

New mathematics and new physics in the four-point energy correlator

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Abstract

Energy correlators are ways to measure and quantify the theory of strong interactions at colliders. Much is known about these correlation functions and even more is measured. The 4-point energy correlator E4C is particularly interesting as it unlocks new mathematics and new physics not present in lower orders. The calculations are challenging however, even at tree-level and even in the $\mathcal{N}=4$ super-Yang–Mills approximation under which similar calculations have been known to simplify. An open question is whether all of the singularities (letters) in the QCD E4C are already present in the $\mathcal{N}=4$ E4C. We begin in the collinear limit, where we can push the calculations far enough to directly compare $\mathcal{N}=4$ to QCD. For the quark-jet correlator we find QCD adds no singularities beyond the $\mathcal{N}=4$ alphabet, at leading singular order, but for the gluon jet it does. We can also compute E4C at generic angles. In $\mathcal{N}=4$, the integrals involve two elliptic curves, and interestingly a new geometric object, a genus-2 hyperelliptic curve. This new geometry is not an artifact of $\mathcal{N}=4$: in QCD the two-holed surface contributes to the quark-pair channel with exact nonzero weight. We then proceed to evaluate the QCD E4C at generic angle. We separate the calculation into exact rational weights, which we compute in closed form for the quark-pair channel, multiplying a fixed set of integrals shared with $\mathcal{N}=4$. The correlator itself we evaluate as a compact integral over the energy fractions, with no phase-space sampling. We then compare the E4C to curves we extract ourselves from CMS Open Data jets and DELPHI events, with good agreement. This sets the stage for precision comparisons to unfolded experimental data.

Contents

1	Introduction	2
2	The observable	4
3	The collinear limit: QCD versus $\mathcal{N} = 4$	7

*Work done with Anthropic, but results and views are not endorsed by Anthropic.

4	Generic angle: the analytic structure	13
5	The $\mathcal{N} = 4$ evaluation	15
6	QCD at generic angle, and the contrast	19
7	Confronting data	21
8	Outlook	25

1 Introduction

Point calorimeter cells at the debris of a high-energy collision, multiply the energies they record, and average over events: that is an energy correlator, one of the oldest observables of perturbative QCD [1]. The two-point energy–energy correlator (EEC), the average product of the energies recorded in two cells at fixed opening angle, has four decades of theory behind it and data from LEP (recently re-analyzed from archived DELPHI events [2]) to inside LHC jets. The shape-resolved three-point correlator, together with the projected (single-angle) correlators up to six points, has been extracted from open collider data [3]. Energy correlators are infrared finite and need no jet algorithm, and in conformal field theory they are correlation functions of light-ray operators [4].

The four-point energy correlator (E4C) is the family’s most differential member: four cells, the product of four energies, and five independent relative angles, so that it resolves the shape of a jet’s energy flow in four directions at once. Every energy-correlator result computed to date is polylogarithmic: the two-point correlator through NNLO in $\mathcal{N} = 4$ super Yang–Mills [5, 6], apart from one residual integral, and at NLO in QCD [7], with its collinear limit in Ref. [8], the three-point correlator at leading order in both theories [9, 10, 11], and, in $\mathcal{N} = 4$ only, the four-point correlator in the collinear limit [12, 13]. Exactly two quantities are known to require more. One is the residual elliptic integral of the NNLO two-point correlator in $\mathcal{N} = 4$, whose underlying curve is the equal-mass sunrise curve and which is solved in closed form in the companion note [14]. The other is the four-point correlator at generic angle, which reduces to master integrals involving two elliptic curves and one genus-2 hyperelliptic curve [15]. None of the integrals associated with those curves had been evaluated. These special geometries, arising in virtual amplitudes, have been the hard cases of Feynman-integral computation for the past decade. The four-point correlator is a collider observable in which they appear, and its two- and three-point counterparts are already measured inside LHC jets. Its computation is challenging even at tree level, and remains so in $\mathcal{N} = 4$ super Yang–Mills, the theory in which similar calculations have been known to simplify.

The $\mathcal{N} = 4$ correlator has long served as a theoretical laboratory for QCD. No QCD prediction for the four-point correlator existed, so the phenomenological study of Ref. [16] used the collinear $\mathcal{N} = 4$ correlator of Ref. [12] in its place. For that substitution to be exact, every singularity (letter) of the QCD E4C must already be present in the $\mathcal{N} = 4$ E4C. If so, the $\mathcal{N} = 4$ function space is big enough to hold the QCD answer. If not, no function built on the $\mathcal{N} = 4$ alphabet can represent the QCD correlator exactly, however closely it approximates the correlator’s dependence on the angles.

We begin in the collinear limit, where the calculations can be pushed far enough to compare the two theories directly. We compute the leading-order collinear E4C in QCD from the known

$1 \rightarrow 4$ splitting amplitudes, and we test whether QCD's singularities lie inside the $\mathcal{N} = 4$ alphabet. The answer depends on the parton channel: for quark jets, QCD adds no singularities beyond the $\mathcal{N} = 4$ alphabet at leading singular order. For gluon jets, the analytic continuation of the correlator has a branch point with nonzero coefficient at every root of twelve letters absent from $\mathcal{N} = 4$ on a one-parameter slice, and on eighteen quartic hypersurfaces of the full shape space of which those are the restrictions, although none of these branch points is visible on the physical sheet. For quark jets the corresponding coefficients vanish identically on the full shape space. The $\mathcal{N} = 4$ function space is thus large enough to hold the QCD collinear correlator for quark jets, at this order, and not for gluon jets. On the one-parameter slice we go further and obtain the leading-order collinear correlator itself in closed form, in QCD for both jets and in $\mathcal{N} = 4$ from the same code, as explicit hyperlogarithms with algebraic arguments and coefficients in the slice parameter; the $\mathcal{N} = 4$ result reproduces the closed form of Ref. [12] on the slice. Comparing the two functions directly, the algebraic letters, including the cube roots of Ref. [12], are common to both theories, and the QCD alphabet exceeds the $\mathcal{N} = 4$ one by exactly the twelve tangency letters of the gluon-jet analysis.

At generic (equivalently, wide) angle we consider a one-parameter slice through the five-dimensional angle space, on which we first determine explicitly an elliptic curve and the genus-2 curve in the $\mathcal{N} = 4$ master integrals of Ref. [15] and then evaluate those integrals. Along the slice we treat the six pairwise angles as independent variables. Four detector directions with those angles exist at only two points of the slice, so elsewhere on it the computed values are analytic continuations of the observable. On the slice the maximal cut of one elliptic sector is a non-isotrivial elliptic surface, whose defining quartic and degree-7 discriminant we give in full, together with the singularities of its exact Picard–Fuchs operator. We bring the sector's differential system to a terminal ε -form by a transformation that introduces the square roots of three quadratics in the slice variable. Every letter of that form is built from the slice variable, the degree-7 discriminant and those three square roots, and the homogeneous solutions of the subsector layer are algebraic. On the same slice we identify the genus-2 curve, and we find that the kernels of three further sectors, non-polylogarithmic at generic angle, degenerate to algebraic functions. For the elliptic sector we express all eight master integrals in terms of the sector's periods and of initial data evaluated numerically at three base points, with the two constants of its boundary (wall) term in closed form at weight two. Ref. [15] gives the $\mathcal{N} = 4$ integrand as a single generator term, to be summed over relabelings of the four detectors, with a numerator weighting that differs from ours. We assemble that generator term over its master integrals along the slice and obtain a generator integral $\mathcal{G}(u)$ with the propagators, masters and geometry of the $\mathcal{N} = 4$ correlator. We evaluate it at five rational points and follow it numerically along the whole slice. At the two physical points of the slice we obtain the correlator itself by direct integration of the form-factor square.

We also compute the wide-angle QCD correlator at leading order, at 46 points of the same slice. To our knowledge its values at the two physical points of the slice are the first wide-angle four-point values in QCD at fixed order (Figure 1). The genus-2 curve is not particular to $\mathcal{N} = 4$: in the quark-pair (n_f) channel of the QCD correlator, the genus-2 sector, together with one polylogarithmic sector that we treat alongside it, contributes through 102 weight functions multiplying a fixed set of integrals shared with $\mathcal{N} = 4$. We obtain all of these weights in closed form as rational functions of the slice variable, and each is nonzero at both physical points of the slice. At those two points the QCD and $\mathcal{N} = 4$ correlators are close: the ratio of the correlator between the two points is 4.1 in QCD and 3.4 in $\mathcal{N} = 4$. Finally, we compare the collinear

prediction with shape-resolved modulation curves extracted from CMS Open Data jets, with nothing fit, and the wide-angle prediction with archival DELPHI data, both at detector level. The shapes and peak positions of the measured modulations follow the leading-order curves.

The methods are those of multi-loop Feynman integrals, applied to energy-fraction integrals: the measurement δ -function and the phase-space on-shell constraints enter as cut propagators, so the Landau analysis, the Griffiths–Dwork reduction (the algorithm that produces the differential equation satisfied by a period integral), and numerical transport of the resulting differential equations carry over unchanged. The only correlator-specific ingredient is the integrand of Eq. (2), which is integrated over a compact region of energy fractions rather than over loop momenta.

The paper is organized as follows. Section 2 defines the observable, contrasts the generic-angle and collinear regimes, and fixes the slice. Section 3 computes the collinear E4C in QCD at leading order and tests whether it lies in the $\mathcal{N} = 4$ function space, with an answer that differs between quark and gluon jets. Section 4 determines the analytic structure at generic angle: the elliptic curve and its Picard–Fuchs operator, the terminal ε -form with its closed alphabet, and the genus-2 and algebraic sectors. Section 5 evaluates the $\mathcal{N} = 4$ generator integral $\mathcal{G}(u)$ on the slice: the transport of the periods of the elliptic sector, the eight masters of that sector in terms of those periods, the assembly of $\mathcal{G}(u)$ from the master integrals at five points, and the formal limit $u \rightarrow 0$. Section 6 treats QCD at generic angle on the same slice, giving the wide-angle values and the exact n_f -channel weight functions. It then compares the two theories at the physical points of the slice and discusses how well the $\mathcal{N} = 4$ correlator can serve as a proxy for QCD. Section 7 compares the predictions with data, CMS Open Data jets in the collinear regime and DELPHI events at wide angle, and Section 8 lists the open problems.

2 The observable

The four-point energy correlator is the average over events of the product of the energies deposited in four calorimeter cells, measured as a function of the relative angles between the cells. Four cells have six pairwise angles, of which five are independent for directions in three-dimensional space, so the correlator is a function of five angle variables. We study it in the collinear limit, where all the angles are small, and at generic angle, where we restrict to a one-parameter family through the five-dimensional angle space.

2.1 Definition

The E4C extends the two-point EEC: the product of four energy fractions is summed over quadruples of detected particles and averaged over events, with δ -functions fixing the five independent detector angles. Throughout we use $\zeta_{ij} = \frac{1}{2}(1 - \cos \chi_{ij})$, with χ_{ij} the angle between detectors i and j , in analogy with the two-point angle variable $\zeta = \frac{1}{2}(1 - \cos \chi)$, so that

$$\text{E4C}(\{\zeta_{jk}\}) = \sum_{i_1, i_2, i_3, i_4} \left\langle \frac{E_{i_1} E_{i_2} E_{i_3} E_{i_4}}{Q^4} \prod_{(jk)} \delta(\zeta_{jk} - \zeta_{i_j i_k}) \right\rangle, \quad (1)$$

where the product runs over five independent pairs (jk) of the four detectors, $\zeta_{i_j i_k}$ is the angle variable between particles i_j and i_k , and the angle brackets denote the average over events.

