Elliptic master integrals and the one-loop wavefunction of the universe

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Abstract

The one-loop triangle wavefunction coefficient of a conformally coupled scalar contains master integrals whose differential equations are solved by elliptic integrals. This suggested that such coefficients fail to be polylogarithmic already at one loop. We compute the coefficient in flat space and find that it is polylogarithmic. The coefficients in de Sitter space and in power-law cosmologies are integrals of it over the external energies. For all external energies the flat-space coefficient is a combination of dilogarithms, which we give in closed form. The periods of the elliptic curve cancel because the residue of the loop integrand at the poles of the elliptic sector is a flat-space amplitude, which is rational in the loop momentum. We obtain the function space from exact differential equations on two lines in kinematic space, and we determine the remaining rational constants numerically. The integrand of the corresponding correlator has no pole on the elliptic curve. The one-loop box and its correlator are not polylogarithmic, even with a single external leg at each site. The discontinuity of the box at threshold is a combination of elliptic integrals, which we give in closed form. For momenta that form a regular tetrahedron the periods of a K3 surface enter the loop integrand as well.

1 Introduction

Inflation predicts the correlation functions of the primordial density perturbations, and surveys of the cosmic microwave background and of the distribution of galaxies measure them. The measurements improve with every survey, but the calculations are less advanced. Beyond the simplest models and the lowest orders of perturbation theory, the integrals behind the predictions have resisted computation, and at loop level even the function space of the result is mostly unknown.

For scattering amplitudes the question of the function space has been productive. Feynman integrals are polylogarithms in the simplest cases, and integrals over elliptic curves or higher-dimensional varieties in others. Once the function space is known, an amplitude can often be constructed from its singularities and symmetries without evaluating any integral. In cosmology the wavefunction of the universe is the analog of the scattering amplitude. Its coefficients, one for each graph of perturbation theory, encode the correlations of the fields at the end of inflation, and equal-time correlators follow from them. Over the past decade the computation of these coefficients has adopted methods from amplitudes, including canonical differential equations, integration-by-parts reduction and twisted cohomology. The cosmological bootstrap [1, 2] aims to construct cosmological correlators from their singularities in the same way. It requires the same input, which is the function space of the result.

At tree level the function space is known for conformally coupled scalars in de Sitter space and in power-law cosmologies. The flat-space wavefunction coefficient is the canonical form of a cosmological polytope [3]. It is a rational function with simple poles where a partial energy vanishes. The wavefunction coefficients in de Sitter space are polylogarithms. In power-law cosmologies they are integrals of the same form with non-integer powers of the energies. They satisfy differential equations of a universal combinatorial form, the kinematic flow [4–8].

The vertices of a graph are called sites in this literature. We call the one-loop graphs with three and four sites the triangle and the box, and the one-loop graph with N sites a polygon. At one loop the graph with two sites is polylogarithmic [9–11]. Ref. [9] expressed the one-loop triangle through a basis of integrals, which are called master integrals, and derived their differential equations. A subset of the master integrals obeys an irreducible second-order equation whose solutions are elliptic integrals. The authors concluded that the triangle wavefunction coefficient involves elliptic functions, and that this "is likely to extend to all polygon graphs with a higher number of sites". They noted that an explicit computation was needed to classify the functions that arise. Cosmological loops would then differ from flat-space amplitudes, where elliptic integrals first appear at two loops [12–14].

The conclusion of ref. [9] rests on the master integrals, and an observable can be simpler than the integrals used to compute it. Three earlier results are of this kind. The one-loop four-point correlator of a conformally coupled scalar with a quartic interaction in de Sitter space, computed in position space, is expressed in terms of single-valued polylogarithms, because an elliptic sector that is present in anti-de Sitter space cancels [15]. The triangle correlator in flat space is a combination of dilogarithms, while the integrals that appear only in the wavefunction were expected to be hyperelliptic [16]. The one-loop triangle correlator of conformally coupled scalars in de Sitter space has been evaluated in dilogarithms [17].

In this paper we compute the wavefunction coefficient of the triangle and examine the conclusion of ref. [9]. The conclusion holds for the master integrals and not for the wavefunction coefficient. The coefficient is a polylogarithmic function of transcendental weight two. With one external leg at each site it is the combination of three dilogarithms in eq. (32). With several legs per site, which is the case that expanding backgrounds require, it is given by eq. (34). The arguments of its dilogarithms contain ten square roots, all of which become rational with one leg per site. On two lines in kinematic space we derive exact differential operators for the coefficient. They factor completely into first-order operators, which determines the function space. We then determine the remaining rational constants from numerical values of the loop integral (section 4). Our method thus combines exact information on the function space with numerical values of the integral, as is standard for multiloop amplitudes [18]. It is complementary to the cosmological bootstrap at loop level [19–23] and to the non-perturbative de Sitter bootstrap [24], which constrain correlators through symmetries, singularities and unitarity.

The periods of the elliptic curve cancel because of a property of the residues of the loop integrand. We derive the elliptic curve of the triangle from the loop measure and give its periods (section 3). The curve lies on three poles of the loop integrand. At these poles the residue of the loop integrand is a flat-space amplitude, which is rational in the loop momentum, so no residue of the wavefunction integrand involves an elliptic curve (proposition 1). The maximal cuts of the master integrals therefore suggest a larger function space than that of the wavefunction coefficient.

The integrand of the triangle correlator has no pole on the elliptic curve. The three poles come from the term of the propagator that imposes the boundary condition of the wavefunction at late times. They are known to be absent from cosmological correlators [25, 26], and ref. [9] asked whether correlators then require elliptic functions. In section 5 we show that the integrand of the triangle correlator has no residue on a line where the loop measure defines an elliptic curve. For

the box and the pentagon the same holds in $d = N$ dimensions, and it does not hold for the box in $d = 3$.

The one-loop box is not polylogarithmic, even with one leg per site, because the residue of its loop integrand at the pole of a single site has two square roots. We give its discontinuity at threshold in closed form in terms of elliptic integrals. For momenta that form a regular tetrahedron the periods of a K3 surface enter the loop integrand as well. On one line in kinematic space we determine the box and its correlator from an exact differential system (section 6).

Section 2 defines the loop integrand, the loop measure and the kinematics, and section 7 discusses open questions. Appendix A gives the construction of the closed forms, and appendix B gives the derivations for the box.

2 Setup

In this section we define the wavefunction coefficient V of the one-loop triangle. We first recall how the wavefunction coefficients of a conformally coupled scalar in an expanding background follow from those of flat space. We then give the loop integrand of the triangle, the loop measure and the external kinematics.

2.1 Wavefunction coefficients and correlators

The late-time state of the fields is described by the wavefunction $\Psi[\phi]$. It is the probability amplitude for the field to have the spatial profile $\phi_{\vec{k}}$ at conformal time $\eta_0 \to 0$, when the field is in the Bunch–Davies vacuum at early times. Its logarithm is expanded in powers of the profile,

$$\ln \Psi[\phi] \;=\; -\sum_n \frac{1}{n!} \int \prod_{i=1}^{n} \frac{\mathrm{d}^3 k_i}{(2\pi)^3}\; \psi_n(\vec{k}_1, \ldots, \vec{k}_n)\, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n}\, (2\pi)^3 \delta^3\Big(\textstyle\sum_i \vec{k}_i\Big)\,, \tag{1}$$

and the functions ψ_n are the wavefunction coefficients. Equal-time correlators, which the literature calls cosmological correlators, follow from the Born rule,

$$\langle \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n} \rangle \;=\; \frac{\int \mathcal{D}\phi\; |\Psi[\phi]|^2\, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n}}{\int \mathcal{D}\phi\; |\Psi[\phi]|^2}\,. \tag{2}$$

Wavefunction coefficients and correlators contain the same information, but they are different functions of the momenta. Sections 2 to 4 concern a wavefunction coefficient. Section 5 shows that the integrand of the corresponding correlator has fewer singularities.

A wavefunction coefficient in an expanding background is an integral of the flat-space wavefunction coefficient over energies. To show this, we consider a conformally coupled scalar with polynomial self-interactions in a spatially flat Friedmann–Robertson–Walker (FRW) background. After the field is rescaled by the scale factor, its free action is that of a massless field in flat space. The scale factor then enters only through a factor $(-\eta)^{-(1+\sigma)}$ at each interaction vertex, where the exponent σ depends on the background and on the interaction. For a cubic interaction in de Sitter space we have $\sigma = 0$. We call the magnitude of the momentum of a leg its energy. After the Wick rotation $\eta = \mathrm{i}t$ with $t > 0$, an external leg of energy X contributes the bulk-to-boundary propagator e^{-Xt}, and the vertex factor is $t^{-(1+\sigma)}$ up to a phase. Because

$$t^{-(1+\sigma)} \;=\; \frac{1}{\Gamma(1+\sigma)} \int_0^\infty \mathrm{d}\omega\; \omega^\sigma\, e^{-\omega t}\,, \tag{3}$$

the factor $e^{-\omega t}$ multiplies e^{-Xt} and replaces the energy X at that vertex by $X+\omega$ [3, 7]. A coefficient in the expanding background is therefore an integral of the flat-space coefficient over shifted energies,

$$\psi_n^{(\sigma)}(X;y) \;\propto\; \int_0^\infty \prod_v \mathrm{d}\omega_v\, \omega_v^\sigma \;\psi_n(X+\omega;y)\,, \tag{4}$$

where ψ_n without a superscript denotes the flat-space wavefunction coefficient. We call it the flat-space coefficient for short. Here X_v is the total energy of the external legs at site v, and y_e is the energy of the internal edge e. The literature also calls them external and internal energies. We call the integrals in eq. (4) the energy integrals. In flat space the vertex factor is absent, so the flat-space coefficient is the integrand of the energy integrals and not the value of $\psi_n^{(\sigma)}$ at some σ. Every coefficient in a background of this type follows from the flat-space coefficient, and sections 3 and 4 study it directly. We return to the energy integrals for de Sitter space in section 4.6.

2.2 The loop integrand

The flat-space coefficients are integrals over the times $t_v > 0$. Each site contributes $e^{-X_v t_v}$, and each internal edge contributes the bulk-to-bulk propagator, which vanishes at the late-time boundary,

$$G_y(t,t') \;=\; \frac{1}{2y}\Big[e^{-y|t-t'|} - e^{-y(t+t')}\Big]\,. \tag{5}$$

We omit the coupling constants and an overall phase, so that every ψ_n below is a positive rational function of the energies. The first term in eq. (5) is the time-ordered propagator of flat space. The second term is fixed by the boundary condition at $t=0$, and it is a function of t multiplied by a function of t'. This second term has no counterpart in a scattering amplitude, and it is the origin of the elliptic curve of section 3.

At tree level every factor of the denominator is a partial energy, which we now define. From here on the subscript of ψ counts the sites of the graph. One site with no edges gives $\psi_1 = 1/X$, and two sites joined by an edge give

$$\psi_2 \;=\; \int_0^\infty \mathrm{d}t_1\,\mathrm{d}t_2\; e^{-X_1t_1-X_2t_2}\, G_y(t_1,t_2) \;=\; \frac{1}{(X_1+X_2)(X_1+y)(X_2+y)}\,. \tag{6}$$

Each factor of the denominator is the sum of the X_v of a connected set of sites and of the y_e of the edges that join the set to the other sites. The literature calls this sum a partial energy. The denominator of every tree graph consists of partial energies [3]. Energy is not conserved because the boundary breaks time translations, and the coefficient is singular where a partial energy vanishes.

We now turn to the one-loop triangle, the graph with three sites and three edges shown in figure 1(a). Its sites have energies X_1, X_2, X_3, and the edge between sites v and w has energy y_{vw}. The time integrals give

$$\psi_3(X;y) \;=\; \frac{2\mathcal{N}(X;y)}{q_{\mathcal{G}}\,\prod_{i=1}^9 q_i(X;y)}\,, \qquad q_{\mathcal{G}} = X_1+X_2+X_3\,, \tag{7}$$

with ten linear factors in the denominator. Besides the total energy $q_{\mathcal{G}}$ they are the nine factors $q_1,\dots,q_9$, which we list row by row,

$$\begin{aligned}
&X_1+y_{12}+y_{31}\,, \quad X_2+y_{12}+y_{23}\,, \quad X_3+y_{23}+y_{31}\,,\\
&X_1+X_2+y_{23}+y_{31}\,, \quad X_2+X_3+y_{12}+y_{31}\,, \quad X_1+X_3+y_{12}+y_{23}\,,\\
&q_{\mathcal{G}}+2y_{12}\,, \quad q_{\mathcal{G}}+2y_{23}\,, \quad q_{\mathcal{G}}+2y_{31}\,.
\end{aligned} \tag{8}$$

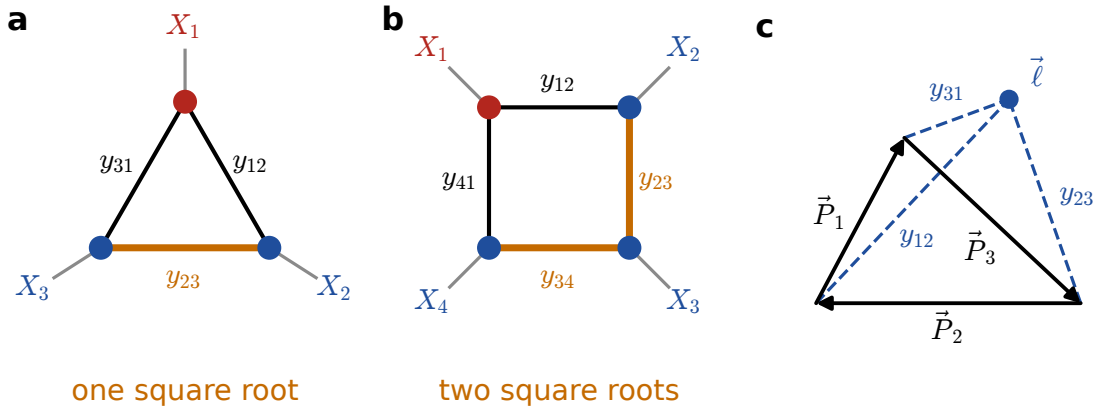

Figure 1: (a) The one-loop triangle and (b) the one-loop box. Each site has the energy X_v of its external legs, and each edge has the energy y_e. At the single-site pole of site 1, marked in red, the residue of the loop integrand is even in the energies of the black edges and contains the square root of a quadratic polynomial for each orange edge (sections 4.1 and 6.1). (c) The loop measure of the triangle. The external momenta form a triangle with sides P_v, and the edge energies are the distances from the loop momentum $\vec{\ell}$ to its corners. In general $\vec{\ell}$ does not lie in the plane of the triangle.

