Supplementary material for "Memory back-reaction in black-hole scattering at fifth post-Minkowskian order"

Matthew D. Schwartz*
Department of Physics, Harvard University

This document collects material that supports the main text and is too long for it: the definitions of the master integrals on which the linear relations of Section 5 of the main text are supported (Section S1), the reduction of the computed boundary integrals to frequency integrals (Section S2), the specification of the continuation of the period bases to bound kinematics and of the accompanying data files (Section S3), the numerical data behind the constant c_M of the memory-region integrals, including the leading terms obtained with Feynman propagators on the two memory gravitons (Section S4), the master-integral lists behind the counts of the main text (Section S5), the rational master ratios of the three-loop (4PM) cut-eikonal family of Appendix B (Section S6), and the sampled values of the 4PM maximal cut of Section 4.2 and Appendix D (Section S7). Equation, table and reference numbers of the form (S$n.m$), Sn refer to this document; unprefixed numbers refer to the main text.

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S1 Supporting master integrals of the PX2 top sectors

S1.1 Family and conventions

The crossed family PX2 of the main text (NP2 of Ref. [1]) has loop momenta $k_1, \ldots, k_4$, worldline velocities u_1, u_2 and momentum transfer q, with

$$u_1^2 = u_2^2 = 1\,, \qquad u_1 \cdot u_2 = \gamma\,, \qquad u_i \cdot q = 0\,, \qquad q^2 = -1\,. \tag{S1.1}$$

Its integrals are

$$I_{n_1 \cdots n_{22}} \;=\; \int \prod_{i=1}^{4} \frac{d^D k_i}{i\pi^{D/2}} \, \frac{\prod_{c=1}^{4} 2\pi\, \delta^{(n_c-1)}(D_c)}{\prod_{i=5}^{22} D_i^{n_i}}\,, \qquad D = 4 - 2\epsilon\,, \tag{S1.2}$$

with the twenty-two propagators

$$\begin{aligned}
&D_1 = 2k_1 \cdot u_1\,, && D_2 = 2k_2 \cdot u_1\,, && D_3 = 2k_3 \cdot u_2\,, && D_4 = 2k_4 \cdot u_2\,,\\
&D_5 = 2k_1 \cdot u_2\,, && D_6 = 2k_1 \cdot u_2 - 2k_2 \cdot u_2\,, && D_7 = 2k_3 \cdot u_1\,, && D_8 = 2k_3 \cdot u_1 - 2k_4 \cdot u_1\,,\\
&D_9 = (k_1 - k_2)^2\,, && D_{10} = (k_1 - k_3)^2\,, && D_{11} = (k_1 - k_4)^2\,, && D_{12} = (k_2 - k_3)^2\,,\\
&D_{13} = (k_2 - k_4)^2\,, && D_{14} = (k_3 - k_4)^2\,, && D_{15} = (k_1 + q)^2\,, && D_{16} = (k_2 + q)^2\,,\\
&D_{17} = (k_3 + q)^2\,, && D_{18} = (k_4 + q)^2\,, && D_{19} = k_1^2\,, && D_{20} = k_2^2\,,\\
&D_{21} = k_3^2\,, && D_{22} = k_4^2\,.
\end{aligned} \tag{S1.3}$$

The first four are cut: $n_c = 1$ denotes the delta function and $n_c = 2$ its first derivative. A negative exponent $n_i < 0$ denotes the numerator $D_i^{|n_i|}$. The uncut linearized propagators $D_5, \ldots, D_8$ enter as $1/(D_i + i\sigma 0)$ with an orientation $\sigma = \pm 1$ that is not part of the family definition, and the graviton propagators $D_9, \ldots, D_{22}$ carry no $i0$ prescription in the family definition. All boundary constants of Section 5 of the main text and all values in this document are computed with Feynman propagators on the graviton lines; no in-in boundary constants are computed. No factor $e^{\gamma_E \epsilon}$ is included, and all values refer to the reference point $\gamma = 5/4$, $q^2 = -1$, that is $x = \gamma - \sqrt{\gamma^2 - 1} = 1/2$.

A sector is labeled by the integer $\sum_i 2^{\,i-1}$ over the propagators D_i present as denominators or cut lines. The three leading top sectors used in the main text have the thirteen denominators $D_1, \ldots, D_9$ together with $\{D_{13}, D_{14}, D_{15}, D_{21}\}$ (S_1, label 1077759), $\{D_{11}, D_{14}, D_{16}, D_{21}\}$ (S_2, label 1091071) and $\{D_{11}, D_{14}, D_{17}, D_{20}\}$ (S_3, label 599551). An integral is called even or odd according to its sign under $u_i \to -u_i$, which reverses every linearized propagator: each power of an uncut $D_5, \ldots, D_8$ and each derivative of a cut delta function contributes a factor -1.

Table S1: The 31 master integrals on which the ten left null vectors of Section 5 of the main text are supported, as exponent vectors $(n_1,\dots,n_{22})$ of Eq. (S1.2) over the propagators (S1.3), grouped as $(n_1\cdots n_4\,|\,n_5\cdots n_8\,|\,n_9\cdots n_{14}\,|\,n_{15}\cdots n_{18}\,|\,n_{19}\cdots n_{22})$; a barred entry $\bar{1}$ is the exponent -1 (numerator). "Parity" is the sign under $u_i \to -u_i$; "linear den." lists the uncut linearized propagators present as denominators; "num., dot" lists numerators and squared propagators. The last column gives the status at the reference point $\gamma = 5/4$: "computed" (value in Table 4 or Section 5.3 of the main text), an identity with another entry of the table, or "not computed". All entries have $n_1 = \cdots = n_4 = 1$.

integral	sector	$(n_1\cdots n_4\,|\,n_5\cdots n_8\,|\,n_9\cdots n_{14}\,|\,n_{15}\cdots n_{18}\,|\,n_{19}\cdots n_{22})$	parity	linear den.	num., dot	value
I_0	S_1	$1111\;0000\;100011\;1000\;0010$	even	–	–	computed
I_2	S_1	$1111\;0000\;1\bar{1}0011\;1000\;0010$	even	–	D_{10}	computed
I_3	S_1	$1111\;0000\;10\bar{1}011\;1000\;0010$	even	–	D_{11}	computed
I_{r0}	S_2	$1111\;0000\;101001\;0100\;0010$	even	–	–	$= I_0$
I_{r1}	S_2	$1111\;\bar{1}000\;101001\;0100\;0010$	odd	–	D_5	$= 0$
I_{r2}	S_2	$1111\;0000\;1\bar{1}1001\;0100\;0010$	even	–	D_{10}	$= I_3$
I_{r3}	S_2	$1111\;0000\;101\bar{1}01\;0100\;0010$	even	–	D_{12}	$= I_2$
I_{r4}	S_2	$1111\;0000\;101002\;0100\;0010$	even	–	D_{14}^2	computed
I_{r5}	S_2	$1111\;1000\;101001\;0100\;0010$	odd	D_5	–	computed
I_{r6}	S_2	$1111\;1000\;101\bar{1}01\;0100\;0010$	odd	D_5	D_{12}	computed
I_{r7}	S_2	$1111\;1000\;1010\bar{1}1\;0100\;0010$	odd	D_5	D_{13}	computed
I_{r9}	S_2	$1111\;\bar{1}010\;101001\;0100\;0010$	even	D_7	D_5	not computed
I_{r10}	S_2	$1111\;001\bar{1}\;101001\;0100\;0010$	even	D_7	D_8	not computed
I_{r18}	S_2	$1111\;1010\;101001\;0100\;0010$	even	D_5,D_7	–	not computed
I_{r19}	S_2	$1111\;1010\;1010\bar{1}1\;0100\;0010$	even	D_5,D_7	D_{13}	not computed
$\tilde{I}_0$	S_3	$1111\;0000\;101001\;0010\;0100$	even	–	–	$= I_0$
$\tilde{I}_1$	S_3	$1111\;\bar{1}000\;101001\;0010\;0100$	odd	–	D_5	$= 0$
$\tilde{I}_2$	S_3	$1111\;0000\;1\bar{1}1001\;0010\;0100$	even	–	D_{10}	$= I_3$
$\tilde{I}_3$	S_3	$1111\;0000\;101\bar{1}01\;0010\;0100$	even	–	D_{12}	$= I_2$
$\tilde{I}_4$	S_3	$1111\;0000\;101002\;0010\;0100$	even	–	D_{14}^2	$= I_{r4}$
$\tilde{I}_5$	S_3	$1111\;1000\;101001\;0010\;0100$	odd	D_5	–	$= I_{r5}$
$\tilde{I}_6$	S_3	$1111\;1000\;101\bar{1}01\;0010\;0100$	odd	D_5	D_{12}	$= I_{r6}$
$\tilde{I}_7$	S_3	$1111\;1000\;1010\bar{1}1\;0010\;0100$	odd	D_5	D_{13}	$= I_{r7}$
$\tilde{I}_9$	S_3	$1111\;\bar{1}010\;101001\;0010\;0100$	even	D_7	D_5	not computed
$\tilde{I}_{10}$	S_3	$1111\;001\bar{1}\;101001\;0010\;0100$	even	D_7	D_8	not computed
$\tilde{I}_{19}$	S_3	$1111\;\bar{1}001\;101001\;0010\;0100$	even	D_8	D_5	not computed
$\tilde{I}_{20}$	S_3	$1111\;00\bar{1}1\;101001\;0010\;0100$	even	D_8	D_7	not computed
$\tilde{I}_{23}$	S_3	$1111\;1010\;101001\;0010\;0100$	even	D_5,D_7	–	not computed
$\tilde{I}_{24}$	S_3	$1111\;1010\;1010\bar{1}1\;0010\;0100$	even	D_5,D_7	D_{13}	not computed
$\tilde{I}_{31}$	S_3	$1111\;1001\;101001\;0010\;0100$	even	D_5,D_8	–	not computed
$\tilde{I}_{32}$	S_3	$1111\;1001\;101\bar{1}01\;0010\;0100$	even	D_5,D_8	D_{12}	not computed

S1.2 The 31 supporting rows

The left null vectors of the three response matrices of Section 5 of the main text are supported on 3, 12 and 16 of their rows. Table S1 lists the master integral of each of these 31 rows by its exponent vector $(n_1,\dots,n_{22})$ in Eq. (S1.2). The names are those of the main text: I_0, I_2, I_3 on S_1, I_{rk} for row k of S_2 and $\tilde{I}_k$ for row k of S_3, with the row index that of the response matrix.