At leading order the correlator is a finite sum of energy-fraction integrals of the form

$$I = \int_{x_i \geq 0} d^n x \frac{N(x) \delta(D_\delta)}{\prod_i D_i}, \quad (2)$$

with x_i the energy fractions of the final state, N a polynomial numerator, D_δ the measurement constraint, and D_i polynomial denominators inherited from the matrix element. Each such integral has the form of a cut loop integral taken over a compact region. The two elliptic curves and the genus-2 curve that appear in the four-point correlator at generic angle are determined by the denominators D_i and the constraint D_δ in Eq. (2), which play the role of the propagators of the cut loop integral.

In $\mathcal{N} = 4$ super Yang–Mills the source of the final state is the scalar half-BPS operator, and at leading order the matrix element is its tree-level form factor into five partons. We take the integrand from the ancillary material of Ref. [15], where it is given as a single generator term g whose sum over the 24 relabelings of the four detectors is the four-point integrand. The integral of g is therefore not the correlator: at the slice point $u = 1/2$ defined below, the orbit sum is 47 times larger. The orbit sum is not the correlator in our normalization either. We weight the $(q^2)^3$ and $(q^2)^2 p_i \cdot p_j$ pieces of each numerator 1:1, as the square of the form factor gives, whereas the generator term g weights them 1:2 when read with the substitution rules that accompany it in the ancillary material.¹ We therefore treat g as a well-defined rational integrand that shares the propagators, and hence the master integrals and the geometry, with the $\mathcal{N} = 4$ correlator. We write $\mathcal{G}(u)$ for its integral on the slice and reserve E4C for the observable. We evaluate the correlator itself, obtained from the form-factor square with the 1:1 weighting summed over the five partons, by direct integration at the two physical points of the slice. When $\mathcal{G}(u)$ is written as a combination of master integrals, the coefficients follow exactly from g alone. The QCD correlator has its own integrand, built from the tree-level matrix elements of QCD, and at leading order it involves the same energy-fraction integrals as $\mathcal{G}(u)$, with different coefficients.

2.2 Generic angle versus collinear

The four-point correlator has been studied in two kinematic regimes, which differ in the experiments that probe them and in what has been computed. In the collinear regime all four detectors collapse into a narrow cone, the configuration measured inside a single LHC jet. The correlator degenerates to a function of the shape of the small-angle configuration. The four-point correlator was first computed in this regime, in $\mathcal{N} = 4$ super Yang–Mills [12], and related integrals were studied in Mellin space in Ref. [13]. The phenomenological program of Ref. [16], which introduced the dipole, tee and tripole families of fixed-shape configurations for comparing four-point predictions with jet-substructure data, is also formulated in the collinear regime.

At generic angle (equivalently, wide angle) the detectors are separated by order-one angles, the configuration probed by e^+e^- event shapes such as the two-point correlator, which was measured at LEP and has recently been re-extracted from archived DELPHI events [2]. In this regime Ref. [15] reduced the leading-order $\mathcal{N} = 4$ correlator, a function of five independent detector angles, to 141 master integrals in 44 sectors and classified them by the factorization of their Picard–Fuchs operators: 31 masters involve two elliptic curves, 14 involve one genus-2

¹In the ancillary notebook of Ref. [15] (arXiv v1), $q^2 \rightarrow 1$ is substituted before the remaining invariants are expanded as $2p_i \cdot p_j$. Expanding q^2 by the same rule would give 2. With the 1:1 weighting the S_5 -summed square is positive at every phase-space point we sampled and has a positive soft coefficient. With 1:2 it is negative at most sampled points.

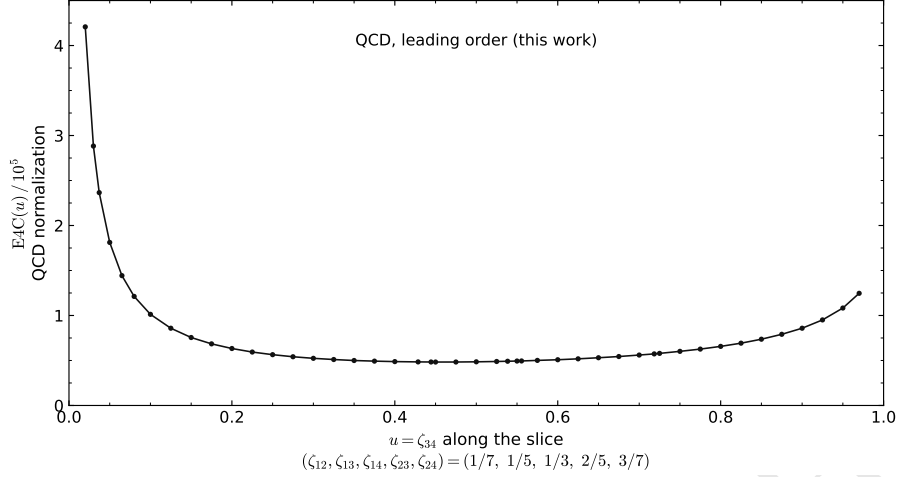


Figure 1: The wide-angle four-point energy correlator in e^+e^- QCD at leading order, computed in Section 6.1, at 46 points along the one-parameter slice $u = \zeta_{34}$. Per-point uncertainties are smaller than the markers. The two physical points of the slice, $u_1^* \simeq 0.0371$ and $u_2^* \simeq 0.7184$, are marked. The other 44 points are the analytic continuation of the same integral off the kinematic manifold. Normalization conventional.

hyperelliptic curve, and the remaining 96 are polylogarithmic. That work supplied the integrand and the master basis, and plotted numerical $\mathcal{N} = 4$ four-point values along coplanar and near-collinear families of configurations. It evaluated none of the masters at generic angle, however, and by its own account the canonical differential equations of the four-point system remain open. In QCD the four-point correlator at generic angle had not been computed before this work.

2.3 A one-parameter slice

We fix $(\zeta_{12}, \zeta_{13}, \zeta_{14}, \zeta_{23}, \zeta_{24}) = (\frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7})$ and vary $u = \zeta_{34} \in (0, 1)$. The six ζ_{ij} are not independent. Four detector directions in $\mathbb{R}^3$ exist only where the 4×4 Gram matrix $G_{ij} = 1 - 2\zeta_{ij}$ is positive semidefinite of rank at most three. With five angles fixed, the slice meets the kinematic manifold $\det G = 0$ at exactly two points: $\det G(u) = -\frac{32}{3675}(225u^2 - 170u + 6)$, with roots $u_{1,2}^* = (17 \mp \sqrt{235})/45 \simeq 0.0371$ and 0.7184 . Between them G has rank four (four directions in $\mathbb{R}^4$, none in $\mathbb{R}^3$), and outside them it is indefinite. Every integrand in this paper is a rational function of the six ζ_{ij} regarded as independent, so each integral is defined at every u . We therefore use the slice as a one-parameter family of the integrals: its only two physical points are u_1^* and u_2^* , and its values at every other u are the analytic continuation of the integrals off the kinematic manifold. In particular the endpoints $u \rightarrow 0$ and $u \rightarrow 1$ are formal limits of that family, not the physical squeezed and back-to-back configurations, in which detectors 3 and 4 would also make equal or supplementary angles, respectively, with each of the other two detectors. The five fixed angles are of order one, so the family stays away from the collinear corner in which all six angles vanish. On this family we evaluate the generator integral $\mathcal{G}(u)$ and the 141 masters of Ref. [15], together with 8 auxiliary masters, 149 in all.

3 The collinear limit: QCD versus $\mathcal{N} = 4$

The four-point correlator is best understood in the collinear limit described in Section 2.2, where the $\mathcal{N} = 4$ correlator is known in closed polylogarithmic form over a fixed alphabet [12]. In this limit the QCD calculation can also be carried far enough to give numerical curves for the correlator and to compare the analytic structure of the QCD and $\mathcal{N} = 4$ functions directly. For that comparison we ask whether the leading-order QCD collinear correlator lies in the function space of its $\mathcal{N} = 4$ counterpart, and we call the affirmative statement the shared-function-space conjecture. The conjecture concerns which singularities the two functions share, and it can be tested exactly, at the level of symbol letters and leading Landau singularities of the splitting-function integrand. We carry out the test on a one-parameter family of dipole configurations, and at leading singular order the conjecture holds for quark jets and fails for gluon jets. On the same family we then integrate both the QCD and the $\mathcal{N} = 4$ correlators in closed form (Section 3.5) and compare their alphabets as functions: they share every algebraic letter, cube roots included, and differ by the twelve tangency letters of the gluon-jet analysis, which QCD carries and $\mathcal{N} = 4$ does not.

3.1 The collinear E4C in QCD at leading order

In QCD the four-point correlator was not known even in the collinear limit. The authors of Ref. [16] write “Although the four-point correlator has not yet been analytically computed in QCD”, and, in their Section 3, “we are not currently able to compute the full four-point correlator in QCD”. In their comparison figures they therefore use the collinear $\mathcal{N} = 4$ correlator of Ref. [12] in place of the QCD curve. Here we compute the QCD curve.

At leading order the collinear E4C is governed by the $1 \rightarrow 4$ splitting functions. We take the tree-level quadruple-collinear splitting amplitudes of Del Duca, Duhr, Haindl, Lazopoulos and Michel [17, 18], for quark and gluon parents and with spin correlations retained, weight them by the four energy fractions, and integrate over the energy fractions and the overall orientation at fixed shape of the four detector directions. For the shapes we use the projections introduced in Ref. [16]: dipole, tee and tripole configurations of size parameter ρ , rotated through an azimuthal angle ϕ , with the observable the modulation of the correlator in ϕ at fixed ρ . We compute the correlator at 588 points across the three projections, for quark and gluon parents, and the resulting modulation curves are compared with CMS Open Data jets in Section 7. The quadrature is accurate to 0.1–0.2% per point, as established by repeating every point at a sequence of increasing quadrature orders.

We check the construction against two known results. First, integrating out one detector reduces the calculation to the analytic QCD collinear EEEEC of Ref. [19], which our result reproduces at five configurations for both parents. Second, in the strongly-ordered limit the $1 \rightarrow 4$ result reduces to the composition of lower-point splittings, as it must. In the course of the first check we found one typographical error in Ref. [19]: the expanded A^2 -form of the p_3 polynomial does not match the product form given there, and the product form is correct.

3.2 The candidate alphabets

We test the shared-function-space conjecture on a one-parameter slice through the collinear shape space. The slice consists of the dipole configurations of Section 3.1 with the two detector pairs at transverse positions $\pm(t/2)e^{i\phi}$ and $\mp\frac{1}{2}$, at fixed $\cos\phi = 3/5$, so the shape reduces to the

single aspect parameter t . On the QCD side the candidate list is the set of irreducible polynomial factors produced by the Landau analysis of the splitting-function integrand, decomposed into the cells of its Cheng–Wu representation. These Cheng–Wu cells are unrelated to the cells of the master-integral assembly at generic angle (Section 5.3). The factors arise from quadric values at linear triple points, from conic–line tangencies, from quadric pairs on a line, and from the deeper resultant layers. The list is a superset of the letters of the function, since Landau-type loci are enumerated and spurious letters are not removed. On the $\mathcal{N} = 4$ side we take the alphabet implied by the ancillary files of Ref. [12], namely all argument differences of all 1,255 distinct polylogarithms that appear in those files, plus the denominators of the rational prefactors. This choice is inclusive, so that the comparison is biased, if at all, toward the conjecture.

