1 , S_2 , S_3 and for the first NP top sector, and the remaining top sectors of the four families have not been analyzed. The master integrals at the reference point (30) are the initial data for the differential equations in the kinematic variable x , and the transport of the 2SF sectors has not been run. We do not perform the reduction of the full 2SF integrands onto master integrals, the step that dominates the cost of the conservative computation [8]; the reductions we do run are the far smaller ones internal to the flow of individual sectors. The ancillary files contain the family definitions, the PX2 reduction setup, and the master-integral lists; the reduction tables and the flow solver are not distributed.

A.2 Tests at lower orders

We tested the use of externally supplied boundary data at $\mathcal{O}(G^3)$, where the master integrals [25] and the radiated energy [6] are known in closed form. For two master integrals of the two-loop soft family of Ref. [25] we derived graviton-cut boundary data from the closed forms given there and supplied them to the flow. The resulting values agree order by order in ϵ with twice the imaginary part of the standard flow solution [44]. We also assembled the $\mathcal{O}(G^3)$ radiated energy at $\gamma = \frac{5}{4}$ through our own cut reduction and Fourier transform. The inputs were the cut integrand of Di Vecchia, Heissenberg, Russo, and Veneziano [5] and the master values of Ref. [25]. The result reproduces the closed form of Herrmann, Parra-Martínez, Ruf, and Zeng [6], with the ϵ^{-4} through ϵ^{-1} poles canceling. In Appendix B we test the auxiliary-mass flow on the 4PM cut-eikonal K3 family [34, 35]. On that family the constraints of two flows with different η insertions are satisfied by a single set of values of the 43 master integrals at each sampled value of ϵ .

A.3 The solver

We carry out the η -evolution with our implementation of the auxiliary-mass flow method [44], extended in three respects for dissipative worldline families. First, we evaluate several classes

of boundary integrals of cut sectors at $\eta \rightarrow \infty$ in closed form. Among them, the two-velocity cusp integrals are of ${}_3F_2$ type in the relative Lorentz factor. Second, uncut linearized worldline propagators introduce half-integer powers of η in the boundary expansion. The local exponents of a coupled block then include both integers and half-integers, as in Eq. (29). Let x now denote the flow variable rather than the kinematic one, and let $d\mathbf{f}/dx = M(x)\mathbf{f}$ be the differential system of such a block. In the double-cover variable $t = \sqrt{x}$ this system takes the form

$$\frac{d\mathbf{f}}{dt} = 2t M(t^2)\mathbf{f} \quad (62)$$

with the integer and half-integer exponent lattices in x combined into a single integer lattice in t . The boundary solutions are then power series in t . Third, the reduction that generates the boundary conditions of the largest sectors is performed in stages. In the most demanding sector this shortens the list of integrals to be reduced from several million to a few thousand.

In the standard enumeration of large- η regions each loop momentum has the scaling $\ell \sim \eta^0$ or $\ell \sim \eta^{1/2}$ [44], whereas along a linearized worldline propagator the region $\ell \sim \eta$ also contributes. With that region omitted, the closed-form boundary integral of a one-loop factor with an odd number of linearized worldline propagators evaluates to $-i(\pi/2)\Gamma(\epsilon)4^\epsilon$. This expression has an ϵ pole, although the full boundary integral is finite for each $i0$ prescription. With the $\ell \sim \eta$ regions and the cusp integrals included, we obtain the correct one-loop and factorized two-loop boundary values. After a further correction to the boundary values with bare worldline propagators, the two-loop product of eikonal vertices takes its exact value $\pi^4/4$.

Three further classes of two-velocity boundary integrals arise in the coupled sectors of the $\mathcal{O}(G^3)$ radiative family and have been derived but not all implemented. Those sectors have therefore not been solved through the flow; the $\mathcal{O}(G^3)$ tests above do not require these classes, since the master values used there are those of Ref. [25]. We have not shown that the implemented classes exhaust the boundary integrals of the coupled 2SF sectors. The response matrices of these sectors, however, are computed from the enumerated large- η regions and the differential equation in η , without using the values of the boundary integrals. Completeness of this enumeration is assumption (ii) of Section 5.4 and is not proved here.