Three changes of integration variables relate entries of the table. (i) The relabeling $k_1 \leftrightarrow k_2$ exchanges $D_1 \leftrightarrow D_2$, $D_{10} \leftrightarrow D_{12}$, $D_{11} \leftrightarrow D_{13}$, $D_{15} \leftrightarrow D_{16}$ and $D_{19} \leftrightarrow D_{20}$, fixes D_9, D_{14}, D_{17}, D_{18}, D_{21}, D_{22}, D_3, D_4, D_7, D_8, and maps $D_5 \to D_5 - D_6$, $D_6 \to -D_6$. It maps the nine-propagator corner sector of S_1 (denominators $D_9D_{13}D_{14}D_{15}D_{21}$ and the four cuts) onto that of S_2

(denominators $D_9D_{11}D_{14}D_{16}D_{21}$), and gives $I_{r0} = I_0$ and $I_{r3} = I_2$. (ii) The reflection $k_i \to -k_i - q$ for all i exchanges $D_{14+j} \leftrightarrow D_{18+j}$, fixes $D_9, \ldots, D_{14}$, and reverses the sign of every linearized propagator, since $u_i{\cdot}q = 0$. It maps the top sector S_2 onto S_3 row by row; the cut delta functions are even, so an entry of S_3 without uncut linearized denominators equals the corresponding entry of S_2 up to the sign $(-1)^m$, with m the number of linearized numerator powers, while for entries with an uncut denominator the orientation σ of that line is reversed. This gives $\tilde{I}_0 = I_{r0}$, $\tilde{I}_2 = I_{r2}$, $\tilde{I}_3 = I_{r3}$, $\tilde{I}_4 = I_{r4}$ and $\tilde{I}_1 = -I_{r1}$; since I_{r5}, I_{r6}, I_{r7} are odd in σ (Section S2.2), it also gives $\tilde{I}_{5,6,7} = I_{r5,6,7}$ at equal σ. (iii) Within the corner sector of S_1 the map $k_1 \to -k_3 - q$, $k_2 \to -k_4 - q$, $k_3 \to -k_1 - q$, $k_4 \to -k_2 - q$ together with $u_1 \leftrightarrow u_2$ reverses the chain of propagators $D_{15}D_9D_{13}D_{14}D_{21}$ carrying the momentum q from one worldline to the other: it exchanges $D_{15} \leftrightarrow D_{21}$ and $D_9 \leftrightarrow D_{14}$, fixes D_{13} and D_{10}, exchanges the numerators $D_{11} \leftrightarrow D_{12}$, and permutes the four cut lines with a sign; composed with the reversal of all loop momenta and of q it is free of sign changes on the linearized propagators and is the automatic self-map of S_1 described in Section S5.1. Combined with (i) it gives $I_{r2} = I_0[D_{12}] = I_0[D_{11}] = I_3$, where $I_0[N]$ is I_0 with the numerator N. With the identifications of the corner triple alone the 31 rows involve at most 25 distinct integrals, the bound quoted in the main text; the reflection (ii) lowers it further. Twelve of them are computed directly in the main text: I_0, I_2, I_3, I_{r1}, I_{r4}, I_{r5}, I_{r6}, I_{r7}, $\tilde{I}_1$, $\tilde{I}_5$, $\tilde{I}_6$, $\tilde{I}_7$, of which I_{r1} and $\tilde{I}_1$ vanish identically because their integrand is odd under the reflection of the in-plane worldline frequencies (Section 5.3 of the main text). The twelve rows marked "not computed" contain an uncut linearized propagator of the second worldline inside the k_3 loop (D_7 or D_8), alone or together with D_5; their reduction classes are described in Section 5.3 of the main text.

A reduction of the corner sectors that seeds numerator integrals, at two numerator powers and two dots beyond the corner, relates the entries further. It returns $I_3 = \frac{1}{6}I_0 + \frac{8}{9}I_2$ with coefficients independent of d and of γ, which is the corner relation $I_0 + \frac{16}{3}I_2 - 6I_3 = 0$ of Section 5.4 of the main text, and it identifies I_{r4} with I_0 with D_9 squared, the image of D_{14} under (iii). The other nine linear relations detected by the left null vectors are likewise reduction identities of the undeformed family at this seeding, obtained from its integration-by-parts and Lorentz-invariance identities together with the automatic sector symmetries, with coefficients rational in d. Raising the seeds of the corner sector of S_1 by one further numerator power and one further dot leaves the master integrals of this reduction unchanged.

S2 Reduction of the boundary integrals to frequency integrals

This section outlines how the boundary constants of Section 5.3 of the main text were obtained: the corner integrals I_0, I_2, I_3 of S_1 (Section S2.1), the static-slice integrals I_{r5}, I_{r6}, I_{r7} of S_2 (Section S2.2) and the dotted integral I_{r4} (Section S2.3). The conventions are those of Section S1.1, at $\gamma = 5/4$, $q^2 = -1$, $x = 1/2$. We give the chain of exact steps and the resulting one- and two-dimensional frequency integrals; the bookkeeping of the Fourier-transform constants and of the signs of the vector insertions, which is mechanical, is stated in outline only.

S2.1 The corner integrals I_0, I_2, I_3

The corner integral of S_1 is

$$I_0 = \int \prod_{i=1}^{4} \frac{d^D k_i}{i\pi^{D/2}} \; \frac{(2\pi)^4\, \delta(2k_1{\cdot}u_1)\, \delta(2k_2{\cdot}u_1)\, \delta(2k_3{\cdot}u_2)\, \delta(2k_4{\cdot}u_2)}{(k_1-k_2)^2\,(k_2-k_4)^2\,(k_3-k_4)^2\,(k_1+q)^2\,k_3^2}\,, \qquad \text{(S2.1)}$$

and $I_2 = I_0[(k_1-k_3)^2]$, $I_3 = I_0[(k_1-k_4)^2]$ carry the indicated numerators. The reduction proceeds in five steps.

(1) Each delta function confines its loop momentum to the Euclidean hyperplane orthogonal to a worldline velocity, $k_1, k_2 \in V_1 = u_1^\perp$ and $k_3, k_4 \in V_2 = u_2^\perp$, of dimension $\tilde{d} = 3-2\epsilon$:

$$\int d^D k \; 2\pi\, \delta(2k\cdot u)\, f(k) = \pi \int_{u^\perp} d^{\tilde{d}}\kappa \; f(\kappa)\,. \tag{S2.2}$$

We write $k_2 = b\,\hat{e}_1 + k_{2T}$ and $k_4 = a\,\hat{e}_2 + k_{4T}$, where $\hat{e}_1 \in V_1$ and $\hat{e}_2 \in V_2$ are the unit vectors in the plane spanned by u_1, u_2, the transverse parts k_{iT} are orthogonal to that plane and live in $T = 2-2\epsilon$ Euclidean dimensions, and a, b are the two worldline frequencies.

(2) The k_1 and k_3 integrations are massless one-loop bubbles inside V_1 and V_2,

$$\int \frac{d^{\tilde{d}}\kappa}{\pi^{\tilde{d}/2}} \, \frac{1}{\kappa^2\,(\kappa+P)^2} = G_{11}\,(P^2)^{-\frac{1}{2}-\epsilon}\,, \qquad G_{11} = \frac{\Gamma(\frac{1}{2}+\epsilon)\,\Gamma(\frac{1}{2}-\epsilon)^2}{\Gamma(1-2\epsilon)}\,, \tag{S2.3}$$

with $P^2 = b^2 + |k_{2T}+q|^2$ for the k_1 bubble (propagators $(k_1+q)^2$, $(k_1-k_2)^2$) and $P^2 = a^2 + |k_{4T}|^2$ for the k_3 bubble (propagators k_3^2, $(k_3-k_4)^2$). The numerators of I_2 and I_3 contain k_1 linearly and quadratically; the vector bubble is $\frac{1}{2}P^\mu$ times the scalar one, with P the momentum flowing through the bubble, and a numerator equal to one of the bubble's own denominators gives a scaleless integral. After this step the numerators become

$$\begin{aligned} (k_1-k_3)^2 \;&\to\; M_2 = p_1\cdot q - 1 + \tfrac{\gamma}{2}\, s\,\frac{w_1 w_2}{r^2} + \tfrac{1}{2}\, p_1\cdot p_3 - q\cdot p_3\,, \\ (k_1-k_4)^2 \;&\to\; M_3 = p_1\cdot q - 1 + \gamma\, s\,\frac{w_1 w_2}{r^2} + p_1\cdot p_3 - 2\, q\cdot p_3 - \big(a^2 + |k_{4T}|^2\big)\,, \end{aligned} \tag{S2.4}$$

where $p_1 = k_{2T}+q$ and $p_3 = k_{4T}$ are the transverse momenta of the two remaining lines, the dot products are Euclidean and transverse, and the in-plane product ab has been written in the rescaled variables of step (4) below, $ab = s\,w_1 w_2/r^2$ with $s = \pm 1$ the relative sign of the two frequencies. The last term of M_3 lowers the power of the k_3-bubble result by one unit.

(3) The one graviton propagator that joins the two hyperplanes, $(k_2-k_4)^2$, contains the indefinite form

$$-(k_2-k_4)^2 = F + |k_{2T}-k_{4T}|^2\,, \qquad F = a^2 + b^2 - 2ab\gamma\,, \tag{S2.5}$$

which vanishes for $\gamma > 1$ on the two rays $a/b = x^{\pm 1}$ (the radiation cone); the Feynman prescription is $F \to F - i0$. What remains is a two-loop chain in the transverse space, with momentum q entering a line of mass b and index $\lambda_1 = \frac{1}{2}+\epsilon$ (the k_1-bubble result), passing through the exchange of mass $\sqrt{F}$ and index one, and leaving through a line of mass a and index $\lambda_3 = \frac{1}{2}+\epsilon$. In transverse position space the chain is a product,

$$\begin{aligned} C &= (4\pi)^T \int d^T x\; e^{-iq\cdot x}\, \alpha_{\lambda_1}(x;b)\, \alpha_1(x;\sqrt{F})\, \alpha_{\lambda_3}(x;a)\,, \\ \alpha_\lambda(x;m) &= \frac{2^{1-\lambda}}{(2\pi)^{T/2}\,\Gamma(\lambda)} \Big(\frac{m}{|x|}\Big)^{\frac{T}{2}-\lambda} K_{\frac{T}{2}-\lambda}(m|x|)\,, \end{aligned} \tag{S2.6}$$

with $\alpha_\lambda(x;m)$ the T-dimensional Fourier transform of $(m^2+p^2)^{-\lambda}$. A vector insertion $p\cdot q$ on a line differentiates its factor, through $\frac{d}{dr}\big[(m/r)^\nu K_\nu(mr)\big] = -(m/r)^\nu\, m\, K_{\nu+1}(mr)$.

(4) At fixed $r = |x|$ we rescale the frequencies, $(b,a) = (w_1,w_2)/r$. The angular integral of $e^{-iq\cdot x}$ gives a Bessel function $J_{T/2-1}(r)$ (or $J_{T/2}(r)$ for a term with one factor of $q\cdot\hat{x}$), and the radial integral is then of Weber–Schafheitlin type,

$$\int_0^\infty r^s J_\nu(r)\, dr = 2^s\, \frac{\Gamma\big(\frac{1+\nu+s}{2}\big)}{\Gamma\big(\frac{1+\nu-s}{2}\big)}\,, \tag{S2.7}$$

understood by analytic continuation in s outside its strip of convergence.