The numerator $\mathcal{N}$ is an irreducible homogeneous quartic polynomial with positive coefficients, which we give in the ancillary files. We call $\psi_3(X;y)$ the loop integrand. It follows from a recursion that cuts one edge at a time [3],

$$\begin{aligned} q_{\mathcal{G}}\,\psi_3(X;y) \;=\; & \psi_{\text{chain}}\big(X_2+y_{12}, X_3, X_1+y_{12}; y_{23}, y_{31}\big) \\ & + \psi_{\text{chain}}\big(X_3+y_{23}, X_1, X_2+y_{23}; y_{31}, y_{12}\big) \\ & + \psi_{\text{chain}}\big(X_1+y_{31}, X_2, X_3+y_{31}; y_{12}, y_{23}\big)\,. \end{aligned} \tag{9}$$

Each term is the coefficient of the chain in which one edge e is cut, with y_e added to the energies of the two sites at its ends. The coefficient of the chain of three sites with energies x_1, x_2, x_3 and edges y, y' is

$$\psi_{\text{chain}}(x_1,x_2,x_3;y,y') \;=\; \frac{(x_1+x_2+y')^{-1}+(x_2+x_3+y)^{-1}}{(x_1+x_2+x_3)(x_1+y)(x_2+y+y')(x_3+y')}\,. \tag{10}$$

The total energy and the first two rows of eq. (8) are partial energies, of the whole graph, of a single site, and of a pair of sites. The factors in the last row are not. The factor $q_{\mathcal{G}}+2y_e$ contains the energy of the edge e twice. It is the total energy of the chain in which e is cut and its two ends are external legs. It arises from the second term of the propagator (5). Every elliptic curve that we find in the triangle involves one of the three factors $q_{\mathcal{G}}+2y_e$. These factors are absent from the correlator integrand, as we show in section 5.

2.3 The loop measure

The edge energies depend on the loop momentum $\vec{\ell}$ through $y_{12}=|\vec{\ell}|$, $y_{23}=|\vec{\ell}+\vec{P}_2|$ and $y_{31}=|\vec{\ell}-\vec{P}_1|$, where $\vec{P}_v$ is the total momentum entering site v and $\vec{P}_1+\vec{P}_2+\vec{P}_3=0$. The integrand depends on $\vec{\ell}$ only through these three lengths, so we use them as integration variables. The three

edge energies are the distances from the point $\vec{\ell}$ to the corners of the triangle formed by the external momenta (figure 1(c)). In three dimensions three distances fix a point up to reflection through the plane of the triangle. For each of the two points the Jacobian is $y_{12}\,y_{23}\,y_{31}/(6\,\mathrm{Vol})$, where Vol is the volume of the tetrahedron with apex $\vec{\ell}$. With $288\,\mathrm{Vol}^2 = B$ and the sum over the two points this gives

$$\mathrm{d}^3\ell \;=\; 4\sqrt{2}\;B(y;P)^{-1/2}\;y_{12}\,y_{23}\,y_{31}\;\mathrm{d}y_{12}\,\mathrm{d}y_{23}\,\mathrm{d}y_{31}\,, \tag{11}$$

where B is the Cayley–Menger determinant of that tetrahedron,

$$B(y;P) \;=\; \det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & P_2^2 & P_1^2 & y_{12}^2 \\ 1 & P_2^2 & 0 & P_3^2 & y_{23}^2 \\ 1 & P_1^2 & P_3^2 & 0 & y_{31}^2 \\ 1 & y_{12}^2 & y_{23}^2 & y_{31}^2 & 0 \end{pmatrix}, \tag{12}$$

and $P_v = |\vec{P}_v|$. The integration region is $B > 0$, where the tetrahedron exists. The polynomial B is quadratic in each y_e^2. This change of variables is the Baikov representation of the Feynman-integral literature, in which B is the Baikov polynomial. Ref. [9] studied the triangle in this representation.

The object of this paper is the integral of the loop integrand over the loop momentum, which is the one-loop wavefunction coefficient of the triangle. We normalize it as

$$V(X;P) \;=\; \int_{B>0} B(y;P)^{-1/2+\varepsilon}\;y_{12}\,y_{23}\,y_{31}\;q_{\mathcal{G}}\,\psi_3(X;y)\;\mathrm{d}y_{12}\,\mathrm{d}y_{23}\,\mathrm{d}y_{31} \;\xrightarrow{\varepsilon\to 0}\; \frac{q_{\mathcal{G}}}{4\sqrt{2}}\int \mathrm{d}^3\ell\;\psi_3\,. \tag{13}$$

The factor $q_{\mathcal{G}}$ removes the pole at vanishing total energy. The loop integrand decreases as $|\vec{\ell}|^{-5}$ at large loop momentum, so V is finite in three dimensions. The exponent ε is the parameter of dimensional regularization, with $d = 3 + 2\varepsilon$. We need it because the individual master integrals of section 3 have poles at $\varepsilon = 0$, which cancel in V. The square root $B^{-1/2}$ is absent at tree level, where the integrands are rational. The elliptic curve of section 3 requires both this square root and the second term of eq. (5).

2.4 Kinematics

The coefficient V depends on the site energies X_v and on the momentum magnitudes P_v, which must satisfy the triangle inequalities. If a single external leg is attached to each site, its energy equals the magnitude of its momentum, so that

$$X_v \;=\; P_v\,, \qquad v = 1,2,3\,. \tag{14}$$

With several legs at a site, X_v exceeds P_v unless the legs are collinear. We refer to eq. (14) as the case of one leg per site.

On a line in kinematic space the coefficient depends on one variable and obeys an ordinary differential equation. Ref. [9] studies the differential equations along lines of the form $X = (\alpha\lambda, \lambda, 1)$. We take $\alpha = 2$ and study the line

$$X \;=\; P \;=\; (2\lambda,\,\lambda,\,1)\,, \qquad \tfrac{1}{3} < \lambda < 1\,, \tag{15}$$

where the range of λ is set by the triangle inequalities. The value of α has no special significance. As a control we also derived the second-order operator of section 3.3 at $\alpha = 3$, where it agrees with ref. [9] as well. We derive the differential equations of sections 3 and 4 on this line. The case of one leg per site does not suffice for expanding backgrounds. The energy integrals of eq. (4) shift X_v at fixed P_v, so every coefficient in an expanding background requires V at $X_v > P_v$. We treat general site energies in section 4.5.

3 The elliptic sector

In this section we describe the elliptic curve of the triangle and the integrals in which it appears. We first express V through a finite set of master integrals. We then derive the curve, its periods and their differential operator, and we describe the master integrals. The section shows that three sectors of master integrals are elliptic. Whether V itself is elliptic is a separate question, which we answer in section 4.

3.1 Master integrals

The integrand of eq. (13) is a polynomial divided by a product of linear factors, multiplied by a power of B. We call it the wavefunction integrand. The integral of eq. (13) belongs to the family

$$I[\mathcal{Q};n] \;=\; \int_{B>0} B(y;P)^{-1/2+\varepsilon}\,\frac{\mathcal{Q}(y)}{\prod_{i=1}^{9} q_i(X;y)^{n_i}}\,\mathrm{d}^3y\,, \tag{16}$$

where $\mathcal{Q}$ is a polynomial, the q_i are the nine factors of eq. (8), and the n_i are non-negative integers. In dimensional regularization the integral of a total derivative over the region $B > 0$ vanishes. The resulting integration-by-parts identities relate the members of the family linearly, with coefficients that are rational functions of the kinematics and of ε. As for Feynman integrals, the family is spanned by a finite number of master integrals. We group them into sectors according to which of the factors q_i appear in the denominator. The derivative of a master integral with respect to λ is a combination of master integrals from its own sector and from sectors with fewer denominators. When we drop the sectors with fewer denominators, we obtain the homogeneous differential equation of the sector. Its solutions determine the function space of the sector.

The sectors built only from the first six factors of eq. (8) produce polylogarithms on the line (15). The sectors with a factor $q_{\mathcal{G}}+2y_e$ do not. Ref. [9] found that the sector with the single denominator $q_{\mathcal{G}}+2y_{12}$ contains master integrals whose homogeneous equation is irreducible and of second order. The same holds for $q_{\mathcal{G}}+2y_{23}$ and $q_{\mathcal{G}}+2y_{31}$ after a permutation of the sites. We call these three sectors the elliptic sectors. Sections 3.2 to 3.4 treat the sector with denominator $q_{\mathcal{G}}+2y_{12}$.

3.2 The curve

The elliptic curve follows from the loop measure. An integrand that contains the square root of a polynomial $p(x)$ is a differential on the curve $w^2 = p(x)$. If p has degree two, the curve has genus zero, and the integral is a logarithm. If p has degree three or four with distinct roots, the curve is elliptic, and the integrals of its holomorphic differential over closed paths are its periods.

The residue of the integrand at the pole of a sector obeys the homogeneous equation of that sector. In the sector with denominator $q_{\mathcal{G}}+2y_{12}$ we therefore take the residue at $y_{12} = -q_{\mathcal{G}}/2$, which is the maximal cut of the sector. It leaves an integral over y_{23} and y_{31} whose integrand depends on these variables only through the square root of B. In terms of $z_e = y_e^2$ the polynomial B is quadratic. Along the ray $z_{31} = r\,z_{23}$ its leading behavior at large z_{23} is

$$B \;=\; -A(r)\,z_{23}^2 + O(z_{23})\,, \qquad A(r) \;=\; 2\big[P_2^2\,r^2 - \big(P_1^2+P_2^2-P_3^2\big)\,r + P_1^2\big]\,, \tag{17}$$

and $A(r)$ does not depend on y_{12}. If the rest of the integrand were a rational function of r, the square root of the quadratic $A(r)$ would define a curve of genus zero. However, the integrands of the elliptic sectors are not functions of the squares y_e^2 alone, although B is. A master integral with

unit numerator has the measure $\mathrm{d}y_{23}\,\mathrm{d}y_{31}$ and not $\mathrm{d}z_{23}\,\mathrm{d}z_{31}$. With $\mathrm{d}y_e = \mathrm{d}z_e/(2\sqrt{z_e})$ the measure at large z_{23} becomes

$$B^{-1/2}\,\mathrm{d}y_{23}\,\mathrm{d}y_{31} \;\longrightarrow\; \frac{\mathrm{d}z_{23}}{z_{23}}\,\frac{\mathrm{d}r}{4\sqrt{-r\,A(r)}}\,, \qquad \frac{\mathrm{d}r}{\sqrt{r\,A(r)}} \;=\; \frac{2\,\mathrm{d}\tau}{\sqrt{A(\tau^2)}}\,, \qquad \tau = \frac{y_{31}}{y_{23}}\,. \tag{18}$$

The residue at large z_{23} is a differential in τ. The square root of the quartic $A(\tau^2)$ defines the elliptic curve

$$w^2 \;=\; P_2^2\,\tau^4 - \big(P_1^2 + P_2^2 - P_3^2\big)\,\tau^2 + P_1^2\,. \tag{19}$$

The discriminant of $A(r)$ is proportional to the Källén function of the P_v^2, so the four branch points are distinct away from the folded configurations, in which the three momenta are collinear.

The curve depends on the momenta P_v only, and not on the site energies X_v. It lies at infinity on the plane $q_{\mathcal{G}} + 2y_{12} = 0$, where y_{23} and y_{31} are large at fixed y_{12}. This is not a region of real loop momentum. An integral therefore depends on the curve only through its integrand at large y_{23} and y_{31} on that plane.

On the line (15) the curve is isomorphic to the curve $w^2 = (1-x^2)(1-m\,x^2)$ with the parameter

$$m(\lambda) \;=\; \frac{\lambda^2 - 1}{9\lambda^2 - 1}\,, \tag{20}$$

which is the parameter found in ref. [9]. The sectors with denominators $q_{\mathcal{G}} + 2y_{23}$ and $q_{\mathcal{G}} + 2y_{31}$ have curves of the same form with the momenta permuted. The three curves are not isomorphic to one another.