Neither candidate alphabet contains the other. In the full four-variable collinear shape space the QCD candidate list has 504 irreducible letters. These are singular loci, and as loci they are all rational: no algebraic extension is needed to define any enumerated layer, and the quadratic fields that occur on the slice arise only from restricting rational four-variable loci to it. The enumeration locates the singularities but does not decide whether the symbol letters attached to them are rational functions of the shape. The $\mathcal{N} = 4$ alphabet has 63 irreducible letters before restriction to the slice, 55 rational and 8 linear in a cube root. The cube-root letters have a cluster-algebraic construction [20]. Whether QCD shares this sector the enumeration cannot say, but the integration can. Cell by cell, the first two energy-fraction integrations of every $1 \rightarrow 4$ channel are linear in the sense of Brown, and the last energy fraction then enters through a cubic which, in one ordering, is the polynomial of Ref. [12] whose roots generate the cube-root letters; the same reduction also admits square roots of the tangency letters found below. We carry out the integrations at two generic rational shapes for the quark jet and at one of them for the gluon jet, with every piece expressed in hyperlogarithms and checked against direct quadrature to at least 28 digits, and reduce the weight-three part of the symbol of each leading-order channel, and of the quark-jet and gluon-jet correlators, exactly in the number fields involved. The cube-root letters survive the sum over cells and assignments in every channel: in the three quark-jet channels at both shapes and in the four gluon-jet channels at the one tested. The square roots of the tangency letters, which only $q \rightarrow \bar{q}qqq$, $g \rightarrow \bar{q}ggq$ and $g \rightarrow gggg$ produce before summation, cancel at weight three: at the first shape they are absent from the symbol of every channel and of both jet correlators, as letters and hence, by Galois invariance of the rational integrand, as coefficients, with every cancellation exact; at the second shape they cancel as letters in the quark-jet correlator. At weight three and at these shapes, therefore, the collinear QCD correlator shares the cube-root letters of its $\mathcal{N} = 4$ counterpart and acquires no square-root letter, for quark and gluon jets alike. The gluon-jet branch points of Section 3.4 leave no trace in the weight-three symbol at the shape tested; the terms beyond the symbol, the lower weights, and the function-level statement remain open. Between the two alphabets, 41 letters are shared, 463 are QCD-only, and 14 rational letters are $\mathcal{N} = 4$ -only. These counts refer to one labeling of the four detectors. Both correlators are symmetric under relabeling, and the $\mathcal{N} = 4$ list is not closed under it, so a QCD letter counts as new only if it lies outside the closure of the $\mathcal{N} = 4$ alphabet under the 24 relabelings. The excess letters identified below lie outside that closure. Restricted to the dipole slice and enumerated over the twelve assignment classes defined below, the 504 letters give 218 distinct slice letters, and every slice letter absent from $\mathcal{N} = 4$ is the image of a QCD-only letter off the slice.

Since a candidate letter need not be a letter of the function, we must determine which of the QCD-only letters are singularities of the observable. At the leading Landau layer (the conic–line

tangency pinches, where the local exponent is $(t - t_*)^{-3/2}$, twelve QCD-only letters occur, which we refer to below as the excess letters. They comprise four linear letters, six quadratics and two palindromic quartics, with 24 roots t_* among them (Table 1).

The twelve assignment classes follow from the automorphism group of the labeled distance matrix, which on the dipole slice has entries $d_{12} = t^2$, $d_{34} = 1$, $d_{13} = d_{24} = q_+(t)$ and $d_{14} = d_{23} = q_-(t)$ with $q_{\pm} = \frac{t^2}{4} \pm \frac{3t}{10} + \frac{1}{4}$. Two of the 24 assignments of detector labels to the four positions give identical integrand summands exactly when an automorphism of that matrix relates them. The automorphism group of the matrix is $\{\text{id}, (12)(34)\}$, the $z \rightarrow -z$ reflection, since any larger group would force polynomial identities among $\{t^2, 1, q_{\pm}\}$ that fail. The assignments therefore fall into $24/2 = 12$ classes of size two. Enumerating every denominator triple in every cell of all twelve classes shows that 77 of the 504 candidate letters occur in some cell, 23 of them shared with the $\mathcal{N} = 4$ alphabet and 54 QCD-only. Every coefficient in this section is computed in exact arithmetic. Letters of the candidate list also occur in the deeper resultant strata. There every candidate letter either has no pinch point on its stratum or has a coefficient sum that vanishes, identically at the $(1, 1, 1, 1)$ signature and at the leading exponent -2 at the higher-power signatures. The orders below the leading exponent -2 at the higher-power signatures are not computed. At the tangency stratum itself the excess configurations carry the $(1, 1, 2)$ signature only, for both jets, at every shape other than the degenerate square. At leading singular order, therefore, no QCD-only letter beyond the twelve of Table 1 enters either observable.

3.3 Quark jets

At a tangency pinch the quadric S restricts on the pinch line to $S|_{\text{line}} = A(t)s^2 + B(t)s + C(t)$ with roots $r_{\pm}$, and the complete local monodromy is controlled by one datum of the physically weighted sum $F = \sum_i N_i/R_i$ of cell numerators over their residual denominators, restricted to the pinch line:

$$V(t) = \frac{(F'(r_+) - F'(r_-))(r_+ - r_-) - 2(F(r_+) + F(r_-))}{A^2}. \quad (3)$$

At a simple root t_* of a letter, $V \neq 0$ forces the $(t - t_*)^{-3/2}$ branch point and $V \equiv 0$ removes it at this layer.

In the quark-jet correlator the coefficient of every branch point at an excess letter vanishes. At the leading Landau layer on the dipole slice, the quark-jet leading-order collinear E4C has $V(t) \equiv 0$ for each of the twelve excess letters at all 24 roots. The cancellation holds pinch by pinch: in each of the 44 pinch groups (the cells of one assignment class that share a given tangency pinch) $V(t)$ is identically zero, although each of the 228 cells in these groups makes a nonzero contribution to it. The cancellation is a property of the quark-channel splitting numerators summed across cells, and is equivalent to a statement about the decomposition of F under the deck transformation $\iota : s \rightarrow r_+ + r_- - s$ that swaps the two branches,

$$V \equiv 0 \iff \text{Res}_{s=r_+} \left[\frac{F_{\text{even}}(s, t)}{S(s, t)^2} \right] \equiv 0 \text{ identically in } t : \quad (4)$$

the ι -even sector contributes a residue-free double pole at the tangency. This residue condition is not a property of the individual cells: F_{even} is a nonzero rational function in every group, the residue is nonzero in each of the 90 cells for which we evaluated it separately, and the vanishing occurs only in the physically weighted sum over cells. The cancellation is not a property of

excess letter	roots	quark jet: $Q(t_*)$	gluon jet: $Q(t_*)$
$t \pm \frac{5}{8}, t \pm \frac{8}{5}$	4	0	$-23814/625$
$t^2 \pm \frac{2}{15}t - \frac{7}{9}, t^2 \pm \frac{6}{35}t - \frac{9}{7}$	8	0	$-135 \mp \frac{135}{4}\sqrt{11}$
$t^2 - \frac{1}{2}, t^2 - 2$	4	0	$-27/2$
$t^4 \pm \frac{4}{15}t^3 - \frac{826}{225}t^2 \pm \frac{4}{15}t + 1$	8	0	$(-214083 \mp 95499\sqrt{5})/178$

Table 1: The channel dichotomy at the leading Landau layer on the dipole slice. At every root t_* of each of the twelve excess letters, the cross-pinch branch coefficient $Q(t_*)$ of Eq. (5) vanishes pinch by pinch in the quark-jet observable, while the analytic continuation of the gluon-jet observable carries an exact nonzero $Q(t_*)$. None of the 24 roots is visible on the physical sheet. At 20 of the roots the slice meets two hypersurfaces of the full shape space, Eq. (6), and $Q(t_*)$ is the sum of their two coefficients, each nonzero. The full branch coefficient is $R^{-3/2}Q$ up to one universal constant. The coefficient Q is constant on the $t \rightarrow -t$ and $t \rightarrow 1/t$ orbits of the letter set, and the $\mp$ signs track the two roots of each disc-11 quadratic and the $\sqrt{5}$ -conjugate root pairs of the quartics. These twelve letters are the complete set of excess letters at leading singular order.

the slice. With the four shape variables free, V is a rational function of the shape for each tangency configuration, and the three nondegenerate excess configurations, all in $q \rightarrow \bar{q}qqq$, have V identically zero on the whole shape space. In each case the numerator of V vanishes identically, as an exact polynomial identity in the shape variables, with structural degree bounds 52, 174 and 388 for the three configurations. The slice identity $V(t) \equiv 0$ is the restriction of these four-variable identities, and every individual cell remains nonzero off the slice as on it. As a check that does not pass through the residue extraction, a direct contour quadrature of the on-line cell sum around the colliding roots reproduces the period $\Pi = 2\pi i V/\Delta^3$, with Δ the root separation, and the exponent $-3/2$ where $V \neq 0$ in the pure-gluon channel, and returns a summed period below 10^{-37} of a single cell's for the quark jet on the same letter. Why the even sector couples to the vanishing cycle only through a residue-free form is open, and is to us the most interesting question the calculation raises.

Of the three quark-jet channels, only $q \rightarrow \bar{q}qqq$ contains excess configurations, that is, tangency pinches of the twelve excess letters. The cancellation $V \equiv 0$ in that channel removes the excess letters from the quark-jet observable at the leading Landau layer, and the QCD-only letters of the deeper strata do not enter it either. With the shape free the same holds beyond the slice: the vertex family contributes no relabeling-genuine QCD-only letter, and in the two-quadric family at the $(1, 1, 1, 1)$ layer every excess letter with a computable datum, eight instances in four $q \rightarrow \bar{q}qqq$ configurations, of which four vanish cell by cell and four have one on-line quadric degenerating to a linear factor, vanishes on its hypersurface, by exact evaluation at rational points of the hypersurface beyond a structural degree bound. Twenty-nine instances have no computable datum at this layer and are not covered, nor are the three-conic family, the strata cut by higher powers of s_{1234} , and the orders below the leading exponent. At the tangency stratum, the vertex family and this layer, therefore, QCD adds no letter beyond the relabeling closure of the $\mathcal{N} = 4$ alphabet for quark jets, on the full shape space.

3.4 Gluon jets

In the gluon-jet correlator none of the branch-point coefficients at the excess letters vanishes once all pinches are summed. Excess configurations occur in two channels: $g \rightarrow \bar{q}ggq$ reproduces the quark cancellation, with $V \equiv 0$ in all 44 of its pinch groups, while in the $g \rightarrow gggg$ channel $V(t)$ is nonzero at all 98 pinches. The two channels with two quark pairs carry no excess configurations, so the gluon-jet branch coefficients come entirely from the $g \rightarrow gggg$ channel. The per-pinch data do not by themselves determine the branch coefficient in the observable, because distinct pinches sharing a root could cancel in the sum over the 24 detector-label assignments. The total $(t - t_*)^{-3/2}$ coefficient is, up to one universal constant,

$$C(t_*) = R(t_*)^{-3/2} Q(t_*), \quad Q(t_*) = \sum_p \frac{V_p(t_*) A_p(t_*)^3}{J_p(t_*)}, \quad (5)$$

where R is the discriminant cofactor of the restricted quadric, identical for all pinches of a given letter, and $J_p = \det[\nabla L_1; \nabla L_2; \lambda]$ is a signed transverse Jacobian that carries all the relative sign content.