(5) The two frequency integrals remain. With $\nu = \frac{T}{2} - \lambda_1 = \frac{1}{2} - 2\epsilon$ the scalar line factors are $w^\nu K_\nu(w)$, a differentiated line gives $-w^{\nu+1}K_{\nu+1}(w)$, and the exchange gives $\hat{F}^{-\epsilon/2}K_\epsilon\big(\sqrt{\hat{F} - i0}\big)$ with $\hat{F} = w_1^2 + w_2^2 - 2s\gamma\, w_1 w_2$. The core of a term with insertion $\mathcal{I}(w_1, w_2, s)$ is

$$W(\epsilon) = 2 \sum_{s=\pm 1} \int_0^\infty \int_0^\infty dw_1\, dw_2\, \mathrm{sh}_1(w_1)\, \mathrm{sh}_3(w_2)\, \hat{F}^{-\epsilon/2} K_\epsilon\big(\sqrt{\hat{F} - i0}\big)\, \mathcal{I}(w_1, w_2, s)\,, \tag{S2.8}$$

where $\mathrm{sh}_{1,3}$ are the two line factors and the factor 2 accounts for the two quadrants with the same relative sign s. For $s = +1$ and $w_1/w_2 \in (x, 1/x)$ the form $\hat{F}$ is negative and the Bessel function is continued through $K_\nu(-iy) = \frac{i\pi}{2} e^{i\nu\pi/2} H_\nu^{(1)}(y)$; this is the only place where the graviton $i0$ enters, and it makes the cores complex. Collecting the constants, the corner integral is

$$I_0(\epsilon) = -\pi\, 2^{5\epsilon} \frac{\Gamma(\frac{1}{2} - \epsilon)^4\, \Gamma(4\epsilon - 1)}{\Gamma(1 - 2\epsilon)^2\, \Gamma(2 - 5\epsilon)}\, W_A(\epsilon)\,, \tag{S2.9}$$

with W_A the core (S2.8) with unit insertion. The factor $\Gamma(4\epsilon - 1)$ gives the simple ultraviolet pole of Table 4 of the main text; W_A is analytic at $\epsilon = 0$ and complex, so the pole coefficient has an imaginary part. The numerator integrals I_2 and I_3 are sums of terms of the same form, one for each term of Eq. (S2.4), with radial factors of the type $\Gamma(4\epsilon - 2)$ and cores with the insertions $s\, w_1 w_2$, one differentiated line, or two. Two identities reduce the number of distinct cores to four: the core with the first line differentiated equals the core with the third line differentiated, by $w_1 \leftrightarrow w_2$ in Eq. (S2.8), and the line factor with its index raised by one unit, $w^{\nu+1}K_{\nu+1}(w)$, which the last term of M_3 produces, is minus the differentiated line factor. At $\epsilon = 0$ the four cores are, with A the unit insertion, B the insertion $s\, w_1 w_2$, C one differentiated line and E two,

$$\begin{aligned} W_A(0) &= \frac{\pi\,(320\ln 2 - 192)}{27} + \frac{4\pi^2}{27}\, i\,, & W_B(0) &= \frac{\pi\,(3056640 - 4409344\ln 2)}{2187} + \frac{352\pi^2}{2187}\, i\,, \\ W_C(0) &= \frac{\pi\,(20480\ln 2 - 14592)}{243} - \frac{68\pi^2}{243}\, i\,, & W_E(0) &= \frac{\pi\,(4014080\ln 2 - 2777088)}{2187} + \frac{1252\pi^2}{2187}\, i\,. \end{aligned} \tag{S2.10}$$

The imaginary parts were obtained in closed form from the cone segment of the $s = +1$ sector; the real parts were identified by an integer-relation search in $\{\pi, \pi\ln 2\}$ at two working precisions and reproduce quadratures of the cores carried to more than 100 digits. The pole coefficients of I_0, I_2, I_3 in Table 4 of the main text follow from Eq. (S2.10) and the residues of the Γ-function prefactors; for instance the residue of Eq. (S2.9) is $\frac{\pi^3}{4} W_A(0)$, which reproduces $I_0\big|_{\epsilon^{-1}} = \pi^4(80\ln 2 - 48)/27 + i\pi^5/27$. The finite parts require the cores at first order in ϵ, where the line factors produce $\ln w$ and $\partial_\nu K_\nu$ and the exchange produces $\ln \hat{F}$; they were computed by quadrature in two independent parametrizations of the (w_1, w_2) quadrant, which agree to 26 digits, and have not been identified in closed form. Table S2 gives them to the accuracy of that agreement; Table 4 of the main text quotes the first ten figures.

S2.2 The static-slice integrals I_{r5}, I_{r6}, I_{r7}

By the relabeling (i) of Section S1.2 the corner sector of S_2 has the same structure as that of S_1, with k_1 and k_2 interchanged: the k_2 and k_3 integrations are the bubbles, and $k_1 \in V_1$ (frequency b) and $k_4 \in V_2$ (frequency a) are joined by the exchange $(k_1 - k_4)^2$. The integrals I_{r5}, I_{r6}, I_{r7} contain in addition the uncut propagator $1/(D_5 + i\sigma 0)$ with $D_5 = 2k_1 \cdot u_2 = -2\sqrt{\gamma^2 - 1}\, b$. We split it as

$$\frac{1}{D_5 + i\sigma 0} = \mathrm{PV}\frac{1}{D_5} - i\pi\sigma\, \delta(D_5)\,, \qquad \delta(D_5) = \frac{\delta(b)}{2\sqrt{\gamma^2 - 1}} = \frac{2}{3}\,\delta(b) \quad (\gamma = \tfrac{5}{4})\,. \tag{S2.11}$$

Table S2: Finite parts of the corner integrals I_0, I_2, I_3 of Eq. (S2.1) at $\gamma = 5/4$, $q^2 = -1$, with Feynman graviton propagators, in the normalization of Section S1.1; the digits shown are those on which two independent quadratures agree, truncated, and the last column is the corresponding absolute accuracy. The pole parts are the closed forms of Table 4 of the main text.

integral	$\mathrm{Re}\, I\big\vert_{\epsilon^0}$	$\mathrm{Im}\, I\big\vert_{\epsilon^0}$	accuracy
I_0	388.8416383979840135592604872	161.9427314980753499103669332	10^{-25}
I_2	−131.69229067090160529225841	−35.15238209960831752412489	10^{-23}
I_3	−52.252874196692980222130731	−4.256106616639279480827641	10^{-24}

The reflection of both in-plane frequencies, $(a,b) \to (-a,-b)$ at fixed transverse momenta, preserves the cut delta functions and every quadratic propagator and reverses D_5; the principal-value part is odd under it and integrates to zero, so only the $\delta(b)$ term survives and the three integrals are linear in σ, purely imaginary for Feynman gravitons and independent of the graviton $i0$: at $b = 0$ the form $F = a^2$ of Eq. (S2.5) is positive and the radiation cone is not reached. The same reflection shows that I_{r1}, whose only linearized factor is the numerator D_5, vanishes.

On the slice $b = 0$ the momentum k_1 is purely transverse and one frequency integral remains,

$$I_{r5} = \sigma\, \frac{2\pi i}{3}\, \pi^{2\epsilon-1}\, G_{11}^2\, J\,, \qquad J = \int_{-\infty}^{\infty} da \int d^T k_1\, d^T k_4\, \frac{\left(|k_1+q|^2\right)^{-\frac{1}{2}-\epsilon} \left(a^2+|k_4|^2\right)^{-\frac{1}{2}-\epsilon}}{a^2 + |k_1-k_4|^2}\,, \quad \text{(S2.12)}$$

with k_1, k_4 transverse and Euclidean. In position space J factorizes as in step (3) above into a massless line, the exchange of mass $|a|$ and a massive line of mass $|a|$; rescaling $a = w/r$ leaves a Weber–Schafheitlin integral (S2.7) in r and a Mellin-type product of two Bessel functions in w,

$$\int_0^{\infty} t^{-\lambda} K_\mu(t) K_\nu(t)\, dt = \frac{2^{-2-\lambda}}{\Gamma(1-\lambda)} \prod_{\pm,\pm'} \Gamma\Big(\frac{1-\lambda \pm \mu \pm' \nu}{2}\Big)\,, \quad \text{(S2.13)}$$

the product running over the four sign choices, here with $\mu = \frac{T}{2}-1$, $\nu = \frac{T}{2}-\lambda_3$ and $t^{-\lambda} = w^{T-1-\lambda_3}$, so that I_{r5} is a product of Γ functions to all orders in ϵ. Its expansion has no pole and starts with $I_{r5} = -\frac{16}{3}\, i\pi^5\, \sigma + O(\epsilon)$, Table 4 of the main text; the integral representation converges for $\frac{1}{8} < \epsilon < \frac{1}{4}$ and the closed form is its continuation.

For $I_{r6} = I_{r5}[(k_2-k_3)^2]$ and $I_{r7} = I_{r5}[(k_2-k_4)^2]$ the numerators are reduced by the two bubbles as in step (2): $\langle k_2 \rangle = \frac{1}{2}(k_1-q)$ in the k_2 bubble (propagators $(k_1-k_2)^2$, $(k_2+q)^2$) and $\langle k_3 \rangle = \frac{1}{2} k_4$ in the k_3 bubble, and a numerator equal to a bubble denominator is scaleless. At $b = 0$ this gives the Euclidean transverse insertions

$$(k_2-k_3)^2 \to N_6 = k_1 q + \tfrac{1}{2}\, k_1 k_4 - \tfrac{1}{2}\, q k_4\,, \qquad (k_2-k_4)^2 \to N_7 = k_1 q + k_1 k_4 - q k_4 - \left(a^2 + |k_4|^2\right), \quad \text{(S2.14)}$$

inserted into J of Eq. (S2.12) with the same prefactor. Each term is again a Weber–Schafheitlin integral times a Bessel product (S2.13) with shifted indices; only the three radial exponent pairs $(s,\nu) = (9\epsilon-2, -\epsilon)$, $(9\epsilon-3, 1-\epsilon)$, $(9\epsilon-4, -\epsilon)$ and two Bessel products, (μ, ν_0) and (μ, ν_0+1) with $\mu = \frac{T}{2}-1$ and $\nu_0 = \frac{T}{2}-\lambda_3$, occur. The results are the closed forms of Table 4 of the main text, $I_{r6} = \frac{128}{81} i\pi^5 \sigma + O(\epsilon)$ and $I_{r7} = \frac{32}{81} i\pi^5 \sigma + O(\epsilon)$, with the order-$\epsilon$ terms given there.

S2.3 The dotted integral I_{r4}

I_{r4} is the corner integral of S_2 with $(k_3-k_4)^2$ squared. In the frame of Section S2.1 the k_3 bubble then has indices $(2,1)$,

$$\int \frac{d^{\tilde d}\kappa}{\pi^{\tilde d/2}}\,\frac{1}{(\kappa^2)^2\,(\kappa+P)^2} = G_{21}\,(P^2)^{-\frac{3}{2}-\epsilon}\,, \qquad G_{21} = \frac{\Gamma(\frac{3}{2}+\epsilon)\,\Gamma(-\frac{1}{2}-\epsilon)\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(-2\epsilon)}\,, \tag{S2.15}$$

which vanishes linearly as $\epsilon \to 0$. The third line now has index $\lambda_3 = \frac{3}{2}+\epsilon$ and line factor $w^{-\frac{1}{2}-2\epsilon}K_{\frac{1}{2}+2\epsilon}(w)$, and the radial integral (S2.7) has $(s,\nu) = (9\epsilon-1,-\epsilon)$, with a pole from $\Gamma(4\epsilon)$ that cancels the zero of G_{21}. The two-frequency core is singular at the endpoint $w_2 \to 0$, where the third line factor behaves as $w_2^{-1-4\epsilon}$. We subtract the endpoint with an e^{-w_2}-damped counterterm, $W' = T_{\rm pole} + T_{\rm reg}$, where the counterterm integrates in closed form,

$$T_{\rm pole} = 4\,A(\epsilon)\,\Gamma(-4\epsilon)\int_0^\infty w^{\frac{1}{2}-3\epsilon}K_{\frac{1}{2}-2\epsilon}(w)K_\epsilon(w)\,dw\,, \qquad A(\epsilon) = 2^{-\frac{1}{2}+2\epsilon}\,\Gamma(\tfrac{1}{2}+2\epsilon)\,, \tag{S2.16}$$

the last integral being again of the form (S2.13), and the regular remainder at $\epsilon = 0$ is elementary in the inner variable because $w^{-1/2}K_{1/2}(w) = \sqrt{\pi/2}\,e^{-w}/w$:

$$\begin{aligned} T_{\rm reg}(0) &= 2\sum_{s=\pm 1}\frac{\pi}{2}\int_0^\infty\!\!\int_0^\infty dw_1\,dw_2\,e^{-w_1}\,\frac{e^{-w_2}}{w_2}\Big[K_0\big(\sqrt{\hat F - i0}\,\big) - K_0(w_1)\Big] \\ &= -\frac{4\pi}{3}\ln 2 + \frac{i\pi^2}{3}\,, \end{aligned} \tag{S2.17}$$

the $s=+1$ sector contributing $\pi(\frac{2}{3}\ln 2 + \frac{i\pi}{3})$ and the $s=-1$ sector $-2\pi\ln 2$. The pole of I_{r4} comes from the endpoint term and is real; the cone enters first through the imaginary part of $T_{\rm reg}(0)$. Assembling the prefactor gives $I_{r4} = \pi^4/\epsilon + \pi^4(4 + \frac{20}{3}\ln 2 - 4\gamma_E) - \frac{2}{3}i\pi^5 + O(\epsilon)$, Section 5.3 of the main text; the value (S2.17) was obtained by two quadratures in different variables agreeing to 38 digits and then identified.