3.3 Periods and their differential equation

The homogeneous equation of the elliptic sector is the differential equation of the periods, whose operator is called the Picard–Fuchs operator. On the line (15) it is [9]

$$\mathcal{L}_2 \;=\; \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} + \frac{45\lambda^4 - 30\lambda^2 + 1}{\lambda\,(\lambda^2-1)(9\lambda^2-1)}\,\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{27\lambda^2 - 10}{(\lambda^2-1)(9\lambda^2-1)}\,. \tag{21}$$

Its solutions are complete elliptic integrals of the first kind, $K(m) = \int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta$, with an algebraic prefactor. On the physical interval $\frac{1}{3} < \lambda < 1$ the two periods are

$$\varpi_0(\lambda) \;=\; \frac{1}{\lambda}\,K\Big(\frac{1-\lambda^2}{8\lambda^2}\Big)\,, \qquad \varpi_1(\lambda) \;=\; \frac{1}{\lambda}\,K\Big(\frac{9\lambda^2-1}{8\lambda^2}\Big)\,. \tag{22}$$

The argument of the first is $m/(m-1)$ with the parameter $m(\lambda)$ of eq. (20). The first period is analytic at $\lambda = 1$ and has a logarithmic singularity at $\lambda = \frac{1}{3}$, and the second has the opposite behavior. The operator $\mathcal{L}_2$ has regular singular points at $\lambda = 0, \pm\frac{1}{3}, \pm 1$ and ∞. At each finite singular point the two local exponents coincide, so that one solution is analytic there and the other has a logarithm. Both ends of the physical interval $\frac{1}{3} < \lambda < 1$ are singular points of this kind. We use this fact in section 4, because a function that contains the periods (22) has their logarithmic singularities.

The operator $\mathcal{L}_2$ is the same for the denominators $q_{\mathcal{G}} + 2y_{12}$ and $q_{\mathcal{G}} + y_{12}$, because the curve (19) does not depend on the position of the pole in y_{12}. Ref. [9] writes $q_{\mathcal{G}} + y_{12}$ in its section on the triangle. The time integrals of section 2.2 give $q_{\mathcal{G}} + 2y_{12}$, which we use.

3.4 The master integrals of the elliptic sector

The master integrals of the elliptic sector can be computed with the methods developed for elliptic Feynman integrals [27–29]. On the line (15) the master integrals of the sector with denominator $q_{\mathcal{G}}+2y_{12}$ couple to master integrals without any linear denominator. At each order in ε the solutions are iterated integrals. Their integration kernels contain rational functions of λ, the periods (22) and algebraic functions, and they involve the single elliptic curve (19). We do not display these solutions, because the periods cancel in the wavefunction coefficient.

The master integrals of the three elliptic sectors are therefore elliptic. Ref. [9] concluded from this that the triangle coefficient involves elliptic functions. In the next section we show that it does not.

4 The triangle wavefunction coefficient

In this section we show that the periods of the elliptic curve of section 3 do not appear in the coefficient V. We first show that the residue of the wavefunction integrand on the factors $q_{\mathcal{G}}+2y_e$ is a flat-space amplitude, so that no residue of the integrand involves an elliptic curve (proposition 1). We then derive a differential equation for V on the line (15) and show that its operator factors completely into first-order operators. In section 4.4 we solve the equation and obtain V in closed form with one leg per site. In section 4.5 we extend the result to general site energies, and in section 4.6 we comment on de Sitter space.

4.1 Why the periods cancel

At the pole $q_{\mathcal{G}} + 2y_{12} = 0$ the residue of the loop integrand is a flat-space amplitude, whereas the residue of a master integral contains the holomorphic differential $\mathrm{d}\tau/w$ of the curve (19). In the recursion (9) only the chain in which the edge y_{12} is cut has this pole, which is its total-energy pole. The residue of a tree-level coefficient at its total-energy pole is the flat-space amplitude [3]. The residue of the loop integrand (7) is therefore

$$\operatorname*{Res}_{y_{12}=-q_{\mathcal{G}}/2} \psi_3 \;=\; \frac{1}{2\,q_{\mathcal{G}}\,\big(y_{23}^2-E_2^2\big)\big(y_{31}^2-E_1^2\big)}\,, \qquad E_1=\tfrac{1}{2}(X_2+X_3-X_1)\,, \quad E_2=\tfrac{1}{2}(X_1+X_3-X_2)\,. \tag{23}$$

Equation (23) is a product of two Feynman propagators with energies E_1 and E_2. It depends on the edge energies y_{23} and y_{31} only through their squares, which are polynomials in the components of $\vec{\ell}$. This property underlies proposition 1. To state the proposition, we write

$$\mu \;=\; q_{\mathcal{G}}\,\psi_3\; y_{12}\,y_{23}\,y_{31}\;\mathrm{d}y_{12}\,\mathrm{d}y_{23}\,\mathrm{d}y_{31} \tag{24}$$

for the rational part of the integrand of eq. (13), which has poles on the nine planes (8).

Proposition 1. *For arbitrary X_v and generic P_v, no residue of the wavefunction integrand on a plane or on a line involves an elliptic curve. More precisely, the following statements hold.*

1. *The residue of μ on the plane $q_{\mathcal{G}} + 2y_{12} = 0$ is*

$$-\frac{q_{\mathcal{G}}}{16}\,\frac{\mathrm{d}z_{23}\,\mathrm{d}z_{31}}{\big(z_{23}-E_2^2\big)\big(z_{31}-E_1^2\big)}\,, \qquad z_e=y_e^2\,, \tag{25}$$

and the polynomial B on this plane is of total degree two in z_{23} and z_{31}. The same holds on the other two planes $q_{\mathcal{G}} + 2y_e = 0$.

2. *The nine planes meet in pairs along 36 lines. Three of the lines lie at infinity. On 12 lines B is a quadratic polynomial in the coordinate along the line. On the remaining 21 lines B is a quartic polynomial with distinct roots, so that $w^2 = B$ is an elliptic curve. Each of these 21 lines lies in one of the planes $q_{\mathcal{G}} + 2y_e = 0$.*

3. *On 9 of the 21 lines the residue of μ vanishes. On each of the other 12 a single edge energy y varies along the line, and $B = b(y^2)$ with b a quadratic polynomial. The residue of $\mu/\sqrt{B}$ is*

$$\pm\frac{q_{\mathcal{G}}}{16}\,\frac{\mathrm{d}Z}{\big(Z-E^2\big)\sqrt{b(Z)}}\,,\qquad Z=y^2\,, \tag{26}$$

with E one of the energies E_i of the corresponding plane. The same is true if $B^{-1/2}$ is replaced by $B^{-1/2+\varepsilon}$.

The form (26) is a differential on the curve $w^2 = b(Z)$, which has genus zero. The wavefunction integrand therefore has no residue, on any plane or line, that involves one of the 21 elliptic curves. A master integral with a constant numerator and the same two denominators has the residue $\mathrm{d}y/\sqrt{b(y^2)}$, which is the holomorphic differential of the elliptic curve. The curve (19) of section 3 lies at infinity on the plane $q_{\mathcal{G}} + 2y_{12} = 0$, and the same argument applies to it. The form (25) depends on y_{23} and y_{31} only through z_{23} and z_{31}. On the ray $z_{31} = r\,z_{23}$ its product with $B^{-1/2}$ is therefore, at large z_{23}, a rational function of r and z_{23} divided by $\sqrt{A(r)}$, which is a differential on a curve of genus zero.

Proof. Part 1 is eq. (23) multiplied by the measure, with $y_{23}\,y_{31}\,\mathrm{d}y_{23}\,\mathrm{d}y_{31} = \frac{1}{4}\,\mathrm{d}z_{23}\,\mathrm{d}z_{31}$, and the statement about B follows from eq. (12). Part 2 follows from the restriction of eq. (12) to each of the 36 lines. The 12 lines with a quadratic B are those on which two partial energies vanish, and all three edge energies vary along them.

Part 3 follows from part 1. The form (25) has poles only at $y_{23} = \pm E_2$ and $y_{31} = \pm E_1$. On the plane $q_{\mathcal{G}} + 2y_{12} = 0$ these are the lines where the energy of site 1, of site 2, or of the pairs of sites $\{2,3\}$ or $\{1,3\}$ vanishes. This gives four lines for each of the three planes. The other planes cross $q_{\mathcal{G}} + 2y_{12} = 0$ along lines where the form (25) is regular. □

In the individual master integrals the holomorphic differential appears, and the contributions cancel in the sum. The decomposition of the residue (23) into partial fractions is

$$\frac{1}{\big(y_{23}^2-E_2^2\big)\big(y_{31}^2-E_1^2\big)} \;=\; \frac{1}{4E_1E_2}\sum_{s,s'=\pm1}\frac{s\,s'}{(y_{23}-s\,E_2)(y_{31}-s'E_1)}\,. \tag{27}$$

Each of the four terms has a non-zero coefficient of the holomorphic differential $\mathrm{d}\tau/w$ of the curve (19). The four coefficients are equal in magnitude and have the signs $s\,s'$, so their sum vanishes. The periods of the elliptic curve therefore appear in the terms into which an integration-by-parts reduction separates V, and they cancel between these terms.

The partial energies of the sites require a separate argument, which also shows where the simplification ends. On the pole of a single site, which we call the single-site pole, the residue of the loop integrand is

$$\operatorname*{Res}_{X_1=-y_{31}-y_{12}}\psi_3 \;=\; \frac{1}{4\,y_{12}\,y_{31}}\sum_{s,s'=\pm1} s\,s'\;\psi_2\big(X_2+s\,y_{12}\,,\,X_3+s'\,y_{31}\,;\,y_{23}\big)\,, \tag{28}$$

with ψ_2 given in eq. (6). The residue is even in y_{12} and in y_{31}, the energies of the two edges that end on site 1, and it is not even in y_{23}. On the pole the loop momentum lies on a spheroid with

foci at the two points where y_{12} and y_{31} vanish, and y_{23} is the square root of a quadratic function on this surface. A single square root of a quadratic can be rationalized, so this residue does not define an elliptic curve. For the polygon with N sites the residue contains $N-2$ such square roots, one for each edge that does not end on the site (figure 1). The box has two, and they define an elliptic curve (section 6).

Proposition 1 holds for arbitrary X_v and generic P_v. It therefore applies with several legs per site, where many more sectors of master integrals contain an elliptic curve (section 4.5). It does not by itself determine the function space of V, because the integral over the region $B > 0$ is not a sum of residues. For the function space we turn to the differential equation.

4.2 The differential equation

We reduce the integrand of eq. (13) to master integrals with the integration-by-parts identities of the family (16), as described in appendix A.1. The result expresses V and its λ-derivatives through the same master integrals. Eliminating the master integrals gives a linear differential operator that annihilates V. At generic ε it has order 20. Its limit as $\varepsilon \to 0$,

$$\mathcal{L}_0 \;=\; \sum_{i=0}^{20} c_i(\lambda)\, \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i}\,, \qquad \mathcal{L}_0\, V \;=\; 0\,, \tag{29}$$

has polynomial coefficients c_i, and we give it in the ancillary files. Its singular points are $\lambda = 0, \pm\frac{1}{3}, \pm 1$ and ∞, which are the singular points of $\mathcal{L}_2$, together with apparent singularities at which all solutions are analytic.

The limit $\varepsilon \to 0$ must be taken after the reduction. The master integrals have poles at $\varepsilon = 0$, so terms of order ε in the reduction contribute to the finite part (appendix A.1).

4.3 Complete factorization

The operator $\mathcal{L}_0$ factors completely over the rational functions of λ,

$$\mathcal{L}_0 \;=\; c_{20}(\lambda) \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{r'_{20}}{r_{20}}\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{r'_{19}}{r_{19}}\Big) \cdots \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{r'_{1}}{r_{1}}\Big)\,, \tag{30}$$

where every r_k is a rational function of λ and a prime denotes $\mathrm{d}/\mathrm{d}\lambda$. We obtain the factorization by right division in the ring of differential operators with coefficients in $\mathbb{Q}(\lambda)$.

The solutions of a product of first-order operators follow from one integration for each factor,

$$V(\lambda) \;=\; r_1 \int \mathrm{d}\lambda\, \frac{r_2}{r_1} \int \mathrm{d}\lambda\, \frac{r_3}{r_2} \;\cdots \int \mathrm{d}\lambda\, \frac{r_{20}}{r_{19}}\,, \tag{31}$$

with one constant of integration at each step. Every integrand is a rational function, so $V(\lambda)$ is an iterated integral of rational one-forms, which is a polylogarithmic function. Equation (30) contains no irreducible factor of second order, which a period of an elliptic curve would require. On the line (15) this excludes the curve (19), its two images under permutations of the sites, and any other elliptic curve. The conclusion holds for every right factor of $\mathcal{L}_0$, so it does not depend on whether $\mathcal{L}_0$ is the operator of lowest order that annihilates V.

The numerical values of V are consistent with this result. The solutions of $\mathcal{L}_2$ have logarithmic singularities at $\lambda = \frac{1}{3}$ or at $\lambda = 1$, whereas the Chebyshev coefficients of V decay as those of a function that is analytic at both points (figure 2). The closed form of section 4.4 is analytic at both points.

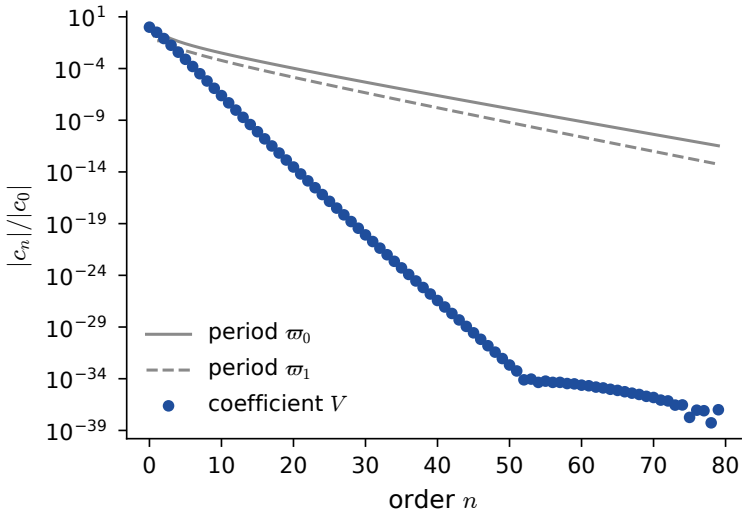

Figure 2: Chebyshev coefficients of the numerical values of $V(\lambda)$ on the interval $0.345 \leq \lambda \leq 0.985$, compared with those of the periods ϖ_0 (solid) and ϖ_1 (dashed) of eq. (22), each normalized by its zeroth Chebyshev coefficient, $|c_n|/|c_0|$. The periods have logarithmic singularities at the ends of the physical interval, which limit the rate of decay. The coefficients of V decay geometrically until they reach the accuracy of the numerical values.