B Beyond the maximal cut at 4PM

In this appendix we ask to what extent the master integrals of the three-loop K3 sector of the conservative 4PM dynamics [14, 18, 34, 35] can be determined beyond the maximal-cut construction of Section 4. We use function-space counting and then the auxiliary-mass flow [43, 44].

B.1 Function space and non-holomorphic components

The construction stops at the maximal cut, where it fixes the normalization of the holomorphic cycle in each Calabi-Yau block and therefore of the canonical basis. For the components along the other cycles the Landau locus (69) determines only the function space in which they lie [67]. In one variable, integrability imposes no condition (all $d\log \wedge d\log = 0$), and the three letters $\{x, 1-x, 1+x\}$ give 3^w symbols at weight w . Tensoring with the Sym^2 -Legendre span (24) of Ref. [35] gives 3 functions per master integral at ϵ^0 (weight 0), 9 at weight 1, and 27 at weight 2. At ϵ^0 the count is consistent with the maximal cut (23), which lies on the holomorphic cycle. The weight-one and weight-two counts, and the non-holomorphic components at weight zero, are untested predictions. The KK' and K'^2 components of the masters are sourced by non-holomorphic

boundary data and would require values of the individual masters, which we do not have. Moreover, the static limit $x \rightarrow 1$ provides no reference values for these components, since $K(1)$ diverges.

The elliptic kernels of the ϵ -form equation (5.40) of Ref. [35] are modular Eisenstein-type kernels on $\Gamma_0(4)$, and the semisimple part of that equation is a uniform power of K'^2 . We therefore expect the remaining components of the elliptic block and their order- ϵ coefficients to lie in the span (24) dressed by the three letters at low weight. A few values per master integral would then fix their rational coefficients. That expectation is not tested here: it requires values of the individual masters at several kinematic points, and none are published for this sector.

B.2 A fixed point of the flow

In the auxiliary-mass flow [43, 44] the boundary recursion at $\eta \rightarrow \infty$ ordinarily ends on vacuum integrals. The 4PM cut-eikonal K3 family, with five graviton propagators and three cut linearized worldline propagators $-2k \cdot u$ [16, 19, 35], is instead a fixed point of the recursion. Cut propagators cannot be deformed. The auxiliary mass is then inserted in the same five graviton propagators at every level, and the $\eta \rightarrow \infty$ rescaling reproduces the family, which we call self-similar under the flow. Successive levels have identical numbers of master integrals, 43 before reduction and 38 or 39 after, for every ordering of the boundary prescriptions [44] that we tried.

At a fixed point the boundary vector b in Eq. (27) coincides with the vector of master integrals $\mathbf{M}$, and the response relation becomes

$$\mathbf{M} = A(\epsilon) \mathbf{M}, \quad (63)$$

which determines only $\text{rank}(A - I)$ independent linear combinations of them. We built $A(\epsilon)$ column by column for the 43 master integrals at twelve values of ϵ . The matrix is sparse, with 88 nonzero entries. Of its rows, 25 are pure identity (master integrals whose η dependence is an exact power of $1 + \eta$), and the 18 coupled rows give

$$\text{rank}(A - I) = 15, \quad \dim \ker(A - I) = 28. \quad (64)$$

The fixed point therefore determines 15 of the 43 master integrals, and the 28 kernel directions are the single-scale cut-eikonal vacuum integrals in $1 + \eta$. The rank is a numerical determination at one kinematic point, $\gamma = 5/3$ ($x = 1/3$); we have not confirmed it symbolically in ϵ .

In the PM literature the master integrals of such cut-eikonal families are obtained from differential equations in x with static-limit boundary values [63, 39], an approach which we have not implemented for this family. Here we constrain the kernel directions with a second flow, β . We call the flow above, with η on the five graviton propagators, flow α . Flow β , with η on all non-cut propagators, either terminates on a pure cut vacuum or reaches a second fixed point whose constraint rows intersect $\ker(A_\alpha - I)$ nontrivially. In the latter case a few analytic inputs may close the system. Among them are the closed-form value of sector 63 (a cube of cut tadpoles),

$$J_{63} = -\frac{i}{8} \pi^{-3/2} \Gamma(\epsilon - \frac{1}{2})^3, \quad (65)$$

and its equality with the integral of sector 111, $J_{111} = J_{63}$.