S3 Continuation of the period bases to bound kinematics

This section specifies the computation behind Table 6 and Section 6.2 of the main text closely enough that the tabulated Frobenius matrices in the ancillary files can be compared entry by entry with an independent integration: the operators, the Frobenius frame, the variables and paths, the file layout, the per-point accuracy, and the two monodromy checks.

S3.1 Operators

With $\theta = z\,d/dz$ the four Picard–Fuchs operators of the main text are

$$\begin{aligned} L_{\rm K3} &= (2\theta-1)^3 + z^2(2\theta+1)^3 - 4z\theta\,(4\theta^2+1)\,, && z = x^2, \\ L_{{\rm CY}_3} &= \theta^4 - 2^8 z\big(\theta+\tfrac{1}{2}\big)^4\,, && z = 2^{-8}x^4, \\ L_A &= \theta^3 - z(2\theta+1)(17\theta^2+17\theta+5) + z^2(\theta+1)^3\,, && z = x^2, \\ L_{{\rm CY}'_3} &= \theta^4 - 2^4 z\,(192\theta^4+128\theta^3+112\theta^2+48\theta+7) && \\ &\quad + 2^{14}z^2(192\theta^4+256\theta^3+208\theta^2+64\theta+7) - 2^{30}z^3\big(\theta+\tfrac{1}{2}\big)^4, && z = \frac{1-\gamma^2}{2^{10}}. \end{aligned} \tag{S3.1}$$

The 4PM operator has indicial exponent $\frac{1}{2}$ at $z = 0$; we work with the conjugated operator $\tilde{L}_{\mathrm{K3}} = \frac{1}{8}\, z^{-1/2} L_{\mathrm{K3}}\, z^{1/2} = \theta^3 - z\big(2\theta^3 + 3\theta^2 + 2\theta + \frac{1}{2}\big) + z^2(\theta+1)^3$, whose holomorphic solution at $z = 0$ is $\varpi_0 = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; z)^2 = (2/\pi)^2 K(z)^2$, so that L_{K3} annihilates $z^{1/2}$ times the solutions tabulated for K3. In d/dz form, obtained from Eq. (S3.1) with $\theta^j = \sum_i S(j,i)\, z^i \partial_z^i$ (S the Stirling numbers of the second kind) and a common power of z removed,

$$
\begin{aligned}
2\, z^{-1} \tilde{L}_{\mathrm{K3}} &= \big(2z^2 - 4z^3 + 2z^4\big)\partial_z^3 + \big(6z - 18z^2 + 12z^3\big)\partial_z^2 \\
&\quad + \big(2 - 14z + 14z^2\big)\partial_z + \big(-1 + 2z\big)\,, && \text{(S3.2)} \\
z^{-1} L_A &= \big(z^2 - 34z^3 + z^4\big)\partial_z^3 + \big(3z - 153z^2 + 6z^3\big)\partial_z^2 \\
&\quad + \big(1 - 112z + 7z^2\big)\partial_z + \big(-5 + z\big)\,, && \text{(S3.3)} \\
z^{-1} L_{\mathrm{CY}_3} &= \big(z^3 - 256z^4\big)\partial_z^4 + \big(6z^2 - 2048z^3\big)\partial_z^3 \\
&\quad + \big(7z - 3712z^2\big)\partial_z^2 + \big(1 - 1280z\big)\partial_z + \big(-16\big)\,, && \text{(S3.4)} \\
z^{-1} L_{\mathrm{CY}_3'} &= \big(z^3 - 3072z^4 + 3145728z^5 - 1073741824z^6\big)\partial_z^4 \\
&\quad + \big(6z^2 - 20480z^3 + 23068672z^4 - 8589934592z^5\big)\partial_z^3 \\
&\quad + \big(7z - 29440z^2 + 38010880z^3 - 15569256448z^4\big)\partial_z^2 \\
&\quad + \big(1 - 7680z + 11796480z^2 - 5368709120z^3\big)\partial_z \\
&\quad + \big(-112 + 114688z - 67108864z^2\big)\,. && \text{(S3.5)}
\end{aligned}
$$

The finite singular points are the zeros of the leading coefficients: $z = 1$ (K3), $z = 17 \mp 12\sqrt{2}$ (K3$'$), $z = 2^{-8}$ (CY$_3$) and $z = 2^{-10}$ (CY$_3'$).

S3.2 Frobenius frame

All four operators (with $\tilde{L}_{\mathrm{K3}}$ in place of L_{K3}) have a point of maximal unipotent monodromy at $z = 0$ with indicial polynomial θ^r, r the order. We fix the Frobenius basis there as

$$\varpi_k(z) = \sum_{j=0}^{k} \frac{\ln^j z}{j!}\, h_{k-j}(z)\,, \qquad k = 0, \ldots, r-1, \qquad h_0(0) = 1\,, \quad h_j(0) = 0 \ \ (j \geq 1)\,, \tag{S3.6}$$

where the power series h_j are defined through the deformed recursion. We write $\varpi(z,\rho) = \sum_{m\geq 0} b_m(\rho)\, z^{m+\rho}$ with $b_0 = 1$, and fix the $b_m(\rho)$ by requiring that $L\,\varpi(z,\rho) = O(\rho^r)$ hold term by term in z. Then

$$h_j(z) = \sum_{m\geq 0} \frac{1}{j!}\, \partial_\rho^j b_m(\rho)\Big|_{\rho=0} z^m\,, \qquad \varpi_k = \frac{1}{k!}\, \partial_\rho^k \varpi(z,\rho)\Big|_{\rho=0}\,. \tag{S3.7}$$

For the Apéry operator $h_0 = \varpi_0 = \sum_n A_n z^n$ with A_n the Apéry numbers. The logarithm is real on the positive real axis inside the disk of convergence, where the series are summed, and is continued along the path. In this frame the local monodromy about $z = 0$ acts by $\ln z \to \ln z + 2\pi i$, that is by the matrix with entries $(2\pi i)^{k-j}/(k-j)!$ for $k \geq j$; its inverse is the matrix M_0^{-1} quoted in Section 6.2 of the main text.

S3.3 Variables, branch and paths

Bound kinematics is $\gamma = \cos\theta \in (0,1)$ with $x = e^{-i\theta}$ on the physical branch (the Feynman $s + i0$, hence $\gamma + i0$, hence the lower arc since $d\gamma/dx < 0$ on $0 < x < 1$). The variables are

$$z_{\mathrm{K3}} = z_{\mathrm{K3}'} = x^2 = 2\gamma^2 - 1 - 2i\gamma\sqrt{1-\gamma^2}\,, \qquad z_{\mathrm{CY}_3} = 2^{-8} x^4\,, \qquad z_{\mathrm{CY}_3'} = 2^{-10}(1-\gamma^2)\,. \tag{S3.8}$$

The twelve energies are those of the main text, $\gamma \in \{\frac{199}{200}, \frac{99}{100}, \frac{49}{50}, \frac{97}{100}, \sqrt{8/9}, \frac{23}{25}, \frac{9}{10}, \frac{\sqrt{3}}{2}, \frac{4}{5}, \frac{\sqrt{2}}{2}, \frac{3}{5}, \frac{1}{2}\}$. Each Frobenius basis is summed at a base point z_b on the positive real axis inside the disk of convergence and transported from there by integrating the differential equation along a polygonal path; every value is computed along two homotopic paths A and B.

- K3 and K3′ ($z = x^2$ on the lower unit semicircle; all finite singular points on the positive real axis, so the open lower half plane is the physical $z - i0$ class): $z_b = \frac{1}{200}$; path A $= \big(z_b,\ \frac{3}{100} - \frac{3}{100}i,\ \frac{55}{100}\,z,\ z\big)$, path B $= \big(z_b,\ \frac{8}{1000} - \frac{6}{100}i,\ \frac{8}{10}\,e^{-i/5}z,\ z\big)$.
- CY_3 ($z = 2^{-8}x^4$ lies on the singular circle $|z| = 2^{-8}$ at angle $\phi = -4\theta \in (-2\pi, 0)$): $z_b = 2^{-8}/50$; the physical class starts inside the disk and winds clockwise, never crossing $\arg z = 0$ (the $z - i0$ continuation past $z_+ = 2^{-8}$ at $\gamma = 1$). Path A runs from z_b through the points $\rho\, 2^{-8}e^{it}$ with $t = -a_0, -a_0-\delta, -a_0-2\delta, \ldots$ down to $t = \phi$, then to z, with $(\rho, \delta, a_0) = (\frac{7}{10}, \frac{1}{2}, \frac{1}{4})$; path B is the same with $(\rho, \delta, a_0) = (\frac{17}{20}, \frac{7}{20}, \frac{3}{20})$.
- CY_3' (z real in $(0, 2^{-10})$, ordinary points): $z_b = 2^{-10}/8$; path A is the real segment from z_b to z, path B the detour $\big(z_b,\ \frac{4}{10}\,2^{-10} - \frac{25}{100}\,2^{-10}i,\ z\big)$.

The transport is carried out in complex ball arithmetic at 600-bit working precision with the Frobenius series truncated at 350 terms at the base point; each transport step carries a rigorous remainder bound, so every matrix entry comes with an enclosure radius.

S3.4 File layout and accuracy

The ancillary files contain, for each geometry, one block per energy with the full Frobenius matrix

$$W_{ik}(z) = \frac{d^{\,i-1}}{dz^{\,i-1}}\,\varpi_{k-1}(z)\,, \qquad i,k = 1,\ldots,r\,, \tag{S3.9}$$

evaluated along path A on the physical branch, in the frame (S3.6) and the variable (S3.8) (for K3 the matrix of $\tilde{L}_{\mathrm{K3}}$, whose $W_{11} = (2/\pi)^2 K(z)^2$). A block begins with a line POINT <geometry> <label> gamma=<value>, followed by the real and imaginary parts of z and by r^2 lines W[i,k] = <re> <im> giving real and imaginary parts as decimal strings of at least 179 significant digits; there are 48 blocks and 600 matrix entries in all. Table 6 of the main text is the entry W_{11} of each block rounded to ten decimals. Table S3 gives, for every point, the number of digits to which the two paths agree (the maximal entrywise difference of the two matrices relative to the largest entry) and the number of digits guaranteed by the enclosure (the largest entrywise enclosure radius of the path-A matrix relative to the largest entry), both rounded down. The smallest values over the 48 points are 176 and 167 digits.

S3.5 Monodromy checks

Two checks compare continuations along inequivalent paths with the exact monodromy matrices in the frame (S3.6). Writing $W^{(P)}$ for the matrix (S3.9) transported along a path P, two paths from z_b to the same z that differ by a loop ℓ based at z_b give $W^{(P')} = W^{(P)}M_\ell$ with M_ℓ constant, so the quotient $(W^{(P)})^{-1}W^{(P')}$ is the monodromy matrix of ℓ. (a) K3′ at $\gamma = \sqrt{3}/2$, $z = e^{-i\pi/3}$: besides the lower-half-plane path we transport along the upper path $\big(z_b,\ \frac{3}{100} + \frac{5}{100}i,\ 1 + \frac{8}{10}i,\ 2 + \frac{3}{10}i,\ 2 - \frac{3}{10}i,\ \frac{12}{10} - \frac{8}{10}i,\ z\big)$, which crosses the real axis between z_+ and z_-, so that the two paths differ by a loop around $z_+ = 17 - 12\sqrt{2}$ only. The quotient $(W^{\mathrm{lower}})^{-1}W^{\mathrm{upper}}$ agrees with the reflection matrix $M(z_+)$ of Section 6.2 of the main text, with rows $(0, 0, \pi^2/3)$, $(0, 1, 0)$, $(3/\pi^2, 0, 0)$, entrywise to 176

Table S3: Accuracy of the tabulated Frobenius matrices at the twelve bound energies γ for the four geometries: "two-path" is the number of digits to which the matrices transported along paths A and B of Section S3.3 agree, "radius" the number of digits guaranteed by the ball-arithmetic enclosure of the path-A matrix, both relative to the largest entry and rounded down.