4.4 The closed form

On the line (15) the solutions of $\mathcal{L}_0$ are polylogarithms of weight at most two. We solve $\mathcal{L}_0\, g = 0$ exactly with an ansatz in multiple polylogarithms $G(l_1, \ldots, l_w; \lambda)$ with letters $l_i \in \{0, \pm 1, \pm\frac{1}{3}\}$ and coefficients that are rational functions of λ. There are 5 solutions of weight zero, 19 of weight at most one and 20 of weight at most two, and there are no further solutions at weights three and four. The coefficient V is a combination of the 20 solutions with constant coefficients. Analyticity at $\lambda = 1$ and the numerical values of V determine these constants (appendix A.3).

Written in terms of the site energies, the result on the line suggests the following formula for one leg per site and arbitrary momenta,

$$V \;=\; \frac{\sqrt{2}\,\pi}{8}\left[\frac{\sum_i X_i \operatorname{Li}_2(u_i)}{\xi_1\xi_2\xi_3} + \frac{8\,X_1X_2X_3}{q_{\mathcal{G}}\,\xi_1^2\xi_2^2\xi_3^2}\;\mathcal{W}\right], \tag{32}$$

$$\mathcal{W} \;=\; \sum_i X_i\xi_i \operatorname{Li}_2(u_i) + 2\sum_{i<j} X_iX_j\,\ln(1-u_i)\ln(1-u_j) - \frac{\pi^2}{12}\sum_{i<j}\xi_i\xi_j\,, \qquad \xi_i = q_{\mathcal{G}} - 2X_i\,, \quad u_i = \frac{\xi_i}{q_{\mathcal{G}}}\,. \tag{33}$$

On the line (15) the function space of eq. (32) is exact and its constants are numerical. For other momenta eq. (32) is an ansatz with 14 rational coefficients. The result on the line determines the 14 coefficients uniquely, and the formula then agrees with numerical values of the loop integral for other triangles of momenta (appendix A.4). We leave an analytic derivation of eq. (32) for future work.

The variables ξ_i are positive by the triangle inequalities, and $\xi_i = 0$ is a folded configuration. The expression is finite there, because the bracket vanishes to second order. It is also finite when one of the momenta becomes soft. Figure 3 shows that $q_{\mathcal{G}}^2\, V$ is nearly constant over all triangles.

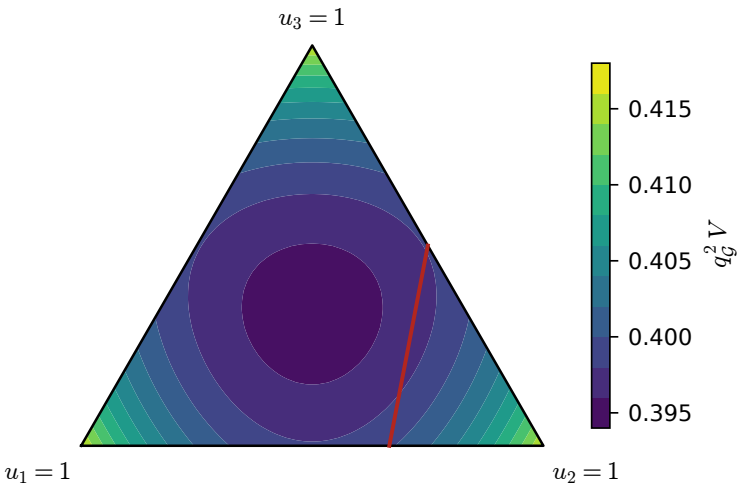

Figure 3: The closed form (32) for all triangles of momenta, in the variables $u_i = (q_{\mathcal{G}} - 2X_i)/q_{\mathcal{G}}$, which are positive and sum to one. The corners are soft limits and the edges are folded configurations. The combination $q_{\mathcal{G}}^2\, V$ is smallest for the equilateral triangle and largest in the soft limits. The red line is the line (15).

With one leg per site, the one-loop triangle coefficient is therefore a function of transcendental weight two with three dilogarithms. We count the weight of the bracket in eq. (32) and not its overall factor of π. A one-loop amplitude in flat space has the same weight. To our knowledge the only earlier closed form for this coefficient is that of ref. [30], for the configuration in which one external momentum vanishes at arbitrary site energies. In that configuration the integration region becomes that of the two-site graph, and the result is again a combination of dilogarithms. Recent work has taken the coefficient at general momenta to be elliptic or hyperelliptic [16, 17].

4.5 General site energies

The energy integrals of eq. (4) require V at $X_v > P_v$. We first state the result and define its terms. We then describe how we obtained it, from an exact differential system on a second line and from the symbol for general kinematics. For these site energies the coefficient is again a polylogarithmic function of weight two, and its letters contain square roots,

$$V(X;P) \;=\; \frac{\sqrt{2}\,\pi}{8}\,\mathrm{Re}\Big[-\frac{1}{4}\,\frac{q_{\mathcal{G}}}{\sqrt{R_0}}\,F_0 - \frac{1}{2}\sum_e \frac{q_{\mathcal{G}}}{\sqrt{R_e}}\,F_e - \sum_{i<j}\frac{\rho_{ij}^-}{\sqrt{T_{ij}^-}}\,F_{ij}^- + \sum_{i<j}\frac{\rho_{ij}^+}{\sqrt{T_{ij}^+}}\,F_{ij}^+\Big]\,. \tag{34}$$

Here R_0 is $-2B$ at the point where the three partial energies of the sites vanish. The three R_e are $-2B$ at the points where $q_{\mathcal{G}} + 2y_e$ and the partial energies of the two sites at the ends of e vanish. The other radicands and prefactors are

$$\begin{aligned} T_{ij}^{\pm} &= \kappa(P) + 4P_i^2X_j^2 + 4P_j^2X_i^2 \pm 4X_iX_j\big(P_i^2 + P_j^2 - P_k^2\big)\,, \\ \rho_{ij}^+ &= \frac{X_k}{X_i + X_j - X_k}\,, \qquad \rho_{ij}^- = \frac{X_k\, q_{\mathcal{G}}}{(X_i - X_j)^2 - X_k^2}\,, \end{aligned} \tag{35}$$

where $\kappa(P)$ is the Källén function of the P_v^2 and k is the third site. Each F is a combination of dilogarithms of the form

$$\mathcal{D}(x,\bar{x}) \;=\; \mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x}) + \tfrac{1}{2}\ln(x\bar{x})\,\ln\frac{1-x}{1-\bar{x}}\,, \tag{36}$$

where $\bar{x}$ is obtained from x by a change of sign of the square root. For a positive radicand, x and $\bar{x}$ are real, and we take the real parts of the dilogarithms and the absolute values of the arguments of the logarithms. For a negative radicand, $\bar{x}$ is the complex conjugate of x, and $\mathcal{D}$ is 2i times the Bloch–Wigner function. With these definitions each term in the bracket of eq. (34) is real. The permutations of the sites relate the three F_e, the three F_{ij}^+ and the three F_{ij}^- to each other, so there are four independent functions. There is no term without a square root and no additive constant.

The function F_0 has a geometric form. At the point where the three partial energies of the sites vanish, the P_v and the values of $-y_e$ are the six edge lengths of a flat tetrahedron, where a negative value is a negative length. Let f_k be the perimeters of its four faces and c_j the perimeters of the three closed paths through all four vertices. Then

$$\mathcal{P}(h) \;=\; \prod_{k=1}^{4}(h-f_k) \;-\; h\prod_{j=1}^{3}(h-c_j) \tag{37}$$

is a quadratic polynomial in the auxiliary variable h, and its discriminant is the radicand R_0. Let $h_\pm$ be the roots of $\mathcal{P}$, and let $\mathcal{D}_a = \mathcal{D}(1-a/h_-, 1-a/h_+)$. Then

$$F_0 \;=\; -2\Big[\sum_{k=1}^{4}\mathcal{D}_{f_k} - \sum_{j=1}^{3}\mathcal{D}_{c_j}\Big]\,. \tag{38}$$

When $R_0 < 0$ each term of eq. (38) is proportional to the volume of an ideal hyperbolic tetrahedron. The function F_e is a sum of nine terms of the same type, and F_{ij}^+ and F_{ij}^- are sums of five and four terms. We give them in appendix A.6.

Equation (34) is exact on a second line, up to one overall constant, which we determine numerically. On the line $X = P + \frac{1}{3}(1,1,1)$ with $P = (2\lambda, \lambda, 1)$ there are 134 master integrals. Fifteen sectors in which two of the factors (8) vanish together have an irreducible homogeneous equation of second order. Their operators are equivalent to the Picard–Fuchs operators of the elliptic curves that B defines on the corresponding lines of proposition 1. At $X = P$ most of these curves degenerate, because B becomes a perfect square there.

The periods of these curves do not enter the coefficient. In the limit $\varepsilon \to 0$ the coefficient is a component of the solution of a first-order system of rank 44, $\mathrm{d}\vec{I} = M\vec{I}$. After a change of basis every entry of the matrix M is a sum of logarithmic forms $\mathrm{d}\ln(\,\cdot\,)$ with rational coefficients, whose arguments are rational or contain a single square root. No entry has a component along the holomorphic differential of any of the elliptic curves, or along its differential of the second kind. The solution is therefore an iterated integral of logarithmic forms, as for the factorization (30). On this line the coefficient is a sum of ten terms, each of which is an iterated integral of weight two divided by a square root. The ten square roots are those of eq. (34), and all ten are perfect squares at $X = P$.

For general kinematics we use the symbol [31]. The symbol of a function of weight two is a sum of ordered pairs of letters. The first entry of a pair determines where the function has branch points, and a sum of pairs is the symbol of a function only if it satisfies an integrability condition. We consider symbols whose first entries are the positive sums of energies that occur on the second line, with i, j and k distinct,

$$q_{\mathcal{G}}\,, \quad X_i + P_i\,, \quad X_i + X_j + P_k\,, \quad X_i + P_j + P_k\,, \quad P_1 + P_2 + P_3\,. \tag{39}$$

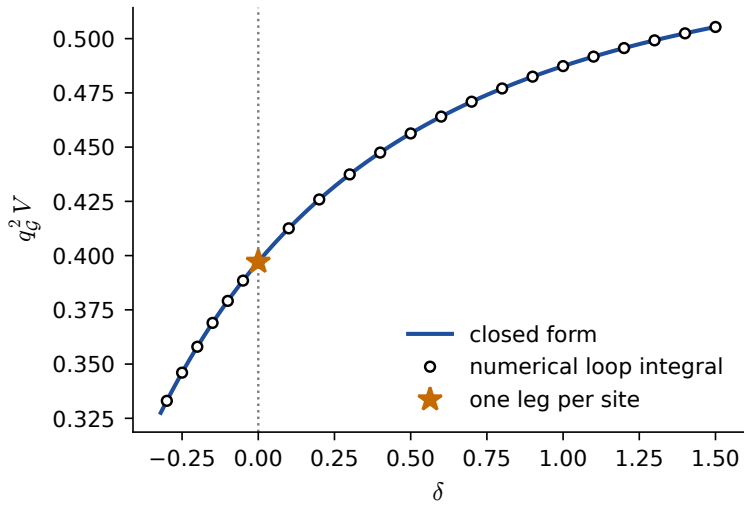

Figure 4: The combination $q_{\mathcal{G}}^2 V$ as a function of δ, for $P = (\frac{7}{5}, \frac{7}{10}, 1)$ and $X_v = P_v + \delta$. The line is eq. (34), continued to $X_v < P_v$ as in section 4.5. The star is eq. (32) at $\delta = 0$, and the points are numerical values of the loop integral. The dotted line marks $\delta = 0$, where all ten square roots of eq. (34) become rational.

The second entries are letters of the form $(p + \sqrt{R})/(p - \sqrt{R})$, where R is one of the ten radicands and p is a polynomial for which $p^2 - R$ is a product of linear functions of the energies. Each symbol is odd under the change of sign of its root. For each of the ten square roots there is exactly one integrable symbol of this kind. The prefactors $q_{\mathcal{G}}/\sqrt{R}$ and $\rho_{ij}^{\pm}/\sqrt{T_{ij}^{\pm}}$ are the leading singularities. The coefficient is therefore fixed up to ten rational numbers. On the second line the symbol of the first-order system determines these ten numbers, which are the coefficients $-\frac{1}{4}$, $-\frac{1}{2}$, -1 and $+1$ of eq. (34). With these values eq. (34) agrees with numerical values of the loop integral throughout $X_v \geq P_v$ (appendix A.4). At $X = P$ the ten square roots become rational, and the symbol of eq. (34) reduces to that of eq. (32).

With the definition (36), eq. (34) holds for $X_v \geq P_v > 0$. Individual terms are singular where the prefactors (35) have poles, where a radicand vanishes, where $P_k = |X_i - X_j|$, and where the leading coefficient of $\mathcal{P}$ vanishes for the primed lengths of appendix A.6. The sum has a finite limit there, and at $X = P$ the limit is eq. (32). The coefficient is defined for all $X_v > 0$, and its integrand has no singularity in that region. For $X_v < P_v$ and a positive radicand, each term of eq. (34) differs from its value with the definition (36) by an integer multiple of $\pi^2/2$. Analytic continuation along the ray $P \to tP$ from $t = 0$ determines the integer. The continued expression is smooth across $X_v = P_v$ (figure 4). The ancillary files contain a numerical implementation of eq. (34) with these branch assignments, including the continuation to $X_v < P_v$.