With the contributions of all pinches summed, the analytic continuation of the gluon-jet leading-order collinear E4C on the dipole slice carries a $(t - t_*)^{-3/2}$ branch point with nonzero coefficient at every one of the 24 roots. None of the 24 is visible on the physical sheet. Every pinch line lies outside the physical domain, because on it a coordinate sum vanishes that is strictly positive throughout that domain. The branch points therefore belong to the analytic continuation of the observable and not to its boundary values on the physical sheet. The shared-function-space conjecture concerns the continuation, since the letters of a function are the letters of its analytic continuation.

The exact gluon-jet branch coefficients $Q(t_*)$ are collected in Table 1. They are constant on the $t \rightarrow -t$ and $t \rightarrow 1/t$ orbits of the letter set, as the symmetry of the shape demands. Pinches that form mirror pairs do cancel against each other, having equal VA^3 and opposite signed J . For every letter, however, there are also pairs of assignments with pinches on the same line, whose two terms are equal and add, and these dominate, so the total is nonzero in the pure-gluon channel. The channel dependence can be seen in a single letter: the quartic $t^4 - \frac{4}{15}t^3 - \frac{826}{225}t^2 - \frac{4}{15}t + 1$ is canceled in $g \rightarrow \bar{q}ggq$ and not in $g \rightarrow gggg$. As in the quark-jet case, the QCD-only letters of the deeper strata do not enter the gluon-jet observable at leading singular order, so on the slice the gluon-jet excess at that order consists of the twelve letters and 24 roots of Table 1.

The slice result determines the excess letters on the full shape space. Each of the twelve slice letters is the restriction of a tangency discriminant of s_{1234} on a cell line, and these discriminants are quadratic forms in the six pairwise distances d_{ab} , hence quartic hypersurfaces in the four shape variables. Eighteen of them pass through the 24 roots. Under relabeling of the detectors they form two orbits, of twelve and of six, with representatives

$$\begin{aligned} \Delta_{12} &= d_{12}^2 + d_{13}^2 + d_{14}^2 + d_{24}^2 + d_{34}^2 - 2d_{12}d_{13} + 2d_{12}d_{14} - 2d_{12}d_{24} + 2d_{12}d_{34} - 2d_{13}d_{14} + 2d_{13}d_{24} - 2d_{13}d_{34} \\ &\quad + 2d_{14}d_{24} - 2d_{14}d_{34} - 2d_{24}d_{34} - 4d_{14}d_{23}, \\ \Delta_6 &= d_{12}^2 + d_{13}^2 + d_{14}^2 + d_{23}^2 + d_{24}^2 - 2d_{12}(d_{13} + d_{14} + d_{23} + d_{24}) + 2d_{13}d_{14} - 2d_{13}d_{23} - 2d_{13}d_{24} \\ &\quad - 2d_{14}d_{23} - 2d_{14}d_{24} + 2d_{23}d_{24} + 4d_{12}d_{34}. \end{aligned} \quad (6)$$

At 20 of the 24 roots the slice passes through the intersection of two of the eighteen hypersurfaces, so that the value of $Q(t_*)$ in Table 1 is the sum of two contributions. Separated, every one of

the eighteen carries a nonzero exact coefficient at some slice root. A branch coefficient that is nonzero at one point of a hypersurface is not identically zero on it, so all eighteen are letters of the analytically continued gluon-jet correlator on the full four-dimensional shape space, and none of them lies in the closure under relabeling of the 55 rational letters of the $\mathcal{N} = 4$ symbol alphabet, which we re-extracted from the ancillary files of Ref. [12] for this check. Computed with the shape free, the datum V of the pure-gluon channel is nonzero as a function of the shape on each of them, while $g \rightarrow \bar{q}ggq$ satisfies the same four-variable identities as the quark jet.

3.5 The exact correlator on the dipole slice

The tests above act on the integrand. On the dipole slice we can also carry out the three energy-fraction integrations in closed form, for QCD and for $\mathcal{N} = 4$, and compare the two functions themselves. Cell by cell the first two integrations are linear in the sense of Brown. With the six pairwise distances d_{ab} rational functions of the single parameter t , they close over an alphabet known in advance from the Landau analysis of Section 3.2. The third integration is carried out in the fibration basis with the algebraic letters of the last energy fraction, the cube roots and square roots identified above, kept exact as elements of $\mathbb{Q}(i)(t)[\rho]/(C_t)$. The result is a finite sum of hyperlogarithms of weight up to three whose arguments and coefficients are explicit algebraic functions of t . We obtain it for each of the seven $1 \rightarrow 4$ channels summed over the 24 assignments, and for the $n_f = 5$ quark-jet and gluon-jet correlators. The quark-jet correlator has 31 117 distinct hyperlogarithm terms built from 4 243 distinct hyperlogarithms, with coefficient degrees in t up to 362; the gluon-jet correlator has 41 034 terms built from 5 167 distinct hyperlogarithms, with degrees up to 438. We give both in machine-readable form in the ancillary files. Evaluated at rational t on either side of every root of Table 1, the files reproduce an independent two-chart quadrature of the same integrand to 34 digits or better, and the numerical curves of Section 3.1 to their quadrature accuracy.

We apply the same procedure to the $\mathcal{N} = 4$ integrand, the square of the tree-level five-point form factor of Ref. [15] in the collinear limit. This gives the $\mathcal{N} = 4$ correlator on the slice as 21 348 terms with coefficient degrees up to 168. It agrees with the closed form of Ref. [12] restricted to the slice to 26 digits, up to the overall factor $5/24$ from the recoiler sum and the symmetry factor. The published representation of Ref. [12] is itself singular on this slice, where several of its denominators vanish identically for parallel-side labelings, and its value there is the limit from off the slice.

With both functions in hand on the same slice, we can compare the alphabets as properties of the functions themselves. They differ in exactly the way Sections 3.3 and 3.4 predict. The genuinely algebraic letters are common to all three functions. They are the same four cubic letters, the slice restriction of the polynomial of Ref. [12] whose roots generate the $\mathcal{N} = 4$ cube-root letters, with discriminant t times the restriction of their q_{23} , together with two square roots, of $25t^2 + 7$ and $7t^2 + 25$, which are the quadratic factors of that polynomial for the parallel-side labelings. These survive in the weight-three symbol of the quark jet, of the gluon jet and of the $\mathcal{N} = 4$ correlator alike, and do not appear at weights one and two. The rational letters of the $\mathcal{N} = 4$ function on the slice are the 34 slice restrictions of the letters of Ref. [12], and only those. The QCD functions carry those 34 and, in addition, the twelve tangency letters of Table 1. Eight of them occur in the quark jet, all from $q \rightarrow \bar{q}qqq$, and all twelve in the gluon jet, from $g \rightarrow gggg$, with eight also from $g \rightarrow \bar{q}ggq$. The channels $q \rightarrow \bar{q}'q'gq$, $q \rightarrow gggq$, $g \rightarrow \bar{q}'q'\bar{q}q$ and $g \rightarrow \bar{q}q\bar{q}q$ carry none. This is the material difference between the two alphabets on the slice, now read off the integrated functions: the extra QCD letters are precisely the tangency letters,

and nothing else.

How the extra letters enter distinguishes the two jets. In every QCD channel they appear only as discriminants of quadratic letters of the last integration and as factors of coefficient denominators. The logarithm of a tangency letter is never a letter of the symbol, and the square roots of the tangency letters cancel exactly from the weight-three symbol of both jet correlators. Both correlators are single-valued and analytic at every root t_* in $(0, 1)$ on the physical sheet, in agreement with Section 3.4. The exact denominators bound the strength of the singularity at t_* on any sheet of the continuation. If a tangency letter occurs to at most the power m in the denominators, the function is no worse than $(t - t_*)^{-m}$ there, with half-odd exponents no worse than $\frac{1}{2} - m$. For the quark jet, and for $g \rightarrow \bar{q}ggq$, $m = 1$ for every tangency letter, which excludes a $(t - t_*)^{-3/2}$ term on every sheet, as the identity $V \equiv 0$ requires. For $g \rightarrow gggg$, and hence for the gluon jet, $m = 2$, which is what the $(t - t_*)^{-3/2}$ branch points of Table 1 need. The channel dichotomy of the Landau analysis is thus visible directly in the closed forms.

4 Generic angle: the analytic structure

We determine the function space of the correlator on the generic-angle slice $u = \zeta_{34}$ of Section 2.3 from the geometry of its master-integral sectors. For one elliptic sector we give the curve and the singularities of its Picard–Fuchs operator, and bring the sector’s differential system to a terminal ε -form. The letters of that ε -form are all built from u , from the discriminant of the curve, and from the square roots of three quadratics in u , and the homogeneous solutions of the system’s subsector layer are algebraic. We then identify the genus-2 curve on the slice and show that the kernels of three further sectors, non-polylogarithmic at generic angle, are algebraic there. The remaining 96 of the 141 masters are polylogarithmic by the classification of Ref. [15], so the non-polylogarithmic content on the slice is confined to the periods of the two elliptic curves and the genus-2 curve.

4.1 An elliptic curve from the maximal cut

In the numbering of the ancillary material of Ref. [15] the elliptic sector we treat has propagators $D_5 = -1 + \sum x_i$, $D_7 = -1 + x_2 + x_3 - x_2x_3\zeta_{23}$, and the measurement constraint $D_\delta = 1 - \sum x_i + \sum_{i < j} x_ix_j\zeta_{ij}$, and D_7 does not depend on u . The eight masters are

$$I_N = \int_{x_i \geq 0} d^4x \frac{N(x) \delta(D_\delta)}{D_5 D_7}, \quad N \in \{1, x_1, x_1^2, x_1^3, x_2, x_1x_2, x_2^2, x_3\}. \quad (7)$$

On the maximal cut $\{D_5 = D_7 = D_\delta = 0\}$ (the integral with all propagators put on shell), x_1 and x_2 are eliminated linearly and the remaining constraint is quadratic in x_4 . Its discriminant is $(2x_3 - 5)^2 P(x_3; u)$, where the square factor is spurious at generic u and

$$\begin{aligned} P(x_3; u) = & (44100u^2 - 47040u + 784)x_3^4 - (220500u^2 - 254100u - 6160)x_3^3 \\ & + (275625u^2 - 404250u - 53975)x_3^2 + (157500u + 28500)x_3 + 22500, \end{aligned} \quad (8)$$

a genus-1 quartic in x_3 , quadratic in u . The leading coefficient of P is $196(225u^2 - 240u + 4)$, a quadratic that reappears in the letter alphabet of the sector. The j -invariant of the curve is a rational function of u with numerator of degree 12 and denominator of degree 7. The

discriminant of the curve is, up to an overall constant, the degree-7 polynomial

$$\begin{aligned} \Delta(u) = & 136744453125 u^7 - 313861078125 u^6 + 234481078125 u^5 - 39091933125 u^4 \\ & - 6770993625 u^3 - 10280927775 u^2 - 346729945 u + 28045969, \end{aligned} \quad (9)$$

irreducible over $\mathbb{Q}$. Since j is not constant in u , the family of curves over the u -line is a non-isotrivial rational elliptic surface, with singular fibers at the seven roots of $\Delta(u)$ and an eighth at $u = \infty$. Two of the roots, $u \simeq 0.0376$ and 0.9796 , lie inside the unit interval. The first lies 5×10^{-4} above the physical point $u_1^* \simeq 0.0371$, and neither is on the kinematic manifold.