	K3		K3′		CY_3		CY'_3	
γ	two-path	radius	two-path	radius	two-path	radius	two-path	radius
199/200	176	171	177	167	177	172	177	176
99/100	176	171	177	167	177	171	177	176
49/50	176	171	177	167	177	171	177	176
97/100	176	171	177	167	177	171	177	176
$\sqrt{8/9}$	176	171	177	167	177	170	177	177
23/25	176	172	177	167	177	170	177	176
9/10	176	172	177	167	177	170	177	176
$\sqrt{3}/2$	176	172	177	167	177	169	177	176
4/5	177	172	177	168	177	169	177	176
$\sqrt{2}/2$	177	173	177	169	177	168	178	175
3/5	177	172	177	168	177	168	178	175
1/2	177	172	178	167	177	167	178	175

digits relative to its largest entry. (b) CY_3 at $\gamma = \frac{1}{2}$: besides the physical clockwise path A we transport along the short counterclockwise path built as path A but with angles $t = +\frac{1}{4}, \frac{3}{4}, \dots$ up to $2\pi - 4\theta$, reaching the same z with one winding fewer around $z = 0$. The quotient $(W^{\text{short}})^{-1} W^{\text{A}}$ agrees with M_0^{-1}, with entries $(-2\pi i)^{k-j}/(k-j)!$ for $k \geq j$ and zero below the diagonal, to 174 digits. We do not tabulate the matrices of the second legs; with the operators (S3.1), the frame (S3.6), the paths above and the tabulated first-leg matrices, both quotients can be formed by any integrator for linear differential equations with a few dozen digits of working precision.

S4 Assembly of c_M

This section gives the numbers behind Table 7 and the value $c_M = 1$ of Section 7 of the main text: the sector integrals and two-frequency cores, the Laurent coefficients of the prefactors and of the integration-by-parts coefficients, the assembled Laurent coefficients of $I_1^{(M)}$, $I_3^{(M)}$ and $I_2^{(M)}$ from two implementations, and the leading terms obtained when Feynman propagators are placed on the two memory gravitons.

S4.1 Cores and assembly

In the notation of Section 7.1 of the main text, with the powers of (γ^2-1) stripped and $\omega_3 = \omega_1 + \omega_2$,

$$\begin{aligned} I_1^{(M)} &= N_1(\epsilon) \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \prod_{j=1,2} (0^+ - i\omega_j)^{1-2\epsilon} \prod_{i=1}^{3} |\omega_i|^{-\epsilon} K_\epsilon(|\omega_i|)\,, \\ I_3^{(M)} &= N_3(\epsilon) \int \frac{d\omega_1 d\omega_2}{(2\pi)^2}\, \omega_1^3 \omega_2^3 \prod_{j=1,2} (0^+ - i\omega_j)^{-1-2\epsilon} \prod_{i=1}^{3} |\omega_i|^{-\epsilon} K_\epsilon(|\omega_i|)\,, \end{aligned} \tag{S4.1}$$

with $N_1 = -2^{7\epsilon-1}\Gamma^2(\frac{2\epsilon-1}{2})\Gamma(4\epsilon-1)/[\Gamma(2-5\epsilon)(4\pi)^{5-4\epsilon}]$, $N_3 = 2^{7\epsilon-3}\Gamma^2(\frac{2\epsilon+1}{2})\Gamma(4\epsilon-2)/[\Gamma(3-5\epsilon)(4\pi)^{5-4\epsilon}]$, and $I_2^{(M)} = r_1(\epsilon) I_3^{(M)} + r_2(\epsilon) I_1^{(M)}$ with the rational functions $r_{1,2}$ of the main text [1].

Table S4: Sector integrals $\mathcal{A}_a^{\text{same}}[w]$ and $\mathcal{A}_a^{\text{opp}}[w]$ of Eq. (S4.4) for the weights $w = 1, A, A^2, R$ of Eq. (S4.2), first implementation, as printed by the computation (25 significant figures shown; the working precision was 35).

weight w	a	$\mathcal{A}_a^{\text{same}}[w]$	$\mathcal{A}_a^{\text{opp}}[w]$
1	1	−0.228017582173806283607356	0.771982417826193716392644
A	1	−0.9363053870563734210357536	1.557195147473730675636082
A^2	1	−8.682056764705753152077277	19.08456911190713455726882
R	1	−0.4102436439914343444220179	1.807157456280905310286605
1	3	−0.113737589951934843818107	−1.44707092328526817715144
A	3	0.05080750060809472258202251	3.250458608298017181504031
A^2	3	−1.378582178693526785358664	−24.77391953826925388689294
R	3	−0.121310185266184224067244	−2.661178318962637097012074

For retarded factors, $(0^+ - i\omega)^{-2\epsilon} = \exp[-2\epsilon(\ln|\omega| - \frac{i\pi}{2}\,\text{sign}\,\omega)]$ and $|\omega|^{-\epsilon} K_\epsilon(|\omega|) = K_0\,[1 + \epsilon\, l + \epsilon^2(\frac{1}{2}l^2 + \kappa_2/K_0) + \ldots]$ with $l = -\ln|\omega|$ and $\kappa_2 = \frac{1}{2}\partial_\nu^2 K_\nu|_{\nu=0}$, so that after the sum over the signs of ω_1, ω_2 the real part of either integrand is

$$
\begin{aligned}
&-\omega_1^p\omega_2^p\, K_0(|\omega_1|)K_0(|\omega_2|)K_0(|\omega_3|)\,\left[1 + \epsilon B_1 + \epsilon^2 B_2 + O(\epsilon^3)\right], \\
&B_1 = A\,, \qquad B_2 = \tfrac{1}{2}A^2 + R - 2\pi^2\,[\,\text{sign}\,\omega_1 = \text{sign}\,\omega_2\,]\,,
\end{aligned}
\tag{S4.2}
$$

with $p = 1$ for $I_1^{(M)}$ and $p = 2$ for $I_3^{(M)}$. Here $A = -(3\ln|\omega_1| + 3\ln|\omega_2| + \ln|\omega_3|)$ and $R = \sum_{i=1}^{3}\kappa_2(|\omega_i|)/K_0(|\omega_i|)$, and the bracket equals one in the two same-sign quadrants and zero otherwise. The $-2\pi^2$ is the order-ϵ^2 term of the real part of the phase $e^{\pm 2\pi i\epsilon}$ of the two retarded factors in the same-sign quadrants; in the opposite-sign quadrants the two phases cancel. The two-frequency cores are

$$
j_a^{(k)} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\omega_1 d\omega_2\,\big(-\omega_1^p\omega_2^p\big)\prod_{i=1}^{3}K_0(|\omega_i|)\;B_k\,, \qquad a = 1\ (p=1),\ a = 3\ (p=2), \quad B_0 = 1\,,
\tag{S4.3}
$$

so that $I_a^{(M)} = N_a(\epsilon)\,[\,j_a^{(0)} + \epsilon j_a^{(1)} + \epsilon^2 j_a^{(2)} + O(\epsilon^3)]$ and $j_1^{(0)} = \frac{1}{30}$, $j_3^{(0)} = -\frac{8}{105}$ are the two moments of the main text divided by $(2\pi)^2$. Each core is the sum of a same-sign piece and an opposite-sign piece. We compute them as two-dimensional quadratures over the sector $\omega_1, \omega_2 > 0$ (weight two for the two same-sign quadrants) and over the sector $\omega_1 > |\omega_2|$, $\omega_2 < 0$ (weight four for the four half-quadrants of opposite sign), with the integrand split by weight,

$$
\begin{aligned}
j_a^{(0)} &= \frac{\mathcal{A}_a^{\text{same}}[1] + 2\mathcal{A}_a^{\text{opp}}[1]}{(2\pi)^2}\,, \qquad j_a^{(1)} = \frac{\mathcal{A}_a^{\text{same}}[A] + 2\mathcal{A}_a^{\text{opp}}[A]}{(2\pi)^2}\,, \\
j_a^{(2)} &= \frac{1}{(2\pi)^2}\Big[\tfrac{1}{2}\big(\mathcal{A}_a^{\text{same}}[A^2] + 2\mathcal{A}_a^{\text{opp}}[A^2]\big) + \mathcal{A}_a^{\text{same}}[R] + 2\mathcal{A}_a^{\text{opp}}[R] - 2\pi^2\mathcal{A}_a^{\text{same}}[1]\Big]\,,
\end{aligned}
\tag{S4.4}
$$

where $\mathcal{A}_a^{\text{same}}[w]$ is twice the integral of $-(\omega_1\omega_2)^p\prod K_0\cdot w$ over the quadrant $\omega_1, \omega_2 > 0$ and $\mathcal{A}_a^{\text{opp}}[w]$ twice the integral of the same function over the half quadrant $\omega_1 > -\omega_2 > 0$. Table S4 lists the sixteen sector integrals of the first implementation (working precision 35 digits, tanh-sinh quadrature with the panels $[0, 0.1, 0.5, 2, 8, \infty)$ in each variable, κ_2 by a fourth-order finite difference in ν at 55 digits); Table S5 the resulting cores together with those of a second implementation of the same expansion and quadrature (working precision 30, one further quadrature level).

Table S5: The six two-frequency cores $j_a^{(k)}$ of Eq. (S4.3) from the two implementations. The exact values $j_1^{(0)} = 1/30$ and $j_3^{(0)} = -8/105$ are reproduced to 28 and 30 digits by the second.

core	first implementation	second implementation
$j_1^{(0)}$	0.0333333333333333333333333	0.0333333333333333333333333333353
$j_1^{(1)}$	0.05517153523525931749972088	0.0551715352352593174997208815679
$j_1^{(2)}$	0.5686271138877558463074322	0.568627113887755846307432245288
$j_3^{(0)}$	$-0.07619047619047619047619048$	$-0.0761904761904761904761904761905$
$j_3^{(1)}$	0.1659571258114707663498443	0.165957125811470766349844334194
$j_3^{(2)}$	$-0.7260115565251557098338957$	$-0.726011556525155709833895696857$

Table S6: Laurent coefficients of the prefactors N_1, N_3 of Eq. (S4.1) and of the integration-by-parts coefficients r_2 (multiplying $I_1^{(M)}$) and r_1 (multiplying $I_3^{(M)}$), at the orders that enter the assembly. These are exact quantities (Γ-function and rational-function expansions) printed to the figures shown.

order	N_1	N_3
ϵ^{-1}	$5.01268664779508554 \times 10^{-6}$	$7.83232288717982116 \times 10^{-8}$
ϵ^{0}	0.0000945100272444485154	$1.51588079013040716 \times 10^{-6}$
ϵ^{1}	0.00100139639567664462	0.000016483082306384574
order	r_2	r_1
ϵ^{-3}	-2.4	-67.2
ϵ^{-2}	42.6933333333333333	971.413333333333333
ϵ^{-1}	-293.924	-4766.272

The prefactors and the integration-by-parts coefficients are expanded exactly; Table S6 lists the Laurent coefficients that enter $I_1^{(M)}$ and $I_3^{(M)}$ through order ϵ^1 and $I_2^{(M)}$ through order ϵ^{-2}. The leading ones are $N_1\big|_{\epsilon^{-1}} = 1/(2048\pi^4)$, $N_3\big|_{\epsilon^{-1}} = 1/(131072\pi^4)$, $r_2\big|_{\epsilon^{-3}} = -\frac{12}{5}$ and $r_1\big|_{\epsilon^{-3}} = -\frac{336}{5}$. Table S7 gives the assembled coefficients. With the exact leading cores the ϵ^{-1} coefficient of $I_1^{(M)}$ is $1/(30 \cdot 2048\,\pi^4) = 1/(15\,(8\pi)^4)$ and that of $I_3^{(M)}$ is $-8/(105 \cdot 131072\,\pi^4) = -1/(1720320\,\pi^4)$. The ϵ^{-4} and ϵ^{-3} coefficients of $I_2^{(M)}$ vanish; the numerical residuals in the table measure the quadrature error, and relative to the ϵ^{-2} coefficient they are -4.812×10^{-33} and -2.693×10^{-34} in the first implementation. The normalized ϵ^{-2} coefficient, $c_M = -\frac{6}{5}(8\pi)^4\, I_2^{(M)}\big|_{\epsilon^{-2}}$, differs from one by -2.06836×10^{-31} in the first implementation and agrees with one to 25 digits in the second.