Evolving unit vectors through the levels of the β recursion gives the linear response $C(\epsilon)$ of the 43 master integrals to the 68 boundary values $\mathbf{v}$ at which the recursion ends,

$$\mathbf{M} = C(\epsilon) \mathbf{v}. \quad (66)$$

We impose the fixed-point condition (63) of flow α on this $\mathbf{M}$ and supplement it with Eq. (65) and the equality $J_{111} = J_{63}$. We solve the resulting linear system for $\mathbf{v}$ at $\gamma = 5/3$ and $\gamma = 13/5$ with

eighteen values of ϵ at each, one run per value. All 43 constraint rows of flow β are then satisfied with residuals of order 10^{-52} . At both points and at every ϵ node the first two master integrals m_0, m_1 of the cut top sector obey

$$\frac{m_1}{m_0} = -\frac{2}{5}, \quad (67)$$

an integration-by-parts identity recovered here numerically.

The flow-invariant directions (the left kernel of $A - I$) pair nondegenerately with the fixed-point space $\ker(A - I)$ whenever the unit eigenvalue is semisimple, as we find it to be. A solution annihilated by every flow-invariant direction would therefore vanish, whereas the cut-eikonal family has nonzero master integrals that are products of Γ functions. The flow-invariant directions thus do not all annihilate the solution, in contrast with the flow-free directions, which annihilate Ab in Eq. (27).

B.3 Master ratios at leading order in ϵ

In three of the eighteen runs at $\gamma = 13/5$, and in none at $\gamma = 5/3$, the eight top-sector master integrals $m_0, \dots, m_7$ are rescaled, relative to the other runs, by a common complex factor that the subsector master integrals do not share and that we have not derived. The factor cancels in ratios of top-sector master integrals from the same run. At the sampled values of ϵ , all near 10^{-6} , each m_r is dominated by its leading Laurent term, so the ratios m_r/m_0 , $r = 1, \dots, 7$, are ratios of leading coefficients. At 200-digit working precision the ratios extrapolate to $\epsilon = 0$ with at least 91 digits stable under changes of the set of runs, at each kinematic point. Each limit is a rational number: rational reconstruction returns a fraction with numerator and denominator below 10^{18} , fixed by at most 36 digits, that reproduces all the stable digits. A separate computation at 60-digit working precision gives the same fractions to its own accuracy of at least 34 digits. The simplest of these fractions is the value of m_1/m_0 in Eq. (67), reproduced at both points to more than 150 digits, and the twelve fractions with $r \geq 2$ are listed in the supplementary material. At leading order in ϵ the eight top-sector master integrals are thus rational multiples of one another at the two points sampled. We have not determined the next order, at which the three K3 periods of Eq. (24) can first enter these ratios, nor derived the ratios from the differential equation at $\epsilon = 0$.

The individual 4PM K3 masters remain undetermined as functions of ϵ . We also tried to evaluate the full 4PM K3 integral directly, without IBP reduction, by integrating out its Baikov variables [38, 39] one at a time. These integrations can be done in closed form until they reach the two variables on which the elliptic period of the maximal cut depends. At that stage a seven-dimensional residual integral remains and has not been evaluated.

C Landau analysis with linear propagators

In this appendix we analyze the Landau equations [67] of post-Minkowskian topologies with linear worldline propagators, through two loops and for the four-loop Apéry sector. We also give the alphabet implied by the leading singularities of Brammer et al. [39] for the non-planar top topology.

C.1 Landau equations and first-type letters

Post-Minkowskian integral families in the worldline formalism [16, 19] contain both quadratic graviton propagators $D_j = \ell_j^2$ and the linear propagators $D_e = 2u_{a(e)} \cdot \ell_e$ of worldline $a(e)$. The integrals depend on the kinematics only through $\gamma = u_1 \cdot u_2 = (1 + x^2)/(2x)$. With loop momenta k_l , line

momenta $\ell_i = \sum_l c_{il} k_l + r_i$ and Feynman parameters α_i , the Landau equations are