4.2 The Picard–Fuchs operator and the period lattice

The eight maximal-cut integrals $M_N(u) = \oint \tilde{N}(x_3; u)/\sqrt{P} dx_3$ (with $\tilde{N}$ the substituted numerator times the cut Jacobian) are periods of the curve (8). By Griffiths–Dwork reduction we obtain the exact order-2 Picard–Fuchs operator

$$L_u = \partial_u^2 - q_1(u) \partial_u - q_0(u) \quad (10)$$

annihilating $\Pi_0 = \oint dx_3/\sqrt{P}$. In the basis $\{1/y, x/y, x^2/y\}$, where $y^2 = P(x; u)$ with $x = x_3$, the 3×3 Gauss–Manin connection (the first-order differential system satisfied by the periods) has rational entries with common denominator $\Delta(u)$ times the quadratic $225u^2 - 240u + 4$, an apparent factor, and Π_0 is cyclic of rank 2. The coefficients $q_{0,1}(u)$ are rational with common denominator $\Delta(u)$ times the irreducible quartic

$$A(u) = 2480625 u^4 - 2570400 u^3 + 536850 u^2 + 43896 u - 23771, \quad (11)$$

whose roots are apparent singularities of the operator, not of the periods.

All eight maximal-cut integrals reduce exactly over $\mathbb{Q}(u)$ onto $\{\Pi_0, \Pi_1, \Pi_2\}$, where $\Pi_k = \oint x^k dx/y$ for $k = 0, 1, 2$, together with one third-kind period $J = \oint dx/[(2x - 5)y]$. The Gauss–Manin connection extends to a 4×4 system with a row for the third-kind period. One of these reductions is $M_1(u)/\Pi_0(u) = \frac{1}{5}$ for all u , so M_1 is a first-kind period.

4.3 Three quadratic curves and a closed alphabet

Beyond the maximal cut we transform the full differential system of the elliptic sector to its terminal ε -form and read off the letters from the transformed connection. On an 11-dimensional frame (the system is exact in four dimensions, and ε is a formal grading parameter) the connection transforms to $\varepsilon \tilde{A}(u) + S(u)$: conjugating by $U = \text{diag}(\varepsilon^L)$ with weight vector $L = (0, 1, 0, 0 | 2, 2, 3, 3 | 4, 4, 4)$ brings the system to $\hat{A} = \hat{S} + \sum_{k=1}^4 \varepsilon^k \hat{A}_k$, with the entire ε -independent part $\hat{S}$ confined to exactly two entries of Eichler type. The transformation to ε -form introduces the square roots of three quadratics in u ,

$$q_1 = 225u^2 - 240u + 4, \quad q_2 = 1225u^2 - 1190u + 1, \quad q_S = 45u^2 - 138u + 5, \quad (12)$$

the first of which is both the quadratic apparent factor of the Gauss–Manin denominators above and the leading coefficient of the quartic (8) up to the constant 196. The letter alphabet is closed over these quadratics: every letter of the transformed system is built from u , $\Delta(u)$, and the square roots of the three quadratics (12). The residue matrices of the transformed connection are nilpotent, and the homogeneous solutions of the subsector layer are algebraic,

because the connection of that layer vanishes identically once its algebraic prefactors are peeled off. The solutions of the sector's system therefore lie in the class generated by the periods of the curve (8), Eichler-type integrals over them, algebraic functions of u , and polylogarithms in the closed alphabet. The maximal cuts themselves, as periods of a non-isotrivial elliptic surface, are not polylogarithms of any argument, so this sector does not degenerate on the slice and is elliptic there, as it is at generic angle [15].

4.4 Genus-2 and algebraic sectors

The classification of Ref. [15] contains further non-polylogarithmic sectors, whose kernels we also identify on the slice. Three sectors, with propagator sets $\{D_5, D_{10}, D_{11}\}$, $\{D_1, D_{10}, D_{11}\}$, and $\{D_6, D_9, D_{10}\}$, have kernels that are algebraic of degree two on the slice. At generic angle Ref. [15] lists the first two as genus-2 hyperelliptic and the third as elliptic, so the kernels of all three sectors degenerate on the slice. The kernel of the sector $\{D_{10}, D_{11}\}$, by contrast, is a genus-2 hyperelliptic period on the slice, with 13 period constants, each a convergent one-dimensional integral. The maximal cut of this sector is one-dimensional: x_1 and x_4 are rational over a plane quartic in (x_2, x_3) whose only singularity at generic u is a node at the origin (a cusp at $u = 3/5$), so the curve has geometric genus 2. Projecting from the node by $x_2 = tx_3$ gives the hyperelliptic model $y^2 = S(t; u)$ with

$$\begin{aligned} S(t; u) = & 50625 t^6 + (236250 u - 54000) t^5 + (275625 u^2 - 31500 u + 149850) t^4 \\ & + (220500 u^2 + 1125600 u - 100800) t^3 + (2050650 u^2 - 285180 u + 441) t^2 \\ & + (555660 u^2 + 30870 u) t + 540225 u^2, \end{aligned} \quad (13)$$

a sextic in t and quadratic in u . The discriminant of S in t is, up to a constant, $u^3(5u-3)^2 L_{10}(u)$ with L_{10} an irreducible polynomial of degree 10. The Gauss–Manin connection in the basis $t^k dt/y$, $k = 0, \dots, 4$, has the letters u , $5u-3$, and L_{10} , and dt/y is annihilated by a Picard–Fuchs operator of order 4. At the ten roots of L_{10} the fibers are nodal, with unipotent monodromy. One of these roots, $u \simeq 0.0363$, lies inside the unit interval, 8×10^{-4} below the physical point $u_1^* \simeq 0.0371$. The curve is non-isotrivial, since its Igusa–Clebsch invariants vary with u . These identifications fix the function class of each of the four kernels, and the corresponding master integrals enter the $\mathcal{N} = 4$ assembly of Section 5 numerically. The integrals of the genus-2 sector also appear in the quark-pair channel of the wide-angle QCD correlator, with the nonzero rational weights computed in Section 6.3.

5 The $\mathcal{N} = 4$ evaluation

The $\mathcal{N} = 4$ generator integral $\mathcal{G}(u)$ is assembled on the slice from 149 master integrals and evaluated at five rational points, none of them on the kinematic manifold. On the maximal cut we transport the eight periods $M_N(u)$ from a base point to other points of the slice by integrating the Picard–Fuchs and Gauss–Manin equations in u . The eight full masters of the elliptic sector are then written exactly, with no fitted constant, in terms of these periods and of initial data taken at three base points of the slice (the anchor constants). The two wall constants K_1 and $K_0(u)$ of the sector's boundary term are in closed form at weight two, and the anchor constants are left as integrals evaluated numerically. The generator integral is then assembled as a sum with exact rational weights over 443 cells (master integrals, their symmetry images, and residual terms of the reduction), nine of the cells evaluated analytically and the rest numerically.

Finally, in the formal limit $u \rightarrow 0$ of the slice we compare $\mathcal{G}(u)$ with the two-particle residue of the generator and with the known three-point correlator.

5.1 Transport on the maximal cut

We compute all eight maximal-cut integrals $M_N(u)$ at the base point $u = 1/5$ and at the two target points $u = 3/7$ and $5/9$ by direct segment quadrature, isolating the roots of P and integrating between the complex-conjugate pair. We then integrate the Picard–Fuchs equation $L_u \Pi_0 = 0$ from the base point to the target points by Taylor-series stepping, with the derivative supplied exactly by the Gauss–Manin connection. At both target points the transported Π_0 agrees with its direct quadrature, and the four-vector $\{\Pi_0, \Pi_1, \Pi_2, J\}$ carried along the slice with the extended connection reproduces all eight maximal-cut integrals M_N there.

5.2 The full elliptic sector

Beyond the maximal cut, the eight masters I_N themselves are obtained exactly on the slice by solving their differential system, including its boundary terms. Because the integration region is bounded, the masters satisfy the differential equations of the closed cycles corrected by a boundary flux: the codimension-one faces of the region contribute source terms, which follow exactly from the boundary reduction. Each master is then a combination, with coefficients rational in u , of the sector’s period functions, an algebraic solution, and subsector functions, where the inhomogeneous part is obtained from the sources by variation of parameters. One piece of the flux is not fixed by the bulk reduction: a corner-localized wall term, an exact rational-in- u combination of a u -independent constant K_1 and a function $K_0(u)$. Both are obtained in closed form at transcendental weight two by term-by-term integration of the local models at the corner, and agree with direct quadrature of the wall terms:

$$\begin{aligned} K_1 = \frac{35}{8} & \left[\text{Li}_2(-1) + \text{Li}_2\left(-\frac{1}{3}\right) + \text{Li}_2\left(-\frac{1}{7}\right) - \text{Li}_2\left(-\frac{19}{21}\right) - \text{Li}_2\left(-\frac{19}{5}\right) \right. \\ & - \text{Li}_2\left(-\frac{3}{2}\right) + \text{Li}_2\left(-\frac{5}{7}\right) + \text{Li}_2\left(-\frac{57}{7}\right) + \text{Li}_2(-8) - 2 \text{Li}_2\left(-\frac{9}{7}\right) \\ & - \text{Li}_2\left(\frac{15}{7}\right) + \text{Li}_2\left(\frac{27}{7}\right) - \text{Li}_2\left(\frac{4}{3}\right) - \text{Li}_2\left(\frac{5}{2}\right) + \text{Li}_2\left(\frac{9}{5}\right) \Big] \\ & + \frac{35}{16} \pi^2 + (\text{explicit log-bilinear remainder}) = 8.30761688\dots, \end{aligned} \quad (14)$$

$$\begin{aligned} K_0(u) = \frac{5}{2(1-u)} & \left[\text{Li}_2\left(-\frac{1}{5u}\right) - \text{Li}_2\left(\frac{5u-1}{u(9-5u)}\right) + \text{Li}_2\left(\frac{5(5u-1)}{5u-9}\right) - \text{Li}_2\left(\frac{4}{5(u-1)}\right) \right. \\ & \left. - \text{Li}_2\left(\frac{5u-1}{u-1}\right) + \text{Li}_2\left(\frac{5u-1}{4u}\right) \right] + (\text{explicit log-bilinear remainder}) \end{aligned} \quad (15)$$

on $u \in (\frac{1}{5}, 1)$, with Li_2 the real part of the principal-branch dilogarithm where an argument exceeds one. The six dilogarithm arguments in $K_0(u)$ are cross-ratios of the roots of the linear forms that define the wall and face pieces of the corner model. The closed forms for K_1 and $K_0(u)$ above are not minimal: dilogarithm identities could shorten them.

The initial data for the differential system of the full elliptic sector are set at $u \in \{\frac{19}{50}, \frac{1}{2}, \frac{13}{20}\}$ by direct quadrature, and these anchor constants remain integrals evaluated numerically. With these data, the exact expressions for the eight masters agree with direct quadrature at four further points, $u \in \{\frac{1}{3}, \frac{21}{50}, \frac{26}{49}, \frac{2}{3}\}$, and at interior points of the slice they satisfy the differential equation only when the boundary-flux term is included.

u	$\mathcal{G}(u)$	uncertainty
1/2	8.7328850	8.5×10^{-9}
3/7	11.726889	1.1×10^{-8}
5/9	6.8697979	2.7×10^{-9}
4/9	10.983988	3.0×10^{-9}
7/13	7.4117061	5.7×10^{-10}

Table 2: The $\mathcal{N} = 4$ generator integral $\mathcal{G}(u)$ at five points of the slice, none on the kinematic manifold, with its absolute uncertainty, obtained by adding weight times cell uncertainty over the 443 cells of the sum (16).