The one-loop triangle coefficient is therefore a polylogarithmic function of weight two. This statement is exact on the two lines, and for other kinematics it rests on the agreement of eq. (34) with numerical values of the loop integral. The elliptic curves of the master integrals, three at $X = P$ and many more for $X_v > P_v$, do not enter. The ten square roots do enter, and for $X_v > P_v$ the letters of the coefficient contain them.

4.6 De Sitter space

For a cubic interaction in de Sitter space the energy integrals (4) have $\sigma = 0$. They change the loop integrand at large loop momentum from $|\vec{\ell}|^{-5}$ to

$$\int_0^\infty \mathrm{d}^3\omega\; \psi_3(X+\omega;y) \;\longrightarrow\; \frac{\zeta(3)}{8\,|\vec{\ell}|^3}\,, \tag{40}$$

so the loop integral of the de Sitter coefficient diverges logarithmically. With a cutoff Λ on the comoving loop momentum, the coefficient of $\ln\Lambda$ is $\pi\,\zeta(3)/2$. With one leg per site, symmetry determines the dependence on the momenta. At late times the coefficient is a three-point function of operators with $\Delta = 2$ in a three-dimensional conformal field theory [32]. If the cutoff breaks the symmetry only through the logarithm, then

$$\psi_3^{\rm dS} \;=\; \int^\Lambda \mathrm{d}^3\ell \int_0^\infty \mathrm{d}^3\omega\; \psi_3(X+\omega;y) \;=\; -\frac{\pi\,\zeta(3)}{2}\,\ln\frac{P_1+P_2+P_3}{\Lambda} + b_0\,, \tag{41}$$

where b_0 is a number that depends on the definition of the cutoff. A numerical evaluation at two kinematic points agrees with this form. We do not derive eq. (41) from eq. (34). For the corresponding correlator the loop integral converges, and symmetry allows only a constant, whose numerical value is $-3\pi^2\zeta(3)$.

With one leg per site the de Sitter triangle is therefore determined by two numbers, the coefficient of the logarithm and b_0, for any function space of V. The function space of V matters when several legs meet at a site, or when the external legs break the de Sitter boosts. In those cases the de Sitter and power-law coefficients are integrals of eq. (34) over the site energies.

5 Correlators

In this section we show that the integrand of the equal-time correlator of the triangle has no pole on the planes $q_{\mathcal{G}} + 2y_e = 0$. These are the planes of the elliptic sectors of section 3. The absence of the factors $q_{\mathcal{G}} + 2y_e$ from correlators is known [25, 26], and the relation between correlators and wavefunction coefficients has been studied from several directions [33–35]. Ref. [9] noted the absence in connection with the elliptic sector and asked whether the correlator requires elliptic functions. We derive the cancellation from the recursion relations. We then consider the other singular loci of the correlator integrand. All statements in this section concern integrands at fixed loop momentum.

5.1 The correlator integrand

The Born rule (2) expresses a correlator through products of wavefunction coefficients. At one loop the integral in eq. (2) includes the profile of the field that runs around the loop. With the Gaussian weight $|\Psi|^2$, this integral joins two external legs of energy y_e and gives a factor $1/(2y_e)$. The correlator integrand of a graph G is therefore a sum over the sets of edges that are cut into such pairs of legs. We write $\hat{\psi}_G = \prod_e (2y_e)\,\psi_G$. For vertex factors that do not depend on time, the expansion of the Born rule at one loop (section 3 of ref. [25] and appendix A of ref. [26]) gives

$$C_G \;=\; \sum_{S\,\subseteq\,\text{edges}} 2^{\,n(S)-1} \prod_{j=1}^{n(S)} \hat{\psi}_{H_j}\,, \tag{42}$$

where the graph with the edges in S cut has $n(S)$ connected components H_j. Each cut edge adds its energy y_e to the two sites at its ends. The correlator integrand is C_G divided by $\prod_e 2y_e$ and by factors that depend only on the external legs. The first terms of eq. (42) are the loop and the chains in which one edge is cut, and they have the same coefficient.

5.2 Cancellation of the factors $q_{\mathcal{G}} + 2y_e$

Consider the polygon with N sites, and write G_e for the chain in which the edge e is cut. The terms of eq. (42) with poles at $q_{\mathcal{G}} + 2y_e = 0$ are the loop and the chains, which appear in the combination

$$\Phi_G \;=\; \hat{\psi}_G + \sum_e \hat{\psi}_{G_e} \,. \tag{43}$$

The recursion of ref. [3], which is eq. (9) for the triangle, gives

$$q_{\mathcal{G}}\, \hat{\psi}_G \;=\; \sum_e 2y_e\, \hat{\psi}_{G_e} \,, \qquad (q_{\mathcal{G}} + 2y_e)\, \hat{\psi}_{G_e} \;=\; \sum_{f \neq e} 2y_f\, \hat{\psi}_{G_e \setminus f} \,. \tag{44}$$

The second relation holds because the total energy of the chain G_e is $q_{\mathcal{G}} + 2y_e$. Here $G_e \setminus f$ is the pair of chains in which a second edge f is cut. We add the first relation to the sum of the second over e. The terms $2y_e \hat{\psi}_{G_e}$ cancel, and we find

$$q_{\mathcal{G}}\, \Phi_G \;=\; \sum_{e \neq f} 2y_f\, \hat{\psi}_{G_e \setminus f} \,. \tag{45}$$

In the tree coefficients on the right-hand side, each site contains the energy of a cut edge at most once. Their poles are therefore partial energies of G, and the poles of $\hat{\psi}_G$ and of $\hat{\psi}_{G_e}$ at $q_{\mathcal{G}} + 2y_e = 0$ cancel in Φ_G. The remaining terms of eq. (42) have two or more edges cut, and their poles are also partial energies of G. For the triangle the denominator of Φ_G consists of $q_{\mathcal{G}}$ and the six partial energies of eq. (8). The argument extends by induction to polygons with trees attached. Ref. [25] derives the cancellation for any number of loops and any time dependence of the background, as a cosmological analog of the Kinoshita–Lee–Nauenberg theorem.

The factors $q_{\mathcal{G}} + 2y_e$ come from the boundary condition of the wavefunction. The propagator (5) is the sum of a time-ordered term and a factorized term, $e^{-y(t+t')}$, which makes the propagator vanish at the late-time boundary. A factor that contains y_e twice requires that both ends of the edge add the energy $+y_e$ to the same set of sites. The time-ordered term adds $+y_e$ at its earlier end only, so the factors $q_{\mathcal{G}} + 2y_e$ come from the factorized term. In the correlator the integral over the profile gives the chains G_e, in which the edge is replaced by the same pair of external legs, and the two contributions cancel.

5.3 Other singular loci of the correlator integrand

The cancellation removes the curve of section 3 from the correlator integrand of the triangle. The other singular loci do not contain elliptic curves either. By proposition 1, every line on which B defines an elliptic curve lies in one of the planes $q_{\mathcal{G}} + 2y_e = 0$. The correlator integrand has no pole on these planes, so it has no residue on any of these lines. The wavefunction integrand has residues on 12 of them, and by proposition 1 these residues do not involve the periods of an elliptic curve. For the triangle neither integrand therefore has a residue that involves an elliptic curve, and the reasons differ. The correlator integrand has no pole on the lines with an elliptic curve, and the residues of the wavefunction integrand on these lines are differentials on curves of genus zero.

For the box and the pentagon the result depends on the dimension. In $d = N$ the N edge energies are independent coordinates on the space of loop momenta, and the measure is the inverse square root of the Cayley–Menger polynomial of the simplex with $N + 1$ vertices. For the triangle this dimension is the physical one. In $d = N$ the wavefunction integrands of the box and the pentagon have residues on lines with an elliptic curve, including lines on which only partial energies vanish, and their correlator integrands have none. We have not studied singular loci of dimension two or higher. In $d = 3$ the exponent of the Cayley–Menger polynomial of the box is an integer, and the curves are of a different kind. Section 6 shows that the correlator of the box in $d = 3$ involves elliptic curves.

The cancellation also holds for time-dependent vertex factors and at two loops. With vertex factors $(-\eta)^{-(1+\sigma)}$ the coefficients in eq. (42) contain a phase for each connected component that depends on its number of sites. The loop and the chains G_e have the same sites, so Φ_G is unchanged, and the energy integrals of eq. (4) apply to Φ_G as a whole. In the sunrise, the kite and five other two-loop graphs with up to four sites, every factor that contains an edge energy more than once is absent from the correlator integrand, in agreement with ref. [25]. We do not know whether the integrand of a two-loop correlator has residues that involve an elliptic curve.

6 The box

The one-loop box is not polylogarithmic. In this section we show that the residue of its loop integrand at the pole of a single site has two square roots, which define an elliptic curve. We give the discontinuity of the box at threshold in closed form in terms of elliptic integrals. We then describe the K3 surface that appears above the threshold, and we compute the box and its correlator on a line. The last subsection summarizes what we know for general momenta. We use the notation of section 2, with sites $v = 1, \ldots, 4$ and edge energies y_{12}, y_{23}, y_{34} and y_{41} (figure 1(b)). The wavefunction coefficient is $V_4 = \int \mathrm{d}^3\ell \; \psi_4$, without a prefactor.

6.1 The single-site pole

For the box the residue on the factors $q_{\mathcal{G}} + 2y_e$ is again a flat-space amplitude, which depends on the other edge energies only through their squares. The residues on the partial energies of two or three sites do not define an elliptic curve either. The elliptic curve comes from the partial energy of a single site. At the pole $X_1 + y_{41} + y_{12} = 0$ the residue of the loop integrand is

$$\operatorname*{Res}_{X_1=-y_{41}-y_{12}} \psi_4 \;=\; \frac{1}{4\,y_{12}\,y_{41}} \sum_{s,s'=\pm 1} s\,s'\;\psi_{\text{chain}}\big(X_2 + s\,y_{12}\,,\; X_3\,,\; X_4 + s'\,y_{41}\,;\; y_{23}\,,\; y_{34}\big)\,, \tag{46}$$

where ψ_{chain} is the coefficient (10) of the chain of the three other sites. This is the analog of eq. (28). The residue is even in y_{12} and in y_{41}. It is not even in y_{23} or in y_{34}, the energies of the two edges that do not end on site 1. The same formula holds for every polygon, with the chain of the other sites in place of eq. (10).

At the pole the loop momentum lies on a spheroid with foci at the two points where y_{12} and y_{41} vanish. Each of the other edge energies is the square root of a quadratic function on this surface. The triangle has one such edge, and a single square root of a quadratic can be rationalized (section 4.1), whereas the box has two. Along a line on the spheroid the two square roots define the curve $Y^2 = z_{23}\,z_{34}$, where $z_e = y_e^2$ is quadratic in the coordinate along the line. This curve has genus one. For the pentagon the three square roots have six branch points, and they define a curve of genus five.

6.2 The threshold discontinuity

With one leg per site, $X_v = P_v$, the discontinuity of V_4 at its threshold is a one-dimensional integral. The discontinuity of V_4 across the branch cut that starts at $X_1 = -P_1$ is the integral of the residue (46) over the spheroid $y_{41} + y_{12} = -X_1$. At the threshold the spheroid becomes the segment between its foci. We parametrize the segment by $\chi = y_{41} - y_{12}$ with $-P_1 \leq \chi \leq P_1$. On the segment

$$y_{23}^2 \;=\; \frac{(P_1+\chi)\,P_2^2 + (P_1-\chi)\,P_{12}^2}{2P_1} - \frac{P_1^2-\chi^2}{4}\,, \qquad y_{34}^2 \;=\; \frac{(P_1+\chi)\,P_{41}^2 + (P_1-\chi)\,P_4^2}{2P_1} - \frac{P_1^2-\chi^2}{4}\,, \tag{47}$$

with $P_{12} = |\vec{P}_1 + \vec{P}_2|$ and $P_{41} = |\vec{P}_4 + \vec{P}_1|$. We define the threshold discontinuity as

$$D_0 \;=\; \frac{\pi}{4P_1}\int_{-P_1}^{P_1} \mathrm{d}\chi \sum_{s,s'=\pm 1} s\,s'\,\psi_{\text{chain}}\Big(P_2 + s\,\tfrac{P_1-\chi}{2}\,,\; P_3\,,\; P_4 + s'\,\tfrac{P_1+\chi}{2}\,;\; y_{23}\,,\; y_{34}\Big)\,. \tag{48}$$

For momenta that do not lie in a plane the four zeros of $z_{23}\,z_{34}$ are distinct and complex, and eq. (48) is an integral on an elliptic curve. Its j-invariant depends on the shape of the quadrilateral of momenta.

The threshold discontinuity is part of the wavefunction coefficient. When the momentum of the leg at site 1 becomes soft, the coefficient with one leg per site has the form

$$V_4 \;=\; -D_0\,\ln 2P_1 \;+\; \text{terms analytic at } P_1 = 0\,. \tag{49}$$

Numerical values of V_4 at two quadrilaterals that do not lie in a plane agree with eq. (49).