S4.2 Feynman propagators on the memory gravitons

The representations (S4.1) descend from integrating out the spatial graviton momentum at fixed worldline frequency. For a retarded propagator raised to the power a this gives, in $3 - 2\epsilon$ spatial dimensions, a factor proportional to $[-(\omega + i0)^2]^{3/2-a-\epsilon} = (0^+ - i\omega)^{3-2a-2\epsilon}$, the whole power carrying one and the same prescription [1]. For a Feynman propagator $1/(\omega^2 - \mathbf{k}^2 + i0)^a$ the same integral has $-\omega^2 - i0 = (0^+ - i|\omega|)^2$ in place of $-(\omega + i0)^2$, so the substitution is

$$(0^+ - i\, s\, \omega)^{a-q\epsilon} \;\longrightarrow\; (0^+ - i|\omega|)^{a-q\epsilon} \tag{S4.5}$$

Table S7: Assembled Laurent coefficients of $I_1^{(M)}$, $I_3^{(M)}$ (through ϵ^1) and $I_2^{(M)}$ (through ϵ^{-2}) under the retarded routing, from the two implementations, as printed by each. The first prints 18 significant figures of a 35-digit computation; the second prints its 30-digit working values. The ϵ^{-4} and ϵ^{-3} entries of $I_2^{(M)}$ are zero up to quadrature error.

coefficient	first implementation	second implementation
$I_1^{(M)}\big\vert_{\epsilon^{-1}}$	$1.67089554926502851 \times 10^{-7}$	0.000000167089554926502851385950
$I_1^{(M)}\big\vert_{\epsilon^{0}}$	$3.42689185949375766 \times 10^{-6}$	0.00000342689185949375765535940
$I_1^{(M)}\big\vert_{\epsilon^{1}}$	0.0000414444926954499726	0.0000414444926954499725596585
$I_3^{(M)}\big\vert_{\epsilon^{-1}}$	$-5.96748410451795898 \times 10^{-9}$	$-5.96748410451795897806965 \times 10^{-9}$
$I_3^{(M)}\big\vert_{\epsilon^{0}}$	$-1.02497381300193386 \times 10^{-7}$	$-0.000000102497381300193386335937$
$I_3^{(M)}\big\vert_{\epsilon^{1}}$	$-1.0611462403126798 \times 10^{-6}$	$-0.00000106114624031267979835865$
$I_2^{(M)}\big\vert_{\epsilon^{-4}}$	$-1.00498883783832961 \times 10^{-38}$	$-2.35345638114070372102871 \times 10^{-35}$
$I_2^{(M)}\big\vert_{\epsilon^{-3}}$	$-5.62492413967696171 \times 10^{-40}$	$-2.45570368418670507767558 \times 10^{-35}$
$I_2^{(M)}\big\vert_{\epsilon^{-2}}$	$-2.08861943658128564 \times 10^{-6}$	$-0.00000208861943658128564232438$

Table S8: Leading terms with Feynman propagators on both memory-graviton lines: the cores (S4.6), the resulting ϵ^{-1} coefficients of $I_1^{(M)}$ and $I_3^{(M)}$, and the ϵ^{-4} coefficient of $I_2^{(M)} = r_1 I_3^{(M)} + r_2 I_1^{(M)}$, from the sector integrals of Table S4 (first implementation). The closed forms in the last column rest on the identification of the quadrant moments given in the text.

quantity	value	closed form
$j_{1,F}^{(0)}$	$-0.0448848389918903682590$7	$-\frac{1}{90} - \frac{1}{3\pi^2}$
$j_{3,F}^{(0)}$	0.0704284625711977396893 4	$\frac{8}{315} + \frac{4}{9\pi^2}$
$I_{1,F}^{(M)}\big\vert_{\epsilon^{-1}}$	$-2.2499363310308107678 \times 10^{-7}$	$-\big(\frac{1}{90} + \frac{1}{3\pi^2}\big)/\big(2048\,\pi^4\big)$
$I_{3,F}^{(M)}\big\vert_{\epsilon^{-1}}$	$5.5161845930527945132 \times 10^{-9}$	$\big(\frac{8}{315} + \frac{4}{9\pi^2}\big)/\big(131072\,\pi^4\big)$
$I_{2,F}^{(M)}\big\vert_{\epsilon^{-4}}$	$1.6929711479424679298 \times 10^{-7}$	$1/(6144\,\pi^6)$

on each of the two lines: the Feynman factor equals the retarded one ($s = +1$) for $\omega > 0$ and the advanced one ($s = -1$) for $\omega < 0$. At leading order in ϵ the product of the two factors in $I_1^{(M)}$ is $-\omega_1\omega_2$ for retarded or advanced propagators and $-|\omega_1\omega_2|$ for Feynman propagators; in $I_3^{(M)}$ it is $-\omega_1^2\omega_2^2$ against $-\omega_1\omega_2|\omega_1\omega_2|$. The two agree in the same-sign quadrants and differ by a sign in the opposite-sign quadrants, so the leading Feynman cores are

$$j_{a,F}^{(0)} = \frac{\mathcal{A}_a^{\text{same}}[1] - 2\mathcal{A}_a^{\text{opp}}[1]}{(2\pi)^2} \tag{S4.6}$$

in place of Eq. (S4.4), from the same sector integrals. At higher orders the Feynman integrand is complex also after the sum over quadrants, because the phase $e^{2\pi i\epsilon}$ of the two factors no longer cancels between the $(+,+)$ and $(-,-)$ quadrants and is present in the mixed ones; we have formed only the leading terms, which are real. Table S8 lists them.

The leading core of $I_1^{(M)}$ changes sign and magnitude, $j_{1,F}^{(0)}/j_1^{(0)} = -1.3465451697567110478$, so the ϵ^{-1} coefficient of $I_1^{(M)}$ is -1.3465451697567110478 times $1/(15\,(8\pi)^4)$ and no longer has the form required of it in the main text (Section 3.3), and $j_{3,F}^{(0)}/j_3^{(0)} = -0.92437357124697033342$. Since the two ratios differ, the cancellation of the ϵ^{-4} pole of $I_2^{(M)}$, which for the retarded routing follows from $r_1\big|_{\epsilon^{-3}} N_3\big|_{\epsilon^{-1}} j_3^{(0)} = -r_2\big|_{\epsilon^{-3}} N_1\big|_{\epsilon^{-1}} j_1^{(0)}$, fails, and

$$I_{2,F}^{(M)}\Big|_{\epsilon^{-4}} = 1.6929711479424679298 \times 10^{-7}\,, \tag{S4.7}$$

nonzero. The sector integrals of the second implementation give the same leading cores in all 22 figures of Table S8, and a separate quadrature of the leading cores at higher working precision gives the coefficient (S4.7) to 18 figures. With an ϵ^{-4} pole in $I_2^{(M)}$ and an ϵ^{-1} coefficient of $I_1^{(M)}$ different from $1/(15\,(8\pi)^4)$, the expansion of $I_2^{(M)}$ does not start at ϵ^{-2} and no constant c_M is defined for this assignment. Driesse et al. report that a calculation with the Feynman prescription yields an ϵ^{-4} pole for $I_2^{(M)}$ [1]; they do not print its coefficient, so the agreement is qualitative.

The quadrant moments behind Eq. (S4.6) are identified by an integer-relation search at two precisions as

$$\begin{aligned}
&\int_{\omega_{1,2}>0} \omega_1\omega_2 \prod_{i=1}^{3} K_0\; d^2\omega = \frac{1}{3} - \frac{\pi^2}{45}\,, \qquad \int_{\omega_1>0>\omega_2} |\omega_1\omega_2| \prod_{i=1}^{3} K_0\; d^2\omega = \frac{1}{3} + \frac{2\pi^2}{45}\,,\\
&\int_{\omega_{1,2}>0} \omega_1^2\omega_2^2 \prod_{i=1}^{3} K_0\; d^2\omega = \frac{16\pi^2}{315} - \frac{4}{9}\,, \qquad \int_{\omega_1>0>\omega_2} \omega_1^2\omega_2^2 \prod_{i=1}^{3} K_0\; d^2\omega = \frac{4}{9} + \frac{32\pi^2}{315}\,,
\end{aligned} \tag{S4.8}$$

with $\prod K_0 = K_0(|\omega_1|)K_0(|\omega_2|)K_0(|\omega_1+\omega_2|)$. Under the retarded sign structure the rational parts cancel between quadrants and the moments of the main text, $-2\pi^2/15$ and $32\pi^2/105$, result; under Eq. (S4.5) they add, giving $j_{1,F}^{(0)} = -\frac{1}{90} - \frac{1}{3\pi^2}$, $j_{3,F}^{(0)} = \frac{8}{315} + \frac{4}{9\pi^2}$ and $I_{2,F}^{(M)}\big|_{\epsilon^{-4}} = r_2\big|_{\epsilon^{-3}} N_1\big|_{\epsilon^{-1}} \big(-\frac{5}{36\pi^2}\big) = 1/(6144\,\pi^6)$, in which the π^2 parts still cancel and the rational parts do not; the value (S4.7) agrees with $1/(6144\pi^6)$ in all 20 figures.

If instead the substitution (S4.5) is applied only to the order-ϵ phase, keeping the retarded factor at order ϵ^0, the ϵ^{-4} and ϵ^{-3} cancellations survive and the normalized ϵ^{-2} coefficient becomes 0.20000000000000000000000: the change relative to $c_M = 1$ is then $-\frac{6}{5}(8\pi)^4\, r_2\big|_{\epsilon^{-3}} N_1\big|_{\epsilon^{-1}} \big[\frac{7}{16}\big(\frac{4}{9} + \frac{32\pi^2}{315}\big) - \frac{1}{3} - \frac{2\pi^2}{45}\big] = -\frac{4}{5}$ exactly by Eq. (S4.8). That phase-only substitution corresponds to no propagator assignment, since the integer and the ϵ-dependent parts of the power in Eq. (S4.5) come from one propagator; it is the origin of the value 1/5 quoted in an earlier version of the main text.

S5 Master-integral lists

S5.1 First-stage counts of the four families

The counts of Table 2 of the main text come from one Laporta reduction per family with the program Kira [2, 3], at the seeding stated in Table S9. In each run the integrals of the listed top sectors with at most r propagator powers in total and at most s numerator powers (and, where stated, at most d dots) were seeded, the corner integrals of those sectors ($r = 13$, $s = d = 0$) and their subsector images were required, and the program's automatically generated sector relations and sector symmetries were included. The last column is the number of master integrals onto which the required integrals were reduced. These are first-stage counts at this seeding: they were not tested for stability under deeper seeding, and for PX2 they cover its four leading top sectors only, not the 36 further top sectors that contain the K3 and Calabi–Yau subtopologies (Table 2 and Section 5.5 of the main text).