5.3 Assembly on the slice

The generator integral is assembled as

$$\mathcal{G}(u) = \sum_{\text{cells}} d_{\text{cell}}(u) I_{\text{cell}}(u) \quad (16)$$

over 443 cells: master-integral cells, their images under the discrete symmetry of the slice, residual cells produced by the reduction, and one combination of divergent cells that is finite only as a group. The weights $d_{\text{cell}}(u)$ are exact rational numbers, given by $d = R^T c$. Here c is the vector of exact coefficients of the integrand in the raw term-integrand basis of Ref. [15] (452 finite and 141 divergent term-integrands), and R is the exact reduction matrix expressing those term-integrands in the master basis.

Nine cells, in five sectors, have analytic evaluations: three of the eight masters I_N of the elliptic sector $\{D_5, D_7\}$ (sector 1), the sector $\{D_3\}$ (sector 2), three masters of $\{D_8, D_{11}\}$ (sector 3), $\{D_4, D_5\}$ (sector 4), and $\{D_1, D_2\}$ (sector 5). The expressions for sectors 2–5 are closed forms with no numerical constants, and the three sector-1 expressions contain the anchor constants of the elliptic sector, evaluated numerically. In the assembly these evaluations replace the numerical values of their cells. Two further sectors, $\{D_3, D_6\}$ and $\{D_1, D_5\}$, have analytic evaluations that we did not substitute, so they enter numerically, as does every other cell.

5.4 Results

The values of the generator integral $\mathcal{G}(u)$ at five rational points of the slice are listed in Table 2, and Figure 2 shows $\mathcal{G}(u)$ along the whole slice. At each of these five points we evaluate the sum (16) with all 443 cells numerical and again with the nine analytic cells substituted, so that the two evaluations differ only in those cells. The point $u = 1/2$ was used in constructing the analytic evaluations, and the other four were not. The two evaluations agree to 30 significant figures or better at four of the five points, and to 29.8 at $u = 4/9$, where the comparison is limited by the accuracy of the numerical values. The 443 terms of the sum cancel among themselves by five to six orders of magnitude, so the totals of Table 2 are accurate to nine to ten significant figures. A direct split-domain cubature of g agrees with the table within the quoted uncertainty at four points. At $u = 7/13$ it differs by 1.8×10^{-9} , three times the uncertainty.

The orbit sum of the generator, obtained by direct integration of the 24 relabeled generators, is about 47 times $\mathcal{G}(1/2)$ at $u = 1/2$ and takes the values 2.73×10^3 and 462 at the two physical points u_1^* and u_2^* . At every sampled point of the slice it is positive, with a shallow minimum near $u = 1/2$. The correlator itself at the two physical points is given in Section 6.5.

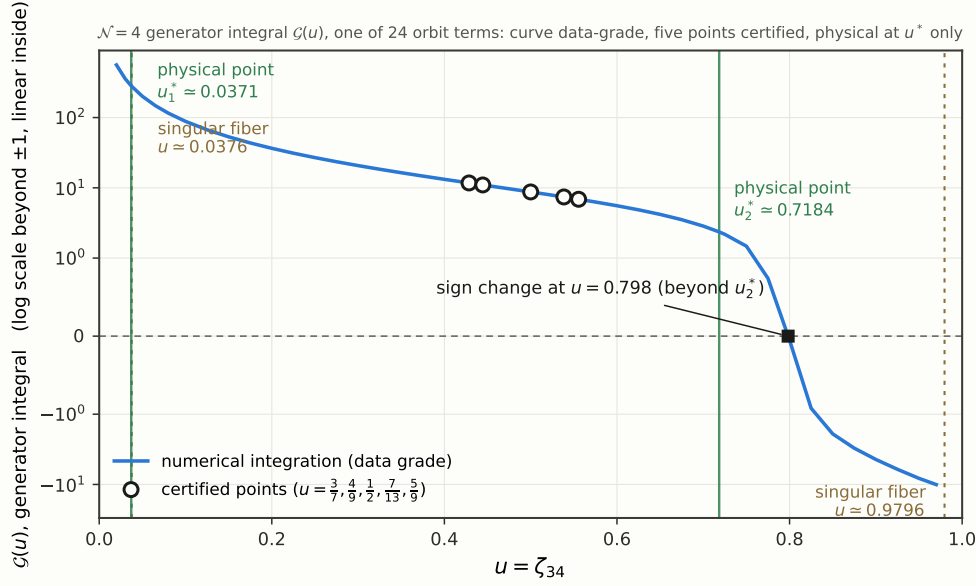


Figure 2: The $\mathcal{N} = 4$ generator integral $\mathcal{G}(u)$ along the slice. It falls by nearly two orders of magnitude from the $u \rightarrow 0$ end and changes sign at $u = 0.798$, beyond the second physical point $u_2^* \simeq 0.7184$, where no detector configuration exists. The curve is a direct numerical integration of the generator at 10^{-4} – 2×10^{-4} relative precision away from the sign change (2.8×10^{-3} at the point bracketing it). The open circles are the five points of Table 2. The two physical points are marked, and dotted vertical lines mark the two singular fibers of the elliptic surface (9) inside the unit interval, $u \simeq 0.0376$ and 0.9796 . The vertical scale is logarithmic beyond ± 1 and linear inside.

5.5 Limits of the generator integral

In the formal limit $u \rightarrow 0$ of the slice, which lies off the kinematic manifold, we compare the generic-angle values of $\mathcal{G}(u)$ with two known quantities, the residue of the generator at its two-particle pole and the closed-form three-point correlator. The identities below are properties of the generator integral, not of the correlator. As $u \rightarrow 0$, detectors 3 and 4 coincide while their angles to the other two detectors stay distinct. There $\mathcal{G}$ develops a $1/u$ pole controlled by the residue at the two-particle pole of the generator, carried by 19 of its 43 terms on the slice. The coefficient of $1/u$, fitted to direct numerical integrations of $\mathcal{G}$ at u down to $1/60000$, matches the exact prediction from that residue. The fitted coefficient of $\ln u/u$ is consistent with zero, so there is no leading logarithm. The residue integral of the generator factorizes over the three-point integrand's family of triangle configurations. At the two endpoints of this family the four-point configuration degenerates to three-point triangles. The three-point correlator there, computed by direct integration of the three-point integrand of Ref. [15] summed over its S_3 orbit, matches the closed form of Yan and Zhang [9] with identical normalization and no free constant.

The full collinear limit, in which all six angles vanish and the $\mathcal{N} = 4$ result of Ref. [12] applies, is a different boundary from the formal endpoint $u \rightarrow 0$. The slice does not reach that regime, since five of its six pairwise angles are held at order-one values. The collinear result and the generic-angle values of this section therefore refer to disjoint regions of the five-dimensional

angle space, and they are values of the same function only once the generator integral is replaced by the correlator.

6 QCD at generic angle, and the contrast

The correlators measured at colliders are QCD observables, whereas the elliptic curves and the genus-2 curve were identified in the master integrals of the $\mathcal{N} = 4$ correlator. At leading order the QCD correlator involves the same energy-fraction integrals with different weights, and the weights of the integrals in which these curves appear could in principle vanish. In the quark-pair channel we find that the genus-2 integrals are multiplied by nonzero rational functions of u , which we obtain in closed form. The genus-2 sector is therefore present in the QCD integrand, although its periods could still cancel among the integrals. Throughout this section we work at leading order in QCD with the e^+e^- wide-angle four-point correlator on the slice. We first obtain it by direct integration along the whole slice, and we then compute exact weights for the quark-pair (n_f) channel only. At the two physical points of the slice, where both theories give the value of an observable, we also compare the QCD correlator with the $\mathcal{N} = 4$ one directly.

6.1 Wide-angle QCD at leading order

We compute the QCD four-point correlator at generic angle on the same slice $u = \zeta_{34}$ as the $\mathcal{N} = 4$ generator integral, with its two physical points $u_1^* \simeq 0.0371$ and $u_2^* \simeq 0.7184$. At leading order the prediction is finite at a configuration only if the four detector directions lie in a common hemisphere. When instead some positive linear combination of the four directions vanishes, four massless partons along these directions can balance momentum among themselves and populate the configuration at $O(\alpha_s^2)$. That four-parton term is then the leading order, and the five-parton integral below is a real-emission correction with an unsubtracted infrared divergence. Both physical points of the slice satisfy the hemisphere condition, with largest pair angles 82° and 116° .

We obtain the correlator by direct integration of the tree-level five-parton matrix elements ($q\bar{q}ggg$ and $q\bar{q}Q\bar{Q}g$, $N_c = 3$, $n_f = 5$) over the energy fractions at fixed detector angles. We omit the interference terms that arise when the two quark pairs in $q\bar{q}Q\bar{Q}g$ share a flavor, since these terms belong to neither of the two matrix-element channels used. The charge interference between the flavors cancels in the correlator by charge conjugation.

We evaluate the correlator at 46 points along the slice by adaptive cubature and by quasi-Monte Carlo integration. The two methods agree to 0.39% or better, so the curve is accurate to two to three significant figures per point. At $u = 1/2$, $1/4$, $3/8$, $5/8$ and $3/4$ we refine the integration to five or six figures. The same integration applied to the $\mathcal{N} = 4$ generator g reproduces the five values of Table 2. At the two physical points the QCD correlator takes the values 2.365×10^5 and 5.731×10^4 in a conventional normalization. We leave unfixed only a single global prefactor (couplings, powers of 2π , the Born normalization, and the angular-measure Jacobian), so shapes and ratios along the slice are convention-independent. The nearest existing results, none of which gives QCD values at generic angle, are the iterated-collinear limits and parton-shower curves of Ref. [16], the collinear $\mathcal{N} = 4$ correlator [12], and the generic-angle $\mathcal{N} = 4$ master integrals of Ref. [15], classified there without evaluation.

6.2 Weights and integrals

The wide-angle QCD curve just computed is numerical throughout. To make part of the correlator exact, we use its layered structure: at leading order the correlator on the slice has the same form as the $\mathcal{N} = 4$ assembly (16), exact rational functions of u , the weights, multiplying a fixed set of energy-fraction integrals. The special geometry is a property of the integrals, fixed by the propagators and the measurement constraint alone, so it is shared between theories. The theory enters through the weights. In particular, whether QCD couples to the genus-2 sector is a property of the weights alone. We carry out this separation of weights from integrals in the n_f channel (the $q\bar{q}Q\bar{Q}g$ final state), with its three color structures N^3 , N^1 and N^{-1} .

6.3 Genus-2 weights in the n_f channel

The n_f channel couples to the genus-2 sector $\{D_{10}, D_{11}\}$ with nonzero exact rational weights at the rational points $u = 1/149$ and $u = 12/149$, neither on the kinematic manifold. We also compute the weights attached to the sector $\{D_8, D_{11}\}$, whose kernels are multiple polylogarithms, not elliptic, and three of whose masters are among the nine analytic cells of the $\mathcal{N} = 4$ assembly. All 34 rows of the weight vector attached to these two sectors (17 each) are nonzero in all three color structures at $u = 1/149$ and at $u = 12/149$, 204 exact rational values in all. With the closed-form weight functions constructed below, the same statement holds at the physical points: evaluated at u_1^* and u_2^* , exactly in $\mathbb{Q}(\sqrt{235})$, all 102 weight functions of the two sectors are nonzero, the smallest in magnitude 0.012 and 0.0060. Genus-2 periods therefore enter the leading-order QCD correlator unless they cancel in the integral layer. The weights in this statement are exact, and it covers only the n_f channel and the two sectors treated here.