The integral (48) has a closed form. The integrand is a rational function of χ, y_{23} and y_{34} with simple poles. Partial fractions reduce it to logarithms and to the integrals of the first and third kind on the curve $Y^2 = z_{23}\,z_{34}$ of section 6.1. The result is

$$D_0 \;=\; \frac{\pi}{4P_1}\Big[\;\frac{1}{4}\sum_{\text{poles}} \rho\,\big(L_0 + \eta\,L_{23} + \eta'\,L_{34} + \eta\,\eta'\,L_\Pi\big) \;+\; 4\,p_0\,J_F\Big]\,, \qquad J_F = \int_{-P_1}^{P_1}\frac{\mathrm{d}\chi}{4\,Y} = 2R_F\,, \tag{50}$$

where L_0, L_{23} and L_{34} are logarithms, and L_Π contains a logarithm and the symmetric elliptic integrals R_F and R_J of Carlson [36]. The sum runs over the poles of the integrand on the four sheets $(\eta, \eta') = (\pm y_{23}, \pm y_{34})$ of the two roots. The coefficients ρ and p_0 are algebraic functions of the six lengths in eqs. (47) and (48). We derive the complete expressions in appendix B.3. The coefficient p_0 does not vanish, so D_0 contains an elliptic integral of the first kind. Figure 5 compares eq. (50) with numerical values of the integral (48).

For momenta in a plane one of the poles becomes a double pole. Equation (50) then contains the additional term of eq. (91). With this term the formula holds for planar quadrilaterals in general position. For momenta in a plane all poles are real, and the result does not depend on the branches of the logarithms. For momenta that do not lie in a plane the branches at the complex poles follow from an analytic continuation, which agrees with numerical values and which we have not proved.

6.3 The K3 surface above the threshold

Above the threshold the discontinuity is an integral over the double cover of the spheroid defined by the two roots, which is a K3 surface fibered by elliptic curves. Its periods obey a differential equation in $y_{41} + y_{12}$. The order of the equation is 5 when the momenta form a regular tetrahedron and 8 for generic momenta. For the regular tetrahedron we have the operator exactly, eq. (95)

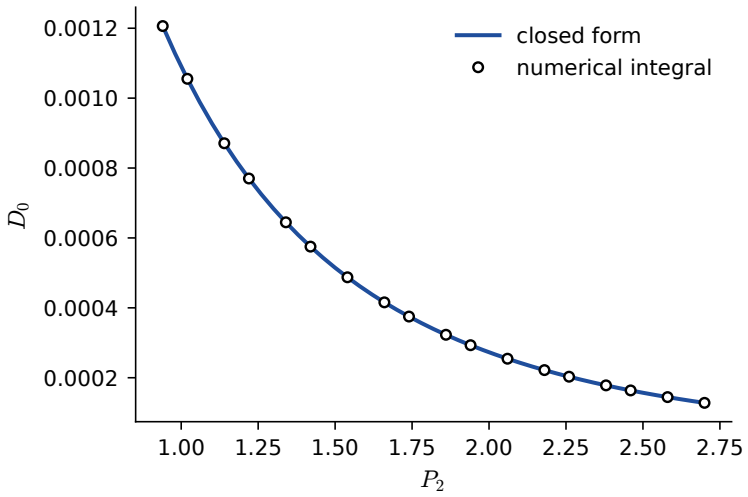

Figure 5: The threshold discontinuity D_0 of the box with one leg per site, for $P_1 = 1$, $P_3 = 2$, $P_4 = \frac{7}{4}$, $P_{12} = \frac{9}{5}$ and $P_{41} = 2$, as a function of P_2. Here P_{12} and P_{41} are the lengths of the two diagonals of the quadrilateral. The line is the closed form (50) and the points are numerical values of the integral (48).

in appendix B.5. It is not a symmetric power of a second-order operator, so its solutions are not products of periods of an elliptic curve. Its five solutions obey the quadratic relation (96), which is a property of the periods of a K3 surface.

The loop integrand has a component along these periods. With several legs per site the leading term of the residue at large site energies generates all five solutions of the fifth-order operator, so the periods of the K3 surface enter the box. With one leg per site the evidence is as follows. The solutions of the fifth-order operator have square-root branch points at complex values of $y_{41} + y_{12}$, and the discontinuity of V_4 has the same branch points with the predicted coefficient. An exact reduction of the loop integrand, for the regular tetrahedron and with three site energies at one leg per site, gives differential equations whose solutions include the period of the holomorphic form of the K3 surface. The same holds for the integrand of the correlator. With one leg per site we have not examined generic momenta.

6.4 The box on a line

On one line in kinematic space we obtain exact differential equations for the box, as in section 4. The momenta form a regular tetrahedron of unit side, and the site energies are

$$X_v \;=\; X \;=\; 1 + t\,, \qquad v = 1, \ldots, 4\,, \tag{51}$$

so that $t = 0$ is the case of one leg per site. We denote by $C_4(t)$ the correlator that corresponds to V_4, integrated over the loop momentum. An integration-by-parts reduction with the regulator of eq. (13) gives exact first-order differential systems in t for V_4 and for C_4. The limit $\varepsilon \to 0$ requires the double poles of the master integrals in ε. Their coefficients are residues at infinite loop momentum, and they do not depend on the kinematics.

The systems determine the function space on the line. An exact rational change of basis brings the system for C_4 to a form with simple poles at the zeros of 19 polynomials in X of degree at most four, which we list in eq. (97). The system has no other singular points at finite X. Its irreducible

parts are the fifth-order operator of section 6.3, the second-order operators of three elliptic curves, and first-order operators with rational and square-root letters. The three curves are those on which two partial energies vanish. The curve of section 6.2 degenerates on this line and does not appear. The periods of the three curves are hypergeometric functions of their j-invariants, eq. (93), and they enter both V_4 and C_4. The monodromy of the solutions indicates that the solutions of the fifth-order operator enter as well, but we have not separated them from the solutions of the first-order operators. The solutions of the systems agree with numerical values of both functions, and they continue to $t = 0$ (appendix B.6). The ancillary files contain the two systems and a program that evaluates their solutions.

6.5 The correlator on the line

On the line the differential system determines the correlator completely. The result is

$$\begin{aligned} C_4(t) &= u(t) + \Omega\, v(t)\,, \\ \Omega &= \int \mathrm{d}^3\ell\, \frac{1}{y_{12}\, y_{23}\, y_{34}\, y_{41}} = 21.4444295139\ldots\,, \end{aligned} \tag{52}$$

where u and v are two solutions of the system for C_4, which we define in appendix B.6. In the integral the four edge energies are the distances from $\vec{\ell}$ to the vertices of the regular tetrahedron. Figure 6 shows the result.

The functions u and v follow from conditions that do not use any value of C_4. The loop integral converges for $t > -1$, so C_4 is single-valued around the singular points of the system in that range. The solutions that are single-valued around the six points listed in appendix B.6 form a space of dimension 4. At large X the correlator has the expansion

$$C_4 = \sum_{n\geq 4} X^{-n}\big(\alpha_n + \beta_n \ln X\big)\,, \qquad \alpha_4 = \Omega\,, \quad \beta_5 = -24\pi\,, \quad \beta_7 = -\tfrac{158}{3}\,\pi\,, \tag{53}$$

with $\beta_n = 0$ for even n. The expansion by regions gives the coefficients β_n exactly. Exactly one solution in this space has no logarithms at large X. This solution is v, normalized to begin with X^{-4}, and it is an elementary function,

$$v = \frac{4\sqrt{10}}{X\,\sqrt{(10X^2-1)\,(16X^4-10X^2+3)}}\,. \tag{54}$$

The coefficients β_n then determine u, and they do not constrain the constant that multiplies v. One further condition determines this constant. The function v changes sign under continuation around the point $X^2 = \frac{1}{10}$, where the loop integral is analytic. The sum $u + \Omega\, v$ is single-valued there only for one value of the constant, which agrees with the integral in eq. (52) (appendix B.6). The differential system therefore determines Ω as well. We have found no integer relation between Ω and known constants, and appendix B.7 reduces it to a one-dimensional integral.

The expression $u + \Omega\, v$ contains no fitted number, and it agrees with the numerical values of C_4. It is a closed form in a weaker sense than eq. (34), because u is defined through a differential system and not through known functions. For the coefficient V_4 the same construction leaves no undetermined constant (appendix B.8).

6.6 The box is not polylogarithmic

An elliptic integral in a discontinuity does not by itself exclude a polylogarithmic coefficient, as the triangle shows. For the box two arguments exclude it. The first concerns the function D_0, and it

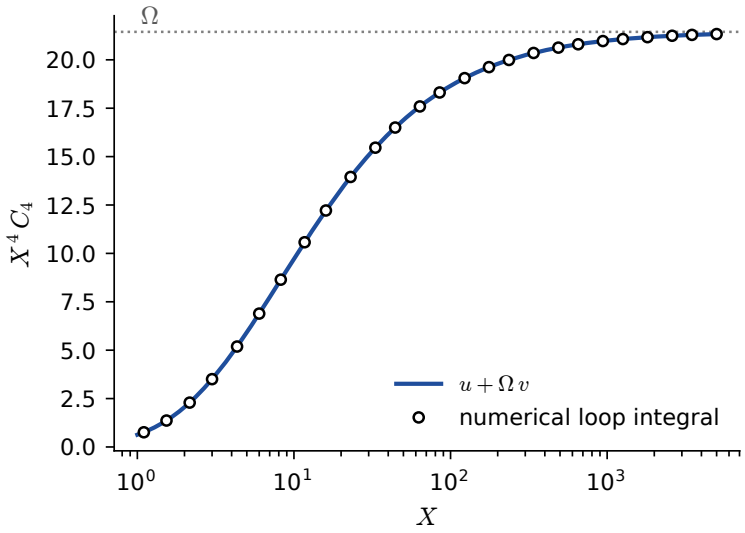

Figure 6: The correlator of the box on the line of section 6.4, multiplied by X^4. The line is eq. (52) and the points are numerical values of the loop integral. The dotted line is the constant Ω, which is the limit at large X.

uses the fact that iterated integrals of logarithmic forms with algebraic letters have quasi-unipotent monodromy, which means that all eigenvalues of the monodromy matrices are roots of unity. We continue D_0 analytically in one of the lengths of the quadrilateral, around the points where the curve degenerates, and we compute the continuation numerically. Its monodromy contains elements with integer traces larger than two, whose eigenvalues are not roots of unity (appendix B.3). The function D_0 is therefore not an iterated integral of this type. For the triangle the same computation gives unipotent monodromy.

The second argument is exact and concerns the loop integrand. As the shape of the quadrilateral varies, the residue (46) and its derivatives obey a linear differential system. With one leg per site the solutions of this system include the periods of the curve $Y^2 = z_{23}\, z_{34}$. For a logarithmic form on the same curve they do not. For the triangle the corresponding curve is a conic and has no periods.

With eq. (49), the two arguments show that the periods of an elliptic curve enter the soft limit of the box with one leg per site. On the line of section 6.4 the periods of three elliptic curves enter V_4 and C_4. The box is therefore the first polygon of this theory that is not polylogarithmic.

6.7 General momenta

The results of sections 6.4 and 6.5 hold on one line, for the regular tetrahedron. They are restrictions of a structure in all variables of the box, which are the four site energies and the six lengths of the quadrilateral of momenta. We state its main properties and leave the rest to a separate paper. The singular points of the master integrals lie on the zeros of 1333 irreducible polynomials in the site energies and the squares of the lengths. Every singular point in this section lies on one of them. The curves on which two partial energies vanish form two families of elliptic curves, one for pairs of partial energies that share an edge and one for pairs that share none. Their invariants are known in all variables, and on the line they reduce to the j-invariants of the three curves of section 6.4. The K3 surfaces of section 6.3 form one family with six moduli. The rank of its transcendental

lattice is 5 for the regular tetrahedron, 6 for tetrahedra whose opposite edges have equal lengths, and 8 for generic momenta. In each case the rank is the order of the differential equation of the periods. The operator (95) and the form of the relation (96) are therefore special to the regular tetrahedron.

For one quadrilateral in general position we have reduced the correlator on the line $X_v = P_v + t$. The reduction contains an operator of order 8 in place of the fifth-order operator. The function v of eq. (54) is a rational function divided by the square root of a polynomial, and on this line the system has at most 16 solutions of this type. We write Ω for the integral of eq. (52) with the edge energies of the given quadrilateral. We have not determined which combination of the 16 solutions multiplies Ω, or whether other solutions multiply it as well. At large site energies the leading term of C_4 is $\Omega/(X_1X_2X_3X_4)$ for all ratios of the site energies.

7 Discussion

The elliptic curve of the one-loop triangle appears in its master integrals and is absent from the wavefunction coefficient. The coefficient is a polylogarithmic function of transcendental weight two. This statement is exact on the two lines that we analyzed, and for other kinematics it rests on the agreement of eq. (34) with numerical values of the loop integral. Proposition 1 supports this result. At the poles of the elliptic sector, the residue of the loop integrand is a flat-space amplitude, which is rational in the loop momentum. At the single-site pole the residue of the triangle has one square root. For the box the residue at the single-site pole has two square roots, and the periods of elliptic curves enter both the coefficient and the correlator (section 6). For momenta that form a regular tetrahedron the periods of a K3 surface enter the loop integrand as well. The box is therefore the first polygon of this theory that is not polylogarithmic. The function space that the maximal cuts of the master integrals suggest can thus be larger than the function space of the wavefunction coefficient.

For the triangle the elliptic curve is absent for two reasons that we cannot yet relate. The curve of the master integrals lies on a factor that the correlator does not contain, and the periods of the curve also cancel in the wavefunction coefficient. Both follow from the factorized term of the propagator (5). For the box neither statement holds, because the curves lie on poles of partial energies, which the correlator also has. In the cosmological polytope the wavefunction integrand is a canonical form and the master integrals are not. This formulation may explain when the master integrals have a larger function space than the wavefunction coefficient.

These results concern the flat-space coefficient. With one leg per site, symmetry determines the de Sitter triangle up to two numbers, the coefficient of the logarithm and a constant (section 4.6). With several legs at a site, the de Sitter and power-law coefficients are integrals of eq. (34) over the site energies. We do not know whether these integrals are polylogarithmic, or whether the ten square roots can be rationalized simultaneously.