$$\sum_j \alpha_j c_{jl} \ell_j + \sum_e \alpha_e c_{el} u_{a(e)} = 0 \quad (l = 1, \dots, L), \quad \alpha_j \ell_j^2 = 0, \quad \alpha_e (2u_{a(e)} \cdot \ell_e) = 0, \quad (68)$$

where j labels graviton lines and e worldline lines. The first Symanzik polynomial U is that of the graviton subgraph, and the second Symanzik polynomial F is at most quadratic in each α_e . The variable γ enters F only through the cross-terms $\alpha_e \alpha_{e'} (u_{a(e)} \cdot u_{a(e')})$. Moreover, U contains no kinematic invariants and the standard second-type condition $U = 0$ is trivial.

At one and two loops we solve the Landau equations by saturated Gröbner-basis elimination of the Feynman parameters. For the 2PM box and the 3PM IY and H topologies [13, 17] the first-type Landau locus is

$$\{x, 1 - x, 1 + x\}, \quad (69)$$

which is the polylogarithmic alphabet at 2PM and 3PM [39]. On one face of the H topology an unsaturated resultant chain produces a spurious letter γ , which saturation removes (the elimination ideal there is $\langle 1 \rangle$). We have not run the three-loop 4PM topologies through the elimination, so at 4PM we note only that the alphabet of Ref. [39] has the same three letters. The 5PM-1SF letter $1 + x^2$ [29] is not a first-type component of any one- or two-loop topology, so it must arise at four loops, plausibly from a second-type face of the Newton polytope of $U + F$; we have not derived it.

The Landau equations above also apply to the four-loop topology 40 (the Apéry K3 sector [37, 39]), for which F has degree one in γ . We take as faces the 544 subtopologies with at least four propagators and a linear propagator of each worldline. These faces are obtained by setting Feynman parameters to zero in F ; we did not examine the mixed faces of the Newton polytope of $U + F$. On the 490 faces with four to seven propagators we eliminate the Feynman parameters from the first-type Landau equations with a lexicographic Gröbner basis at symbolic γ . The elimination, run with a time limit on each face, terminated on all but 19 of these faces and gave only $\gamma = \pm 1$, that is, the letters $\{x, x \pm 1\}$. On the full ten-propagator topology a resultant-chain elimination at symbolic γ did not terminate. On all 544 faces we also test whether the Landau ideal, saturated by the product of the Feynman parameters, has a solution at fixed rational values of γ . The fixed values are the zeros in γ of $1 + x^2$, $x^2 \pm x + 1$ and $x^2 \pm 6x + 1$, namely $\gamma_0 \in \{0, \pm \frac{1}{2}, \pm 3\}$, together with the generic value $\gamma = 7/11$; no other values of γ are probed. A face yields a letter vanishing at γ_0 only if its equations have a solution there and none at generic γ . The test terminated at all six values except on eight faces with eight to ten propagators, among them the full ten-propagator topology. Elsewhere the outcome was the same at all six values: twelve faces with seven or eight propagators are nonempty throughout, and the rest have no solution or no γ dependence. The test cannot detect a component of the Landau variety confined to γ_0 on the twelve generically nonempty faces, and neither computation constrains the eight unresolved ones. With these limitations we find no first-type letter of topology 40 beyond $\{x, x \pm 1\}$, a result we use in Section 4.1.

C.2 The topology-41 alphabet

Topology 41 of Ref. [39] is the thirteen-propagator top topology of the non-planar family. Both the $d \log$ structure of the six-dimensional leading singularities computed there and the factorization over $\mathbb{Q}[x]$ of the algebraic ones lead to the alphabet

$$\mathcal{A}_{\text{top.41}} = \{x, x - 1, x + 1, x^2 + 1\}. \quad (70)$$

This alphabet coincides with the multiple-polylogarithm alphabet of the 5PM-1SF result [29] and is a subset of the kernels of the 5PM-2SF observables [8].

The letter $x^2 + 1$ does not arise from a first-type Landau pinch, which on each face we analyzed would require orthogonal worldlines, $u_1 \cdot u_2 = 0$. In topology 41 it enters instead through zeros of the leading singularities, such as $\text{LS}_{\text{even}}^{(6D)} = q^2(x^2 + 1)/(x^2 - 1)$, which become poles of the canonically normalized master integrals. The alphabet (70) is a prediction: no master of this topology has been computed, and the absence of the cyclotomic letters $x^2 \pm x + 1$ in particular remains to be tested.