The 34 rows, across the three color structures, define 102 functions $c_j(u)$, and all 102 are rational functions of u in closed form, with numerator-plus-denominator degree between 4 and 15 and coefficients of at most 76 bits. Among the other weights of the channel the degrees reach 406, so these two sectors are low-degree outliers. We obtain these functions by exact linear algebra modulo 59 primes, recombining the modular images by the Chinese remainder theorem and rational reconstruction. The closed forms reproduce the weights computed at four further values of u , including $u = 1/500$ and $u = 1/1000$. As $u \rightarrow 0$, 68 of the 102 functions have poles, of order up to u^{-4} , 32 are regular and nonzero, and 2 vanish. Individual weight functions are steeper at small u than the assembled correlator, so the excess must cancel in the assembly.

6.4 The full weight vector

Beyond the sectors $\{D_{10}, D_{11}\}$ and $\{D_8, D_{11}\}$ we compute the full weight vector of the n_f channel exactly at ten rational values of u . Expanding the channel over its integrand basis gives 13,601 independent rows per color structure. The ten values of u are the six points $u = k/149$ with $k \in \{1, 4, 8, 12, 16, 20\}$ (among them $u = 1/149$ and $u = 12/149$, used above) and the four points $u = 1/100, 1/200, 1/500$ and $1/1000$ on the wide-angle curve. We reconstruct the rational values from their images modulo 59 primes by the Chinese remainder theorem. The reconstruction is stable under removal of primes and reproduces the images at further primes not used in it. Two entries of the weight vector at $u = 1/500$ and $u = 1/1000$, of about 3,550 bits, exceed the range of the 59-prime reconstruction, and for these we use 62 primes.

6.5 The two theories at the physical points

Along the whole slice the $\mathcal{N} = 4$ curve is the generator integral $\mathcal{G}(u)$, while the QCD curve is summed over all 120 parton assignments, and away from $u_{1,2}^*$ neither is a value of an observable. In particular the sign change of $\mathcal{G}$ at $u = 0.798$ lies beyond u_2^* , where no detector configuration exists, and the orbit sum of the generator is positive at every sampled point of the slice. We can therefore compare the predictions of the two theories for the same observable only at the two physical points. To obtain the $\mathcal{N} = 4$ correlator itself at these two points, we integrate the form-factor square with the 1:1 weighting directly, summing over the five partons with the energy-fraction measure of Eq. (2). The correlator is 1.221×10^5 at u_1^* and 3.613×10^4 at u_2^* , each value converged well beyond the figures shown. With the leading-order QCD values of Section 6.1 at the same two configurations, the ratio of the correlator between the two points is 4.13 in QCD and 3.38 in $\mathcal{N} = 4$, with each theory's overall normalization conventional. By this measure the two theories are close: from u_1^* to u_2^* the QCD correlator falls 22% further than the $\mathcal{N} = 4$ one. Two physical points determine only this one ratio, and a comparison of the shapes of the two correlators would need a physical one-parameter family of configurations, with three detectors fixed, the fourth moving on the sphere, and ζ_{34} solved from $\det G = 0$.

6.6 $\mathcal{N} = 4$ as a proxy for QCD

In the decomposition into weights and integrals above, the two theories share their integrals, so at generic angle they differ only through their weights, the rational coefficients with which the same functions are summed. We have carried out that decomposition for the n_f channel only. In the collinear limit, with the leading-order QCD correlator now computed, one can test directly whether the $\mathcal{N} = 4$ shape [12] used in the comparison figures of Ref. [16] is a good quantitative proxy for the QCD one. We have not made that comparison here. Whatever its outcome, the proxy cannot be exact for gluon jets, because the analytic continuation of the QCD collinear correlator has singularities that the $\mathcal{N} = 4$ function space does not contain. For quark jets no QCD-only letter contributes at leading singular order, so at that order the QCD collinear correlator lies in the $\mathcal{N} = 4$ function space. At generic angle the only evidence in this paper comes from the two physical configurations, between which the QCD and $\mathcal{N} = 4$ correlators fall by similar factors. That similarity does not exclude the $\mathcal{N} = 4$ correlator as a proxy for QCD at generic angle, but two configurations are too few to justify the substitution quantitatively.

7 Confronting data

We compare the collinear QCD correlator of Section 3.1 with CMS Open Data jets and the wide-angle correlator of Section 6.1 with archival DELPHI events. Both comparisons are a preliminary proof of principle made with public open-data releases. The CMS curves are extracted from charged tracks in the 2011A dataset and the DELPHI curves from the archival events, in both cases at detector level and without unfolding. We leave unfolded four-point measurements on the full datasets to the collaborations.

7.1 Curves from CMS Open Data

Shape-resolved energy correlators have been measured only up to the three-point correlator. Projected (single-angle) correlators have been extracted up to $N = 6$ from CMS Open Data [3],

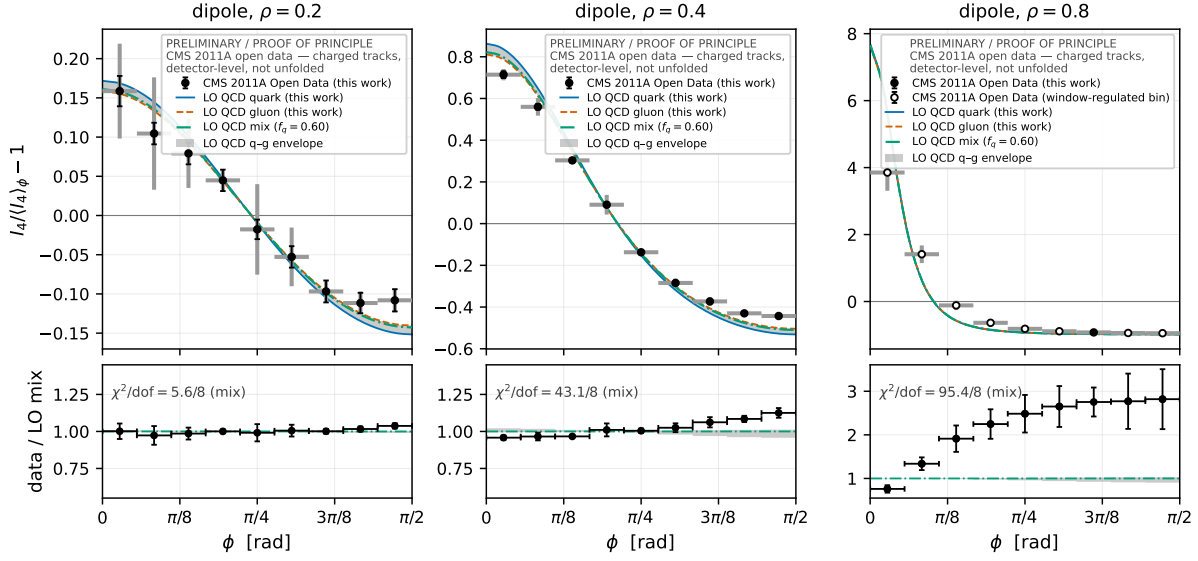


Figure 3: Leading-order collinear QCD describes the shape of the measured four-point dipole modulation, with agreement degrading as the configuration grows toward the jet boundary (χ^2/dof in Table 3). Top row: CMS 2011A open-data modulation $I_4/\langle I_4 \rangle_\phi - 1$ (points), with I_4 the measured four-point correlator, against the LO QCD quark, gluon and $f_q = 0.60$ mixture curves of this paper. The gray band is the pure-quark/pure-gluon envelope, which bounds every mixture. Bottom row: data over the mixture curve. Inner error bars are statistical (block bootstrap), and outer bars add the window-halving systematic. Open symbols mark window-regulated bins, where that systematic exceeds three times the halved-sample statistical error. Detector-level open data.

but we know of no shape-resolved four-point measurement on any dataset. We therefore extract the curves ourselves. From the CMS 2011A Open Data we select 136,740 AK5 jets (anti- k_T , $R = 0.5$) with corrected $p_T \in [500, 550]$ GeV and $|\eta| < 1.9$, and compute the correlators from each jet's charged constituents with $p_T > 1$ GeV. As a calibration of the extraction, we reproduce the ratios of projected correlators measured in Ref. [3] (the three- through six-point correlators, each normalized to the two-point correlator) to within 0.5σ with no free parameters. The absolute two-point correlator requires one global normalization factor, 0.99, and after that fit still deviates by up to 1.5σ . The modulation observables compared below cancel all overall normalizations by construction. We then measure the dipole, tee and tripole modulation curves at the same shape settings as the leading-order curves (Figures 3, 4 and 5). The error bars on the measured curves combine 1–4% per-bin statistical errors with a finite-search-window systematic, and we assign no other experimental systematic uncertainty. The shapes are matched in data only within a finite window of nearby configurations, and the systematic is the shift in the modulation when that window is halved.

7.2 Comparison with the CMS curves

The theory is per-flavor while the data are a quark/gluon mixture. We show three curves on every figure: pure quark, pure gluon, and a mixture with quark fraction $f_q = 0.60 \pm 0.05$, the

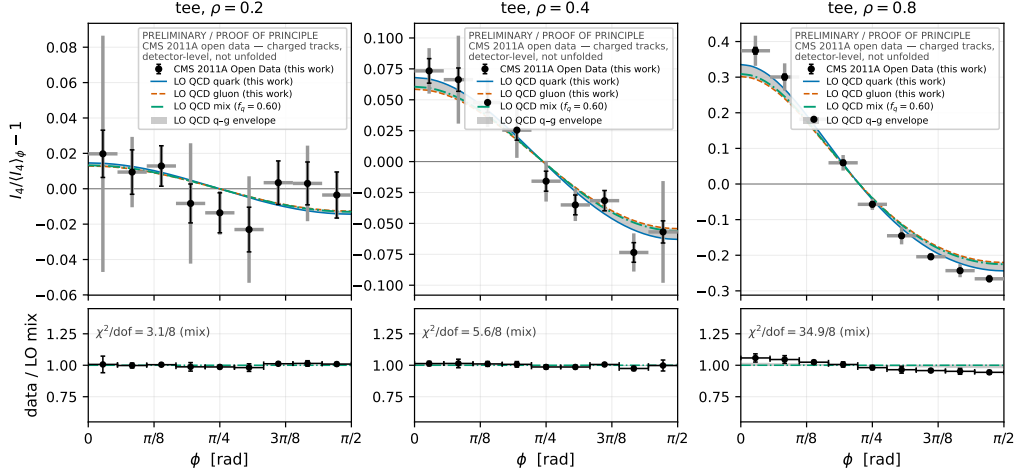


Figure 4: The tee modulation, the best-described of the three projections, with no window-regulated bins (χ^2/dof in Table 3). Symbols and bands as in Fig. 3.

	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.8$
dipole	0.70	5.39	11.93
tee	0.38	0.69	4.36
tripole	–	9.24	5.07

Table 3: The shapes and peak positions follow leading-order QCD. The residuals are at the percent level, growing with ρ as the configurations approach the jet boundary. Entries are χ^2/dof of the CMS open-data modulation curves against the LO QCD mixture ($f_q = 0.60$), with the full bootstrap covariance and window systematics, with $\text{dof} = 8$ for dipole and tee and 11 for tripole. The tripole grid is $\rho = 0.4, 0.8$ and 1.2: the $\rho = 0.2$ setting was not extracted, and the $\rho = 1.2$ setting is excluded on the geometric criterion described in the text, leaving the eight kept curves. No parameter was fit. Detector-level data.

standard value for central jets at $p_T \sim 500$ GeV at 7 TeV [21]. We do not fit the fraction, and the comparison is insensitive to its precise value because the leading-order (LO) modulation is nearly flavor-blind, with quark and gluon curves differing by less than 5% of the modulation at every setting compared here.