For the triangle the main open problem is a derivation of the closed forms. We obtained eqs. (32) and (34) from an exact function space on two lines and a small number of constants that we determined numerically. The ten constants of eq. (34) are $-\frac{1}{4}$, $-\frac{1}{2}$ and ± 1, which suggests that a direct derivation exists. Such a derivation may also give a form that is manifestly analytic at $X_v = P_v$.

For the box the open problems concern the functions themselves. On one line exact conditions on a differential system determine the correlator and the coefficient, with no fitted number. One of the two functions in the correlator is elementary, and the other is defined through the system and not through known functions. The number Ω is a one-dimensional integral of elliptic integrals, and

we have not related it to known constants. For generic momenta the differential equation of the periods of the K3 surface has order 8. The reduction on a line contains this operator (section 6.7), and we have not constructed its solutions. For the pentagon the residue at the single-site pole has three square roots and a curve of genus five, which we have not examined further. Beyond one loop we have examined only integrands. In the two-loop graphs of section 5.3 the correlator integrand contains partial energies only.

All fields in this paper are conformally coupled, so the loop integrand is a rational function. For fields of general mass the singular loci of the integrand are the same, but the integrand is not rational, and our methods do not apply directly. The leading cosmological collider signals of a triangle loop of massive fields have recently been obtained with cutting rules [37], and one-loop signals of this kind can be integrated numerically [38]. The function space of such loops is an open problem.

Acknowledgments

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A Construction of the closed form

In this appendix we construct the closed form (32). We first describe the operator $\mathcal{L}_0$ and its comparison with the integral. We then construct its solutions, select the solution that equals V on the line (15), and extend the result to a general triangle. Finally we give three limits of the closed form and the four functions of eq. (34).

A.1 The operator and its singular points

The integration-by-parts identities of the family (16) follow from the vector fields that preserve B up to a factor. These are the polynomials n_e and n_0 that solve the syzygy equation $\sum_e n_e\, \partial B/\partial y_e = n_0\, B$ [9]. For a rational function g of the edge energies they give

$$\int \mathrm{d}y_{12}\, \mathrm{d}y_{23}\, \mathrm{d}y_{31}\; B^{-1/2+\varepsilon} \Big[\sum_e \frac{\partial (n_e\, g)}{\partial y_e} + \big(\varepsilon - \tfrac{1}{2}\big)\, n_0\, g \Big] \;=\; 0\,. \tag{55}$$

We solve these identities over finite fields at numerical values of λ and ε. Modulo the identities, the integrand of V and its derivatives with respect to λ span a space of finite dimension. They therefore obey a linear relation, which is a differential operator of order 20 with coefficients that depend on λ and ε. We normalize its leading coefficient to one and reconstruct the other coefficients as rational functions of ε at fixed λ. All of them are finite at $\varepsilon = 0$, and their values at $\varepsilon = 0$ are the coefficients of $\mathcal{L}_0$. We then reconstruct the coefficients as rational functions of λ, and we lift them to the rational numbers with the Chinese remainder theorem.

The operator $\mathcal{L}_0$ of eq. (29) has singular points at $\lambda = 0$, $\pm\frac{1}{3}$ and ± 1 and at infinity, and its local exponents are integers at each of them. The other zeros of its leading coefficient are apparent singularities, at which every solution is analytic. On the line (15) the finite singular points are the zeros of $q_{\mathcal{G}}$, of the ξ_i and of X_2,

$$q_{\mathcal{G}} = 3\lambda + 1\,, \qquad \xi_1 = 1 - \lambda\,, \qquad \xi_2 = 1 + \lambda\,, \qquad \xi_3 = 3\lambda - 1\,, \qquad X_2 = \lambda\,. \tag{56}$$

The points $\lambda = 1$ and $\lambda = \frac{1}{3}$ are the two folded configurations at the ends of the physical interval, and $\lambda = 0$ is the limit in which two of the momenta become soft. The points $\lambda = -\frac{1}{3}$ and $\lambda = -1$ lie outside the physical interval.

If the limit $\varepsilon \to 0$ is taken before the reduction, the reduction has additional relations and gives an operator of order 14. This operator is a right factor of $\mathcal{L}_0$ and does not annihilate V. When the reduction is applied to a master integral of the elliptic sector, its operator has a second-order factor equivalent to $\mathcal{L}_2$. A second reduction, which does not use the master integrals of section 3.1, gives an operator of order 9 that divides $\mathcal{L}_0$ from the right and annihilates V. The structure of eq. (32) indicates that the operator of lowest order that annihilates $V(\lambda)$ has order 7. We have not constructed this operator.

We compare $\mathcal{L}_0$ with numerical values of V without differentiating them. We obtain the values from eq. (13) at $\varepsilon = 0$. The integral over y_{12} is elementary, and we evaluate the remaining two-dimensional integral numerically to about 30 digits or better. All comparisons with numerical values in section 4 and in this appendix hold to this accuracy. Let φ be a function on an interval $a \leq \lambda \leq b$ that vanishes at both ends together with its first 19 derivatives. Integration by parts then moves all derivatives from V to φ without boundary terms,

$$\int_a^b \varphi\, \mathcal{L}_0 V \, \mathrm{d}\lambda \;=\; \int_a^b V\, \mathcal{L}_0^\dagger \varphi \, \mathrm{d}\lambda\,, \qquad \mathcal{L}_0^\dagger \varphi \;=\; \sum_{i=0}^{20} (-1)^i \, \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} \big(c_i\, \varphi\big)\,. \tag{57}$$

The right-hand side contains only the values of V, and it vanishes for every such φ if $\mathcal{L}_0 V = 0$. We take $\varphi_m = [(\lambda - a)(b - \lambda)]^{20} \, \mathrm{Leg}_m$, where Leg_m is the Legendre polynomial of degree m on the interval. The right-hand side of eq. (57) vanishes to the accuracy of the numerical values for $\mathcal{L}_0$, and it does not vanish for the operator of order 14.

A.2 The space of solutions

We now construct all solutions of $\mathcal{L}_0$. Since the singular points are 0, $\pm\frac{1}{3}$ and ± 1, we look for solutions among the multiple polylogarithms with singular points at these five values. They are defined recursively by $G(;\lambda) = 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, G(a, w; \lambda) \;=\; \frac{G(w;\lambda)}{\lambda - a}\,, \tag{58}$$

where a is a letter and w is a word in the letters. The number of letters of a word is its weight. We make the ansatz

$$g(\lambda) \;=\; \sum_w R_w(\lambda)\, G(w;\lambda)\,, \tag{59}$$

where the R_w are rational functions of λ and the sum runs over all words up to a given weight. To apply $\mathcal{L}_0$ to the ansatz we use the behavior of the multiple polylogarithms under a shift of the argument,

$$G(w; \lambda + \tau) \;=\; \sum_{w = w' w''} h_{w'}(\tau)\, G(w''; \lambda)\,, \tag{60}$$

where the sum runs over all ways to split the word w into a first part w' and a second part w''. The function $h_{w'}(\tau)$ is the iterated integral of the word w' along the segment from λ to $\lambda + \tau$. It is a power series in τ whose coefficients are rational functions of λ. The i-th derivative of g is $i!$ times the coefficient of τ^i in $g(\lambda + \tau)$, which we denote by $[\tau^i]$. We obtain

$$\mathcal{L}_0\, g \;=\; \sum_{w''} G(w''; \lambda) \, \sum_{i=0}^{20} c_i \; i! \, \big[\tau^i\big] \sum_{w'} R_{w' w''}(\lambda + \tau)\, h_{w'}(\tau)\,. \tag{61}$$

The multiple polylogarithms are linearly independent over the rational functions, so the coefficient of each $G(w'';\lambda)$ must vanish. The term with empty w' is $\mathcal{L}_0 R_{w''}$, and the other terms contain only the coefficients of words that are longer than w''. The coefficients of the longest words are therefore rational solutions of $\mathcal{L}_0$. The equations for the shorter words are linear equations for rational functions, which we solve exactly, starting from the highest weight.

There are 5 rational solutions, 14 further solutions of weight one and one solution of weight two. Since $\mathcal{L}_0$ has order 20, these are all of its solutions. The apparent singularities therefore do not contribute letters.

A.3 The coefficient on the line

The coefficient V is a combination of the 20 solutions with constant coefficients. We determine the constants in two steps. First, the integral (13) is analytic at the folded configuration $\lambda = 1$. The solutions of $\mathcal{L}_0$ that are analytic at $\lambda = 1$ form a subspace of dimension 14, which we determine exactly. Second, we fit the 14 constants of this subspace to numerical values of V at half of the points of our sample, and we identify them with an integer-relation algorithm [18]. Up to an overall factor of $\sqrt{2}\,\pi$ they are combinations of 1, $\ln 2$, $\ln^2 2$ and π^2 with rational coefficients. The individual constants depend on the choice of basis in the subspace. The logarithms of 2 cancel in the final result. The result agrees with the numerical values at the other half of the points.

All multiple polylogarithms of weight two in the result combine into the three dilogarithms $\mathrm{Li}_2(u_i)$ and into products of the logarithms $L_i = \ln(1-u_i) = \ln(2X_i/q_{\mathcal{G}})$. With the variables of eq. (56) we obtain

$$\begin{aligned}\frac{4\sqrt{2}\,q_{\mathcal{G}}\,\xi_1^2\,\xi_2^2\,\xi_3^2}{\pi}\,V \;&=\; 2\lambda(1-\lambda)\,p_1\,\mathrm{Li}_2(u_1) + \lambda(1+\lambda)\,p_2\,\mathrm{Li}_2(u_2) \\ &\quad + (3\lambda-1)\,p_3\,\mathrm{Li}_2(u_3) + \frac{4\pi^2}{3}\,\lambda^2\big(\lambda^2-6\lambda+1\big) \\ &\quad + 32\lambda^3\big[\,2\lambda\,L_1L_2 + 2\,L_1L_3 + L_2L_3\big]\,,\end{aligned} \tag{62}$$

where the three polynomials are

$$p_1 = 9\lambda^3+25\lambda^2-\lambda-1\,, \qquad p_2 = -9\lambda^3+25\lambda^2+\lambda-1\,, \qquad p_3 = -3\lambda^3+15\lambda^2+3\lambda+1\,. \tag{63}$$

The right-hand side of eq. (62) vanishes to second order at $\lambda = 1$ and at $\lambda = \frac{1}{3}$, so V is finite at both ends of the physical interval. The ancillary files contain the 20 solutions in terms of multiple polylogarithms, the 14 constants in one basis, and the reduction of the result to eq. (62).

A.4 General triangles

For general momenta we have no differential operator, and we infer the result from the structure of eq. (62). In terms of the site energies $X = (2\lambda, \lambda, 1)$ the prefactors of the dilogarithms are $X_i\,\xi_i$, and the three polynomials are given by one expression,

$$p_i \;=\; q_{\mathcal{G}}\,\xi_j\,\xi_k + 8X_1X_2X_3\,, \tag{64}$$

where i, j and k are distinct. For $i = 1$ this expression is $(3\lambda+1)(1+\lambda)(3\lambda-1) + 16\lambda^2$, which equals p_1. The coefficient of $L_i\,L_j$ is $16\,X_1X_2X_3\,X_iX_j$. This suggests the ansatz

$$\begin{aligned}\frac{4\sqrt{2}\,q_{\mathcal{G}}\,\xi_1^2\,\xi_2^2\,\xi_3^2}{\pi}\,V \;&=\; -\sum_i \xi_i\,b(X_i;X_j,X_k)\,\mathrm{Li}_2(u_i) + \pi^2\,c(X) \\ &\quad + 16\,X_1X_2X_3\sum_{i<j} X_iX_j\,L_i\,L_j\end{aligned} \tag{65}$$

for an arbitrary triangle. Here $b(x;y,z)$ is a homogeneous polynomial of degree four that is symmetric in y and z, and c is a symmetric homogeneous polynomial of degree five. The polynomial b has 9 coefficients and c has 5. We restrict eq. (65) to the line and compare it with eq. (62). The three dilogarithms are linearly independent over the rational functions of λ, so the coefficient of each of them must agree as a polynomial in λ. This gives linear equations for the coefficients of b and c. Their unique solution is

$$b(x;y,z) \;=\; -x\,\big[(x{+}y{+}z)(x{-}y{+}z)(x{+}y{-}z){+}8\,x\,y\,z\big]\,, \qquad c(X) \;=\; -\frac{2}{3}\,X_1X_2X_3\sum_{i<j}\xi_i\,\xi_j\,. \tag{66}$$

With these polynomials eq. (65) becomes eq. (32).

The comparison on the line determines the 14 coefficients, but it does not test the form of the ansatz. We therefore compare eq. (32) with numerical values of the loop integral at 25 other triangles, which include two nearly folded and two soft configurations. A fit of the 14 coefficients to numerical values gives the polynomials of eq. (66), and the formula agrees with the numerical values at all 25 triangles. Equation (34) agrees with the numerical values on the line $X = P + \frac{1}{3}(1,1,1)$ and at other points with $X_v \geq P_v$.