D Numerical checks

In this appendix we collect the numerical evaluations underlying Tables 1 and 4.

D.1 Maximal cuts and period identities

We evaluated each contour integral of Table 1 at three to five rational points and divided it by the Frobenius series of its Picard–Fuchs operator [37]. The ratio is constant across the points to at least 34 significant figures in each case and equals the value in Table 1. That value is unity by construction for the Apéry row (Table 8) and is otherwise found by an integer-relation search (for the 4PM row the sampled values are tabulated in the supplementary material). For topology 37 the residue series and a trapezoidal quadrature of the T^3 integral give the same constant.

The identity $L_A \varpi = 0$ for the Apéry row would follow from a Griffiths–Dwork reduction of the integrand of Eq. (19). The corresponding Griffiths–Dwork system, of rank 163, is consistent at the three rational values $x = 1/7, 2/11, 3/17$ at which we solved it, and remains unsolved at symbolic x .

We evaluate no master integral beyond the maximal cut, and two further restrictions apply to the three 5PM rows. First, only the 4PM surface is derived from the Gram determinant. At four loops the nested reduction of the 7×7 Gram determinant at symbolic d stops on factors with exponent $(2 - d/2)$. The loop-by-loop construction of Refs. [38, 39] takes one further residue and reaches the surfaces Q_6, P_8, Q_8 , which we use as input in Section 4.2. The Gram-determinant reduction still reproduces the residual dimension at $d = 4$ (three for topology 3, four for topology 37). Second, all sampled kinematic points lie inside the Frobenius disk around the MUM point [37] ($z = x^2 < (3 - 2\sqrt{2})^2$ for topology 40, $|z| < 2^{-8}$ for topology 3, $|z| < 2^{-10}$ for topology 37). For topology 3 the product-of-circles cycle is obstructed for $x \gtrsim 0.28$, and points outside the disks would require period transport or a deformed contour, which we have not attempted; in particular no point crosses the singular point $\gamma = 3$ of the Apéry operator discussed in Section 7.

Table 8: The torus period of the Apéry sector equals the holomorphic Frobenius solution of L_A with no adjustable normalization. Listed is $\varpi(x^2)$ of Eq. (19), evaluated by trapezoidal quadrature on T^2 with $Q_6 = P_+ P_-$, at three rational points inside the disk $z < (3 - 2\sqrt{2})^2$ in which the Frobenius series of L_A [37] converges. The series $\varpi_0 = \sum_n A_n z^n$, with A_n the Apéry numbers of Eq. (13), reproduces every digit shown, and $\varpi(0) = 1$ on both sides. Conversely, fitting a third-order operator with cubic polynomial coefficients to these contour data and their z -derivatives alone returns the coefficients $(0, 0, 0, 1; -5, -27, -51, -34; 1, 3, 3, 1)$ of L_A .

x	$\varpi(x^2)$
1/11	1.04731834954215233249142889358089...
1/13	1.03248859719863337655413910339274...
3/20	1.18676654372101512485005499381289...

D.2 Boundary constants

We derived each constant in Table 4 and in Eq. (35) at least twice, by routes sharing no code and no term-level algebra. We obtained the closed forms from the numerical values by integer-relation search, and the finite parts of I_0, I_2, I_3 are known to about 26 significant figures. Where the two routes use the same integral table or analytic continuation, the complete integral is in addition evaluated directly at one kinematic point at which it reduces to a sum of residues. The integer-relation fits are run at two numerical accuracies, with constants of known value in the search basis. A relation is kept only if both runs return it and it continues to hold when re-evaluated against the data at full accuracy. The real parts of the pole coefficients in Eq. (32) rest on numerically identified closed forms whose direct derivation reduces to a logarithm–dilogarithm cancellation that we have not carried out. Finally, the constants I_{r5}, I_{r6}, I_{r7} satisfy the instances of the relation (38) that connect them (Table 5), and agree through order ϵ with their S_3 counterparts $\tilde{I}_5, \tilde{I}_6, \tilde{I}_7$.

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