Table 3 collects the χ^2 values. For all eight curves in the table the measured modulation has the shape and the peak position of the leading-order prediction. The residuals in the ratio panels are at the few-percent level at $\rho = 0.2$ and at the ten-to-twenty-percent level at $\rho = 0.4$ and 0.8, apart from the window-regulated bins. At $\rho = 0.8$ most bins of the dipole modulation and the peak bin of the tripole modulation are window-regulated (the window-halving systematic exceeds three times the halved-sample statistical error) and drawn with open symbols. With detector-level data and no unfolding, the shape agreement carries more weight than the χ^2 values.

We exclude the tripole $\rho = 1.2$ setting on a geometric criterion: there the configuration's maximum pairwise opening is $2.2R$, where R is the large-pair separation that sets the overall scale of the configuration. As R ranges over the extraction's finite search window, the opening

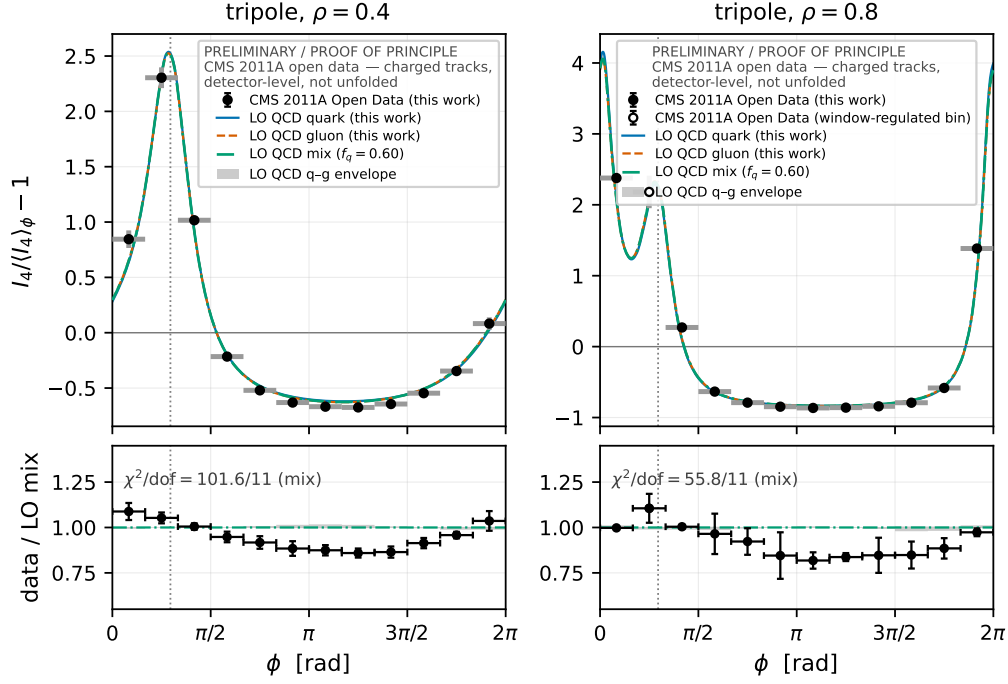


Figure 5: The tripole modulation, the most structured of the three projections: the peak position and the full non-sinusoidal shape of the data follow the LO QCD curve, with residuals of ten to twenty percent visible in the ratio panels (χ^2/dof in Table 3). Symbols and bands as in Fig. 3. The dotted vertical line marks the azimuth $\phi = \arctan(4/3)$ at which the rotated fourth direction becomes collinear with the second fixed direction of the tripole. The tripole grid is $\rho = 0.4, 0.8$ and 1.2 ($\rho = 0.2$ was not extracted for this projection). The $\rho = 1.2$ setting is not shown: its configurations extend beyond the jet cone and the comparison would measure detector acceptance rather than QCD.

spans 0.44–0.66, straddling the AK5 radius parameter 0.5. The LO modulation is scale-free (independent of R at fixed ρ), so a genuine QCD modulation cannot depend on which half of the R window it is measured in. In the data, however, the modulation in the four bins whose opening exceeds the jet radius shifts coherently between the two halves of the window, by 0.028–0.060 at significances reaching 8.3σ , while the $\rho = 0.4$ control shows no shift beyond 2σ . We therefore attribute this R dependence of the $\rho = 1.2$ modulation to jet-boundary sculpting, a detector acceptance effect. The excluded curve in fact agrees with the prediction better ($\chi^2/\text{dof} = 5.22$) than the retained $\rho = 0.4$ tripole curve (9.24).

7.3 Comparison with DELPHI

We compare the wide-angle prediction with archival e^+e^- data. From the public DELPHI 1994 and 1995 datasets we select 1.8 million hadronic Z events at detector level, of which 1.6 million lie in the Z -peak window used below. The two-point EEC computed from the 1994 events agrees with the published measurement on the same dataset [2]. We then scan fixed-shape four-point windows at three largest-pair-angle targets, $\chi_L \approx 1.0, 1.5$ and 2.0 . Five of the six pairwise angles are held within ± 0.05 rad of the target configuration while the largest angle is scanned

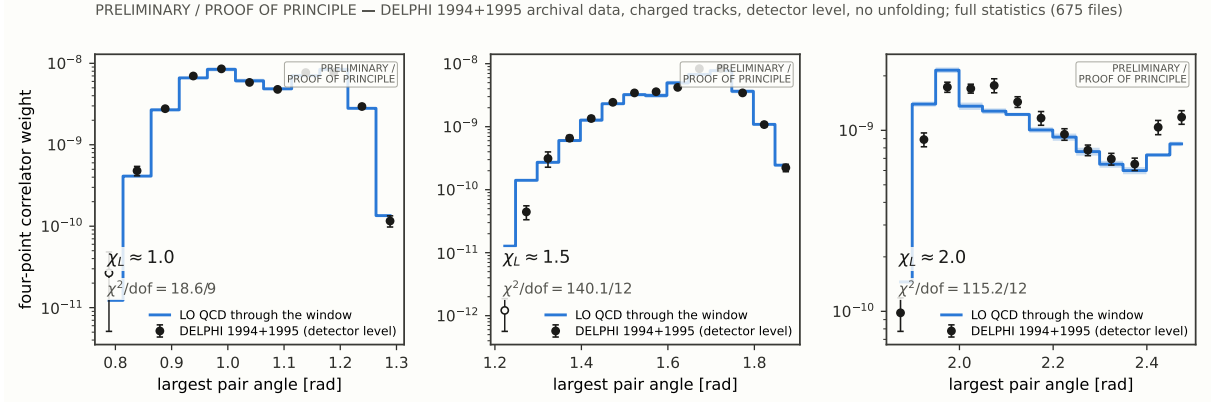


Figure 6: LO QCD follows the shape of the DELPHI four-point data. Fixed-shape slice scans of the DELPHI 1994 and 1995 detector-level data at largest pair angle $\chi_L \approx 1.0$, 1.5 and 2.0 , against the wide-angle LO QCD prediction of this paper sampled through the same windows with per-target normalization. Filled points enter the normalization fit. Open points, of relative error above 0.5 , are excluded from it. Shape $\chi^2/\text{dof} = 18.6/9$, $140.1/12$ and $115.2/12$. Detector-level data, from the full 1994 and 1995 sample.

in bins of 0.05 rad, and the LO QCD prediction is sampled through the same windows with the same sorted-angle cut. Every configuration in the three windows satisfies the hemisphere condition under which the leading-order prediction is finite (all four directions in a common hemisphere), including the $\chi_L \approx 2.0$ window, whose largest angles reach 2.5 rad. The data follow the predicted shape across two decades of correlator weight. The shape χ^2/dof is 2.06 , 11.7 and 9.6 at the three targets, and the per-target normalizations spread by a factor of 2.71 . Figure 6 shows the three windows.

At the two wider-angle targets the statistical errors are small enough that the residuals reflect detector effects and the difference between the parton-level leading-order prediction and the hadron-level data, while the smallest-angle target remains statistics-limited.

8 Outlook

On the slice we have assembled the $\mathcal{N} = 4$ generator integral $\mathcal{G}(u)$ from its master integrals, but have evaluated the correlator itself only by direct integration at the two physical points. The first open problem is therefore to evaluate the correlator from the master integrals on a physical one-parameter family of configurations. Doing so requires rebuilding the integrand from the form-factor square with the 1:1 weighting, deriving the weight vector, and assembling the result over the same master basis as $\mathcal{G}(u)$. Of the 443 cells of $\mathcal{G}(u)$, nine enter the assembly in analytic form, and the cells of two further sectors have analytic evaluations that we have not yet substituted (Section 5.3). The analytic evaluation of all other cells, which are known numerically, is open. For the elliptic sector, the eight masters are expressed in terms of the sector's periods together with anchor constants that are still left as integrals evaluated numerically. Reducing those anchor constants to closed form would complete the analytic solution of that sector. The full masters could also be evaluated by a domain-decomposed cubature, independent both of their representation over the sector's periods and of the direct quadratures of Section 5.2, but

that cubature is not yet complete. An origin blow-up $x_i = r\omega_i$, whose r^2 measure cancels the $1/r$ pole, cures the $1/D_5$ corner divergence, and the cubature then converges monotonically as the quadrature degree is increased. The interior logarithmic surface $D_7 = 0$ must still be rectified before the full masters can be integrated by such a cubature. It would also be interesting to compare the four-point elliptic surface of Eq. (8)–(9) with the equal-mass sunrise curve, the modular curve of $\Gamma_1(6)$, which underlies the elliptic sector of the two-point correlator at NNLO [14]. The monodromy of the four-point surface around its two singular fibers inside the unit interval remains to be computed.

For the genus-2 sector, Ref. [15] provides a semi-analytic differential-equation matrix. In the assembly of Section 5.3 the genus-2 integrals that enter $\mathcal{G}(u)$ are evaluated at five rational points only. Transporting the sector’s master integrals numerically with that matrix would give them along the whole slice, in particular at the two physical points. A closed form may need the Siegel-modular ε -factorization of Ref. [22]. Combined with the exact QCD weights that we computed in the quark-pair (n_f) channel, the transported values would complete the genus-2 part of the assembly in that channel. In QCD we have built neither the integrand basis of the $q\bar{q}ggg$ channel nor the integral layer of the wide-angle assembly.

In the collinear limit we have not computed the orders below the leading exponent at the higher-power signatures of the Landau analysis (Section 3.2), which cannot alter the leading-layer coefficients, nor, with the shape free, the pinch families of the gluon jet other than the conic–line tangency. The closed forms of Section 3.5 are for one one-parameter family of shapes; any other one-parameter family is within reach of the same method, whereas the correlator as a closed-form function of all four shape variables is not, its rational coefficients being about twice the degree of the $\mathcal{N} = 4$ ones and correspondingly larger. The value of the gluon-jet branch coefficient on the pinch sheet, which would test Table 1 against the closed form directly, requires the monodromy of the hyperlogarithms along a loop in t and is left open. We do not understand the origin of the quark-jet cancellation of Section 3.3: summed across the Cheng–Wu cells, the quark splitting numerators cancel at every tangency pinch and the pure-gluon numerators at none. The gluon-jet branch points belong to the analytic continuation of the observable, and it is an open question whether this difference between the quark-jet and gluon-jet correlators has any counterpart on the physical sheet. Once the $\mathcal{N} = 4$ correlator is assembled at generic angle, that assembly could be tested further by transporting it into the collinear limit and comparing it with the known collinear correlator [12]. An unfolded four-point measurement, from any experiment, would allow quantitative tests of the leading-order predictions in place of the detector-level comparisons of this paper.

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