Table S9: Seeding and outcome of the four first-stage reductions behind Table 2 of the main text: the number of top sectors reduced, the seed range (r, s, d) (total propagator powers, numerator powers, dots; a dash means unrestricted), the range of required integrals, the numbers of sector relations and sector symmetries generated by the reduction program, and the resulting number of master integrals.

family	top sectors	seeds (r, s, d)	required (r, s, d)	relations	symmetries	masters
P	56	$(14, 0, 1)$	$(13, 0, 0)$	1885	43	195
PX1	44	$(14, 0, 1)$	$(13, 0, 0)$	1463	40	256
PX2	4	$(14, 1, –)$	$(13, 0, 0)$	48	4	37
NP	9	$(14, 1, –)$	$(13, 0, 0)$	176	38	243

Each of the 52 automatic identifications of the PX2 reduction (48 relations between sectors and 4 self-maps of sectors) is realized by a relabeling of the loop momenta, $k_i \to k_{\pi(i)} + c_i\, q$, possibly combined with $q \to -q$ or with the worldline exchange $u_1 \leftrightarrow u_2$ or with both, under which the linearized propagators $D_1, \ldots, D_8$ are mapped into one another without sign changes. Each identification maps every uncut linearized propagator to itself or exchanges uncut propagators of the two worldlines, so it holds when the uncut propagators carry a common orientation σ. The self-map of S_1, for example, is realized by $u_1 \leftrightarrow u_2$, $q \to -q$, $k_1 \to k_3 + q$, $k_2 \to k_4 + q$, $k_3 \to k_1 + q$, $k_4 \to k_2 + q$, which permutes the quadratic propagators of the sector ($D_9 \leftrightarrow D_{14}$, $D_{15} \leftrightarrow D_{21}$, D_{13} fixed), exchanges $D_5 \leftrightarrow D_7$ and $D_6 \leftrightarrow D_8$, and reverses the momentum of the graviton line D_{13}; composed with the reversal of all loop momenta and of q it is the map (iii) of Section S1.2. Thirty of the 52 identifications, among them the four self-maps, reverse the momentum of at least one graviton line, which is immaterial for graviton propagators that are even in their momentum, such as the Feynman propagators used for every value in this document, and would not hold line by line for retarded ones [4]; the other 22 preserve the orientation of every line. The PX2 count of Table S9 is therefore a count for the family as defined, with a common σ on the uncut worldline propagators and graviton propagators even in their momenta. With the automatic sector relations and symmetries switched off (empty symmetry tables read in and none generated, all else unchanged) the four top sectors of PX2 share no nonvanishing subsector, and each top sector together with its subsectors (its tree) reduces independently; at the same seeding the tree of S_1 alone then reduces to 30 master integrals (8 even, 22 odd) and the tree of S_2 alone to 28 (8 even, 20 odd). The trees of S_3 and of the fourth top sector are the images of these two under the relabeling $k_i \to k_i + q$, $q \to -q$ of the family, so that the four trees reduced separately carry $2 \times (30 + 28) = 116$ master integrals, against 37 for the reduction of Table S9 with the identifications.

S5.2 The 37 PX2 master integrals

Table S10 lists the 37 master integrals of the PX2 run of Table S9 as exponent vectors of Eq. (S1.2), in the order returned by the reduction, with their sector, parity, the uncut linearized propagators they contain and the top sector under which they lie (S_4 denotes the fourth top sector, with denominators $D_1, \ldots, D_9, D_{13}, D_{14}, D_{17}, D_{19}$). Twelve are even and twenty-five odd; two contain no uncut linearized propagator (the corner integral I_0 of S_1 and its partner with a differentiated delta function), and the response-matrix rows of Section S1 are not in general members of this list, since the auxiliary-mass flow works with the master integrals of the η-deformed system.

S5.3 The eleven Apéry-sector master integrals

The K3$'$ sector is topology 40 of Ref. [5]. We use the family with loop momenta $k_1, \dots, k_4$, the kinematics (S1.1) and the propagators

$$
\begin{aligned}
&P_1 = k_1^2\,, && P_2 = (k_1 - k_2)^2\,, && P_3 = (k_2 - k_3 - q)^2\,, && P_4 = (k_2 + k_4)^2\,,\\
&P_5 = k_3^2\,, && P_6 = k_4^2\,, && P_7 = 2u_1\cdot(k_1 - k_3)\,, && P_8 = 2u_1\cdot(k_2 - k_3)\,,\\
&P_9 = 2k_3\cdot u_2\,, && P_{10} = 2u_2\cdot(k_3 + k_4)\,, && P_{11} = k_2^2\,, && P_{12} = (k_2 - q)^2\,,\\
&P_{13} = 2k_2\cdot u_1\,, && P_{14} = 2k_2\cdot u_2\,, && P_{15} = 2k_1\cdot u_2\,, && P_{16} = 2k_4\cdot u_1\,,\\
&P_{17} = (k_1 + q)^2\,, && P_{18} = (k_3 + q)^2\,, && P_{19} = (k_4 + q)^2\,, && P_{20} = (k_1 - k_3)^2\,,\\
&P_{21} = (k_1 - k_4)^2\,, && P_{22} = (k_3 - k_4)^2\,. &&
\end{aligned}
\tag{S5.1}
$$

in which all four worldline lines $P_7, \dots, P_{10}$ are linearized propagators; the sector has the ten denominators $P_1, \dots, P_{10}$ and $P_{11}, \dots, P_{22}$ serve as numerators. On the maximal cut of the sector (all ten lines cut, so that integrals in which one of them is absent are set to zero) we generated the integration-by-parts and Lorentz-invariance identities of the even-parity block, in which an integral and its image under $u_i \to -u_i$ are identified, for all seeds with at most 4 dots on $P_1, \dots, P_{10}$ and at most 4 powers of the numerators $P_{13}, \dots, P_{16}$, without any momentum-relabeling symmetry. The system has 70,070 seeds and 1,092,014 equations in 951,974 integrals; reduced at $\gamma = 23/17$ it returns the eleven master integrals of Table S11, and the same eleven at a second value of γ. At one dot and numerator power fewer the count was 28.

For the conservative problem Klemm et al. [6] report nine master integrals in this sector, and Duhr et al. [4] give an explicit nine-element basis on the maximal cut, obtained with Feynman propagators and all symmetry relations, in the ancillary files of their paper. Their family has velocities $v_1 = u_1$, $v_2 = u_2$, the same q, and propagators

$$
\begin{aligned}
&\hat{D}_1 = k_1\cdot v_1\,, && \hat{D}_2 = k_2\cdot v_1\,, && \hat{D}_3 = k_3\cdot v_2\,, && \hat{D}_4 = k_4\cdot v_2\,,\\
&\hat{D}_5 = k_1\cdot v_2\,, && \hat{D}_6 = k_2\cdot v_2\,, && \hat{D}_7 = k_3\cdot v_1\,, && \hat{D}_8 = k_4\cdot v_1\,,\\
&\hat{D}_9 = (k_1 - k_3)^2\,, && \hat{D}_{10} = (k_2 - k_3)^2\,, && \hat{D}_{11} = (k_1 - k_4)^2\,, && \hat{D}_{12} = (k_2 - k_4)^2\,,\\
&\hat{D}_{13} = (k_1 - k_2)^2\,, && \hat{D}_{14} = (k_3 - k_4)^2\,, && \hat{D}_{15} = k_1^2\,, && \hat{D}_{16} = k_2^2\,,\\
&\hat{D}_{17} = k_3^2\,, && \hat{D}_{18} = k_4^2\,, && \hat{D}_{19} = (k_1 + q)^2\,, && \hat{D}_{20} = (k_2 + q)^2\,,\\
&\hat{D}_{21} = (k_3 + q)^2\,, && \hat{D}_{22} = (k_4 + q)^2\,. &&
\end{aligned}
\tag{S5.2}
$$

with delta functions on $\hat{D}_1, \dots, \hat{D}_4$. Their loop momenta K_i are related to ours by $K_1 = k_2 - k_3 - q$, $K_2 = k_1 - k_3 - q$, $K_3 = -k_3 - k_4 - q$, $K_4 = -k_3 - q$, under which their ten sector lines $\{\hat{D}_1, \hat{D}_2, \hat{D}_3, \hat{D}_4, \hat{D}_9, \hat{D}_{12}, \hat{D}_{13}, \hat{D}_{14}, \hat{D}_{15}, \hat{D}_{22}\}$ map onto $\{P_8, P_7, P_{10}, P_9, P_4, P_1, P_2, P_6, P_3, P_5\}$ up to the normalization of the linear lines ($\hat{D} = \pm\frac{1}{2}P$). Their nine basis elements are built from the 8 integrals of Table S12 and the first and second x-derivatives of the first of them, with prefactors rational in x and ϵ. Two of their numerators, $\hat{D}_{11}$ and $\hat{D}_{18}$, equal to $P_{11} = k_2^2$ and $P_{18} = (k_3 + q)^2$ in our variables, lie outside the numerator set $P_{13}, \dots, P_{16}$ that we seeded.

The sector has a relabeling symmetry that the reduction above does not impose: the worldline exchange $u_1 \leftrightarrow u_2$ with $k_1 \to k_2 + k_4$, $k_2 \to k_2$, $k_3 \to k_2 - k_3 - q$, $k_4 \to k_1 - k_2$, which permutes the ten sector lines as $(P_1P_4)(P_2P_6)(P_3P_5)(P_7P_{10})(P_8P_9)$ without sign changes on the linear ones, since $u_i\cdot q = 0$. It exchanges entries 2 and 5 of Table S11, both of which are master integrals of that system. Adding to the system the relations that equate every even seed integral of the same range to its image under this map, and reducing again, we obtain nine master integrals, all among the eleven: entries 1, 2, 3, 4, 6, 7, 8, 9 and 11 of Table S11, that is, the corner integral, the three integrals with one dot on P_4, P_5 or P_6, and five integrals with one dot on P_9 or P_{10} and one numerator power. Nine is also the number of elements of the maximal-cut basis of Ref. [4]; we have not mapped that basis, two of whose numerators lie outside our seeded set, onto these nine integrals.

Table S10: The 37 first-stage master integrals of PX2 (seeds $(r,s)=(14,1)$ on the four top sectors, required integrals $(13,0,0)$, automatic sector symmetries included) as exponent vectors $(n_1,\ldots,n_{22})$ of Eq. (S1.2), grouped as in Table S1; an entry 2 in the first group is a differentiated delta function. Columns: sector label, parity under $u_i \to -u_i$, uncut linearized denominators, top sector.

#	$(n_1\cdots n_4 \,\vert\, n_5\cdots n_8 \,\vert\, n_9\cdots n_{14} \,\vert\, n_{15}\cdots n_{18} \,\vert\, n_{19}\cdots n_{22})$	sector	parity	uncut linear den.	under top sector
1	1111 1111 100011 0010 1000	340479	even	D_5,D_6,D_7,D_8	S_4
2	2111 1111 100011 0010 1000	340479	odd	D_5,D_6,D_7,D_8	S_4
3	1111 1111 101001 0010 0100	599551	even	D_5,D_6,D_7,D_8	S_3
4	2111 1111 101001 0010 0100	599551	odd	D_5,D_6,D_7,D_8	S_3
5	1111 0000 100011 1000 0010	1077519	even	–	S_1
6	1112 0000 100011 1000 0010	1077519	odd	–	S_1
7	1111 1000 100011 1000 0010	1077535	odd	D_5	S_1
8	1111 1000 200011 1000 0010	1077535	odd	D_5	S_1
9	1111 1000 100021 1000 0010	1077535	odd	D_5	S_1
10	1111 0100 100011 1000 0010	1077551	odd	D_6	S_1
11	1111 0100 100021 1000 0010	1077551	odd	D_6	S_1
12	1111 0100 100012 1000 0010	1077551	odd	D_6	S_1
13	1111 1000 101001 0100 0010	1090847	odd	D_5	S_2
14	1111 0100 101001 0100 0010	1090863	odd	D_6	S_2
15	1111 0100 102001 0100 0010	1090863	odd	D_6	S_2
16	1111 0100 101002 0100 0010	1090863	odd	D_6	S_2
17	1111 1010 100011 1000 0010	1077599	even	D_5,D_7	S_1
18	2111 1010 100011 1000 0010	1077599	odd	D_5,D_7	S_1
19	1111 1100 100011 1000 0010	1077567	even	D_5,D_6	S_1
20	1111 0110 100011 1000 0010	1077615	even	D_6,D_7	S_1
21	1111 0210 100011 1000 0010	1077615	odd	D_6,D_7	S_1
22	1111 0120 100011 1000 0010	1077615	odd	D_6,D_7	S_1
23	1111 0101 100011 1000 0010	1077679	even	D_6,D_8	S_1
24	1111 0102 100011 1000 0010	1077679	odd	D_6,D_8	S_1
25	1111 1010 101001 0100 0010	1090911	even	D_5,D_7	S_2
26	1111 1100 101001 0100 0010	1090879	even	D_5,D_6	S_2
27	1111 0110 101001 0100 0010	1090927	even	D_6,D_7	S_2
28	1111 0210 101001 0100 0010	1090927	odd	D_6,D_7	S_2
29	2111 0110 101001 0100 0010	1090927	odd	D_6,D_7	S_2
30	1111 1001 101001 0100 0010	1090975	even	D_5,D_8	S_2
31	1111 0101 101001 0100 0010	1090991	even	D_6,D_8	S_2
32	1111 0102 101001 0100 0010	1090991	odd	D_6,D_8	S_2
33	1111 0201 101001 0100 0010	1090991	odd	D_6,D_8	S_2
34	1111 1110 100011 1000 0010	1077631	odd	D_5,D_6,D_7	S_1
35	1111 1110 101001 0100 0010	1090943	odd	D_5,D_6,D_7	S_2
36	1111 1011 101001 0100 0010	1091039	odd	D_5,D_7,D_8	S_2
37	1111 1101 101001 0100 0010	1091007	odd	D_5,D_6,D_8	S_2

Table S11: The eleven master integrals of the even block of the Apéry sector on its maximal cut at the seeding stated in the text, as exponent vectors $(n_1, \ldots, n_{22})$ over the propagators (S5.1) (first group: the ten cut denominators, an entry 2 denoting one dot; second group: the numerators, $\bar{1}$ denoting one power).

#	$(n_1 \cdots n_{10} \,\vert\, n_{11} \cdots n_{22})$	raised denominators	numerators
1	1111111111 | 000000000000	–	–
2	1111121111 | 000000000000	P_6	–
3	1111211111 | 000000000000	P_5	–
4	1112111111 | 000000000000	P_4	–
5	1211111111 | 000000000000	P_2	–
6	1111111112 | 00$\bar{1}$000000000	P_{10}	P_{13}
7	1111111112 | 000$\bar{1}$00000000	P_{10}	P_{14}
8	1111111112 | 0000$\bar{1}$0000000	P_{10}	P_{15}
9	1111111112 | 00000$\bar{1}$000000	P_{10}	P_{16}
10	1111111121 | 00$\bar{1}$000000000	P_9	P_{13}
11	1111111121 | 0000$\bar{1}$0000000	P_9	P_{15}

Table S12: The distinct integrals entering the nine-element maximal-cut basis of Ref. [4] for the Apéry sector, as exponent vectors over their propagators (S5.2) (first group: the four cut lines and the four linear numerators; second and third groups: the graviton propagators), in the order in which they appear in their basis. Basis elements 2 and 3 are the first and second x-derivatives of element 1; element 7 combines entries 2, 5 and 6.

#	$(\hat{n}_1 \cdots \hat{n}_8 \,\vert\, \hat{n}_9 \cdots \hat{n}_{14} \,\vert\, \hat{n}_{15} \cdots \hat{n}_{22})$
1	11110000 | 100111 | 10000001
2	11110000 | 100211 | 10000001
3	11110000 | 100111 | 10000002
4	1111$\bar{1}$00$\bar{1}$ | 100111 | 20000002
5	1112000$\bar{1}$ | 100211 | 10000001
6	1112000$\bar{1}$ | 200111 | 10000001
7	12210000 | 10$\bar{1}$111 | 10000001
8	11110000 | 200111 | 100$\bar{1}$0001

S6 Master ratios of the three-loop (4PM) cut-eikonal family

Appendix B.3 of the main text states that the ratios m_r/m_0, $r = 1, \ldots, 7$, of the eight top-sector master integrals of the cut-eikonal 4PM family, at leading order in ϵ, are rational numbers at the two kinematic points $x = 1/3$ ($\gamma = 5/3$) and $x = 1/5$ ($\gamma = 13/5$). Table S13 gives the fractions and the evidence. The masters were obtained at eighteen values of ϵ between 1.1×10^{-6} and 1.4×10^{-6} at each point with boundary data carried at 200 digits; in that window each m_r is dominated by its leading Laurent term: $|m_r|$ varies across the window in proportion to $1/\epsilon$ while $|\epsilon\, m_r|$ is constant to about one part in 10^5 (apart from the three runs at $x = 1/5$ that carry the common factor described in Appendix B.3), so the ratios are ratios of leading coefficients. The ratios were extrapolated to $\epsilon = 0$; the column "stable digits" is the number of digits of the limit that do not change when the set of runs used in the extrapolation is varied. Rational reconstruction of the limit returns the fraction p/q shown, whose height fixes it with the number of digits in the fourth column (about $2\log_{10}$ of the larger of $|p|$, q); the column "digits reproduced" is the number of digits to which p/q agrees with the extrapolated value, and "60-digit run" the same for an independent set of runs with boundary data carried at 60 digits. For every $r \geq 2$ the fraction is fixed by at most 36 digits, at least 91 digits are stable, so at least 55 stable digits are reproduced beyond those that fix the fraction, and the 60-digit runs reproduce the same fraction to at least 34 digits; the continued-fraction expansion of each extrapolated value terminates in the sense that the partial quotient following p/q exceeds 10^{68}. The ratio $m_1/m_0 = -2/5$ holds at every ϵ and is reproduced to 163 and 154 digits at the two points. The denominators contain 3^{13} at $x = 1/3$ and 5^{10} to 5^{12} at $x = 1/5$, times divisors of the common factor $7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$; we have not derived the ratios as rational functions of x from the differential equation at $\epsilon = 0$, and we have not determined the next order in ϵ.

Table S13: Leading-order master ratios m_r/m_0 of the cut top sector of the 4PM family of Appendix B of the main text at $x = 1/3$ and $x = 1/5$: the reconstructed fraction, the number of decimal digits that fix it, the number of digits of the $\epsilon \to 0$ extrapolation stable under changes of the run set (200-digit data), the number of digits to which the fraction reproduces the extrapolated value, the same for independent 60-digit data, and the prime factorization of the denominator. Digit counts are rounded down, except the number of digits that fix the fraction, which is rounded up.

x	r	m_r/m_0	fix	stable	repr.	60-digit	prime factors of q
1/3	1	$-2/5$	2	162	163	37	5
1/3	2	$\frac{51155172278951}{159600939515325}$	29	91	103	38	$3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 17 \cdot 19 \cdot 23$
1/3	3	$\frac{352510632927424}{2074812213699225}$	31	91	103	37	$3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/3	4	$\frac{1450584560982016}{10374061068496125}$	32	91	103	37	$3^{13} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/3	5	$\frac{1332838778273792}{10374061068496125}$	32	91	103	38	$3^{13} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/3	6	$\frac{1309993823371264}{10374061068496125}$	32	91	103	39	$3^{13} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/3	7	$\frac{231391363072}{1693724256081}$	25	91	103	37	$3^{13} \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/5	1	$-2/5$	2	154	154	37	5
1/5	2	$\frac{953885155266247}{6900673828125}$	30	91	100	34	$3 \cdot 5^{10} \cdot 7^2 \cdot 11 \cdot 19 \cdot 23$
1/5	3	$\frac{532143919437359552}{7625244580078125}$	36	91	101	34	$3 \cdot 5^{11} \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/5	4	$\frac{724435141921990656}{12708740966796875}$	36	91	100	34	$5^{12} \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/5	5	$\frac{648070583586914304}{12708740966796875}$	36	91	100	34	$5^{12} \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/5	6	$\frac{619263636963065856}{12708740966796875}$	36	91	100	34	$5^{12} \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
1/5	7	$\frac{506052911038464}{10374482421875}$	30	91	100	34	$5^{10} \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

S7 Sampled values of the 4PM maximal cut

Section 4.2 of the main text evaluates the maximal cut $\mathrm{LS}_{\mathrm{shape}}(x)$ of the three-loop (4PM) K3 sector, Eq. (22) of the main text, at rational values of x and compares it with the closed form $x\,K(x^2)^2$; the ratio is the constant 8 of Eq. (23) of the main text, found by the integer-relation search described in Appendix D.1 there. Table S14 lists the sampled values.

Table S14: The 4PM maximal cut is $8\,x\,K(x^2)^2$ at every sampled point, Eq. (23) of the main text. Listed are $\mathrm{LS}_{\mathrm{shape}}(x)$ of Eq. (22) of the main text and the closed form $x\,K(x^2)^2$, where $K(z)$ is the complete elliptic integral of the first kind with parameter z. In $\mathrm{LS}_{\mathrm{shape}}$ the b_2 integral is taken once along the segment between the real branch points, and the c_2 integral is evaluated by tanh–sinh quadrature. The last column is the value that an integer-relation search over $\{1, \pi, 1/\pi, \pi^2, 1/\pi^2, 1/\pi^3, \ln 2\}$ returns for the ratio, with no π or $\ln 2$ term. On the closed cycle enclosing the segment the constant would read 16.

x	$\mathrm{LS}_{\mathrm{shape}}(x)$	$x\,K(x^2)^2$	ratio
1/2	11.36701703500317380207...	1.42087712937539672526...	8
1/3	6.97583960686620827930...	0.87197995085827603491...	8
1/4	5.09597846343242818029...	0.63699730792905352254...	8
1/5	4.02903930445411041210...	0.50362991305676380151...	8
2/3	17.46611436917707621125...	2.18326429614713452641...	8

References

[1] M. Driesse, G.U. Jakobsen, G. Mogull, C. Nega, J. Plefka, B. Sauer et al., *Conservative Black Hole Scattering at Fifth Post-Minkowskian and Second Self-Force Order*, *Phys. Rev. Lett.* **137** (2026) 081402 [2601.16256].

[2] J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, *Integral reduction with Kira 2.0 and finite field methods*, *Comput. Phys. Commun.* **266** (2021) 108024 [2008.06494].

[3] F. Lange, J. Usovitsch and Z. Wu, *Kira 3: integral reduction with efficient seeding and optimized equation selection*, *Comput. Phys. Commun.* **322** (2026) 109999 [2505.20197].

[4] C. Duhr, S. Maggio, C. Nega, B. Sauer, L. Tancredi and F.J. Wagner, *Aspects of canonical differential equations for Calabi–Yau geometries and beyond*, *JHEP* **06** (2025) 128 [2503.20655].

[5] D. Brammer, H. Frellesvig, R. Morales and M. Wilhelm, *Classification of Feynman integral geometries for black-hole scattering at 5PM order*, *JHEP* **10** (2025) 212 [2505.10274].

[6] A. Klemm, C. Nega, B. Sauer and J. Plefka, *Calabi-Yau periods for black hole scattering in classical general relativity*, *Phys. Rev. D* **109** (2024) 124046 [2401.07899].