A.5 Limits

We give three limits of eq. (32). The first two are the smallest and the largest value of $q_{\mathcal{G}}^2\,V$ in figure 3. For an equilateral triangle with $X_v = X$ all three arguments are $u_i = \frac{1}{3}$, and

$$V \;=\; \frac{\sqrt{2}\,\pi}{8X^2}\Big[\,11\,\mathrm{Li}_2\big(\tfrac{1}{3}\big) + 16\ln^2\tfrac{2}{3} - \frac{2\pi^2}{3}\Big]\,. \tag{67}$$

When the momentum at site 1 becomes soft, the other two site energies approach a common value X. The result is finite and has no logarithm of X_1,

$$V \;\longrightarrow\; \frac{3\sqrt{2}\,\pi}{128\,X^2}\,, \qquad X_1 \to 0\,. \tag{68}$$

For a folded configuration with $X_1 = X_2{+}X_3$ the denominator of eq. (32) vanishes to second order in ξ_1. The numerator vanishes to the same order by the identity $\mathrm{Li}_2(x){+}\mathrm{Li}_2(1{-}x) = \frac{\pi^2}{6} {-} \ln x\,\ln(1{-}x)$. With $X_{23} = X_2 + X_3$ and $x = X_3/X_{23}$ the limit is

$$V \;\longrightarrow\; \frac{\sqrt{2}\,\pi}{384\,X_{23}^2\,x^2(1-x)^2}\;N(x)\,, \qquad \xi_1 \to 0\,, \tag{69}$$

$$\begin{aligned} N(x) \;=\;& 6\,(2x^3 - 3x^2 + 3x - 1)\,\mathrm{Li}_2(x) + 6\,x\,(x^2 - x + 1)\ln x\,\ln(1-x) - 3\,x^2(x-2)\ln x \\ & + 3\,(x^3 - x^2 - x + 1)\ln(1-x) - \pi^2x^3 - 9\,x^2 + 9\,x\,. \end{aligned} \tag{70}$$

A.6 The functions of the general formula

Here we give the four functions of eq. (34). For a square root w and two polynomials x and y we write

$$\mathcal{D}[x;y] \;=\; \mathcal{D}\Big(\frac{x-w}{y-w}\,,\,\frac{x+w}{y+w}\Big)\,, \tag{71}$$

with $\mathcal{D}$ defined in eq. (36). The functions F_0 and F_e are defined by tetrahedra. Vertices 1, 2 and 3 are the corners of the triangle of momenta, and vertex 4 is the loop momentum. For six lengths $\ell_{12}, \ell_{13}, \ell_{23}, \ell_{14}, \ell_{24}, \ell_{34}$ the perimeters of the faces and of the closed paths are

$$\begin{aligned} f_1 &= \ell_{23} + \ell_{24} + \ell_{34}\,, & f_2 &= \ell_{13} + \ell_{14} + \ell_{34}\,, \\ f_3 &= \ell_{12} + \ell_{14} + \ell_{24}\,, & f_4 &= \ell_{12} + \ell_{13} + \ell_{23}\,, \\ c_1 &= \ell_{12} + \ell_{13} + \ell_{24} + \ell_{34}\,, & c_2 &= \ell_{12} + \ell_{14} + \ell_{23} + \ell_{34}\,, \\ c_3 &= \ell_{13} + \ell_{14} + \ell_{23} + \ell_{24}\,. \end{aligned} \tag{72}$$

The polynomial $\mathcal{P}(h)$ of eq. (37) is quadratic, its discriminant is the radicand R, and $w = \sqrt{R}$. We write $p_a = -\mathcal{P}'(a)$, so that $\mathcal{D}[p_a; p_0]$ is the function $\mathcal{D}_a$ of section 4.5. For F_0 the lengths are

$$(\ell_{12}, \ell_{13}, \ell_{23}) = (P_2, P_1, P_3)\,, \qquad (\ell_{14}, \ell_{24}, \ell_{34}) = \tfrac{1}{2}\big(X_1 + X_2 - X_3,\ X_2 + X_3 - X_1,\ X_1 + X_3 - X_2\big)\,, \tag{73}$$

and F_0 is given in eq. (38). For the edge $e = 12$ the lengths are

$$(\ell_{12}, \ell_{13}, \ell_{23}) = (P_2, P_1, P_3)\,, \qquad (\ell_{14}, \ell_{24}, \ell_{34}) = \big(\tfrac{1}{2}q_{\mathcal{G}},\ \tfrac{1}{2}q_{\mathcal{G}} - X_2,\ \tfrac{1}{2}q_{\mathcal{G}} - X_1\big)\,. \tag{74}$$

A prime denotes the same quantities with $\ell_{24} \to -\ell_{24}$ and $\ell_{34} \to -\ell_{34}$, which has the same radicand. Then

$$\begin{aligned} F_{12} \;=\; & -\mathcal{D}[p_{f_1}; p_0] + \mathcal{D}[p_{c_1}; p_0] + \sum_{k=2}^{4} \mathcal{D}[p'_{f_k}; p'_0] - \mathcal{D}[p'_{c_2}; p'_0] - \mathcal{D}[p'_{c_3}; p'_0] \\ & + \mathcal{D}[p'_0; p_{f_4}] - \mathcal{D}[p'_{f_4}; p_0]\,. \end{aligned} \tag{75}$$

The functions $F_{12}^{\pm}$ have $w = \sqrt{T_{12}^{\pm}}$,

$$\begin{aligned} F_{12}^{-} &= \mathcal{D}[a_1; a_2] + \mathcal{D}[a_3; a_7] + \mathcal{D}[a_4; a_7] - \mathcal{D}[a_5; a_6]\,, \\ F_{12}^{+} &= -\mathcal{D}[b_1; b_2] + \mathcal{D}[b_1; b_5] + \mathcal{D}[b_1; b_6] + \mathcal{D}[b_3; b_4] - \mathcal{D}[b_3; b_5]\,, \end{aligned} \tag{76}$$

with the polynomials

$$\begin{aligned} a_1 &= P_2^2 - (P_1 + P_3)^2 + 2X_1(P_1 + P_3) - 2P_1X_2\,, \\ a_2 &= P_2^2 - P_1^2 - P_3^2 + 2X_1(X_1 - X_2)\,, & a_3 &= P_3^2 - (P_1 - P_2)^2 + 2P_1X_2 - 2P_2X_1\,, \\ a_4 &= P_2^2 + P_3^2 - P_1^2 + 2X_2(X_1 - X_2)\,, & a_6 &= P_3^2 - (P_1 - P_2)^2 - 2P_1X_2 + 2P_2X_1\,, \\ a_5 &= (P_2 + P_3)^2 - P_1^2 + 2X_1P_2 - 2X_2(P_2 + P_3)\,, \\ a_7 &= P_2^2 - (P_1 + P_3)^2 - 2X_1(P_1 + P_3) + 2P_1X_2\,, \\ b_1 &= P_3^2 - P_1^2 - P_2^2 - 2X_1X_2\,, & b_2 &= P_3^2 - (P_1 - P_2)^2 - 2P_1X_2 - 2P_2X_1\,, \\ b_3 &= P_2^2 - (P_1 + P_3)^2 + 2X_1(P_1 + P_3) + 2P_1X_2\,, \\ b_4 &= P_2^2 - P_1^2 - P_3^2 + 2X_1(X_1 + X_2)\,, & b_5 &= P_2^2 + P_3^2 - P_1^2 - 2X_2(X_1 + X_2)\,, \\ b_6 &= (P_2 + P_3)^2 - P_1^2 + 2X_1P_2 + 2X_2(P_2 + P_3)\,. \end{aligned} \tag{77}$$

The other functions follow from relabelings of the sites. The functions F_{23} and $F_{23}^{\pm}$ follow from F_{12} and $F_{12}^{\pm}$ under $(1,2,3) \to (2,3,1)$, applied to the X_v and to the P_v. The functions F_{31} and $F_{13}^{\pm}$ follow under $(1,2,3) \to (1,3,2)$.

B Derivations for the box

In this appendix we derive the results of section 6. We first derive the residue at the single-site pole and the geometry of the threshold. We then reduce D_0 to Carlson integrals. The remaining subsections concern the line of section 6.4. They give the periods of its elliptic curves, the fifth-order operator, the conditions that determine u and v, and the expansions of C_4 and V_4 at large site energy.

B.1 The residue at the single-site pole

The pole at $X_1 + y_{41} + y_{12} = 0$ comes from the region of the time integrals in which t_1 is larger than the times of the two neighboring sites. For $t_1 > t_2$ the propagator (5) factorizes,

$$G_{y_{12}}(t_1, t_2) \;=\; \frac{e^{-y_{12}t_1}}{2y_{12}} \left(e^{\,y_{12}t_2} - e^{-y_{12}t_2}\right), \tag{78}$$

and the same holds for $G_{y_{41}}(t_1, t_4)$. The integral over t_1 from the larger of t_2 and t_4 to infinity is then

$$\int_{\max(t_2,t_4)}^{\infty} \mathrm{d}t_1 \; e^{-(X_1+y_{41}+y_{12})\,t_1} \;=\; \frac{e^{-(X_1+y_{41}+y_{12})\,\max(t_2,t_4)}}{X_1 + y_{41} + y_{12}}\,. \tag{79}$$

At the pole the exponential on the right-hand side equals one, and the residue is an integral over the times of the other three sites. Each of the two brackets of the form (78) adds $\pm y$ to the energy of a neighboring site, with the sign as a factor. The remaining integral is that of the chain of sites 2, 3 and 4, which gives eq. (46). The recursion for ψ_4 gives the same residue. For the triangle the same steps give eq. (28).

B.2 The threshold

At the pole the sum of the distances from $\vec{\ell}$ to two fixed points is constant. The points are those at which y_{12} and y_{41} vanish, and they are a distance P_1 apart. The loop momentum therefore lies on a spheroid with these points as foci. At the threshold $y_{41} + y_{12} = P_1$ the spheroid becomes the segment between the foci, on which $y_{12} = \frac{1}{2}(P_1 - \chi)$ and $y_{41} = \frac{1}{2}(P_1 + \chi)$ with $\chi = y_{41} - y_{12}$. The distance from a point of the segment to a third point follows from Stewart's theorem. Let C be a point at distance a from the focus at which y_{12} vanishes and at distance b from the other focus. The theorem gives

$$|\vec{\ell} - \vec{C}|^2 \;=\; \frac{(P_1 + \chi)\,a^2 + (P_1 - \chi)\,b^2}{2P_1} - \frac{P_1^2 - \chi^2}{4}\,. \tag{80}$$

With the two remaining vertices of the quadrilateral in place of C this is eq. (47). Both $z_{23} = y_{23}^2$ and $z_{34} = y_{34}^2$ are quadratic in χ with leading coefficient $\frac{1}{4}$. We write

$$z_{23} \;=\; \tfrac{1}{4}\,(\chi - e_1)(\chi - e_2)\,, \qquad z_{34} \;=\; \tfrac{1}{4}\,(\chi - e_3)(\chi - e_4)\,, \qquad 16\,Y^2 \;=\; \prod_{i=1}^{4}(\chi - e_i)\,, \tag{81}$$

so that $Y^2 = z_{23}\,z_{34}$. The roots are

$$e_{1,2} \;=\; -m \pm \sqrt{m^2 - n}\,, \qquad m = \frac{P_2^2 - P_{12}^2}{P_1}\,, \qquad n = 2\big(P_2^2 + P_{12}^2\big) - P_1^2\,, \tag{82}$$

and $e_{3,4}$ follow with $P_2 \to P_{41}$ and $P_{12} \to P_4$. For momenta that do not lie in a plane the roots form two complex-conjugate pairs.

B.3 Reduction of D_0 to Carlson integrals

The integrand of eq. (48) is a rational function of χ, y_{23} and y_{34}. We separate it according to its parity under $y_{23} \to -y_{23}$ and $y_{34} \to -y_{34}$. The part that is even in both square roots is rational in χ. The parts that are odd in one square root contain a single square root of a quadratic polynomial, and they integrate to logarithms. The part that is odd in both square roots is a rational function of χ divided by Y, and it gives the elliptic integrals.

The chain coefficient (10) has poles where one of six linear factors vanishes. Each zero lies at a point $\chi = \alpha$ on one of the four sheets of the two square roots. We label the sheets by the values $\eta = \pm y_{23}(\alpha)$ and $\eta' = \pm y_{34}(\alpha)$ of the roots at the pole. For momenta that do not lie in a plane all these poles are simple. Let ρ be the residue of the integrand at such a pole. The partial-fraction decomposition then gives eq. (50),

$$D_0 \;=\; \frac{\pi}{4P_1}\Bigg[\,\frac{1}{4}\sum_{(\alpha,\eta,\eta')}\rho\,\Big(L_0 + \eta\,L_{23} + \eta' L_{34} + \eta\,\eta'\,L_\Pi\Big) + 4\,p_0\,J_F\Bigg]\,, \tag{83}$$

with the integrals

$$L_0 = \int_{-P_1}^{P_1}\frac{\mathrm{d}\chi}{\chi-\alpha}\,,\qquad L_{23} = \int_{-P_1}^{P_1}\frac{\mathrm{d}\chi}{(\chi-\alpha)\,y_{23}}\,,\qquad L_\Pi = \int_{-P_1}^{P_1}\frac{\mathrm{d}\chi}{(\chi-\alpha)\,y_{23}\,y_{34}}\,, \tag{84}$$

and L_{34} is defined as L_{23} with y_{34} in place of y_{23}. The part that is odd in both square roots also contains a constant divided by Y, whose integral is proportional to the integral of the first kind $J_F = \int \mathrm{d}\chi/(4Y)$. The coefficient of J_F follows from the partial fractions at the branch point $\chi = e_1$,

$$p_0 \;=\; \frac{1}{4}\sum_{(\alpha,\eta,\eta')}\frac{\rho\,\eta\,\eta'}{\alpha-e_1}\,. \tag{85}$$

The first two integrals in eq. (84) are elementary,

$$\begin{aligned} L_0 &= \ln\frac{P_1-\alpha}{-P_1-\alpha}\,,\\ L_{23} &= -\frac{1}{\sqrt{z_{23}(\alpha)}}\left[\ln\frac{2\,z_{23}(\alpha) + z'_{23}(\alpha)\,(\chi-\alpha) + 2\sqrt{z_{23}(\alpha)\,z_{23}(\chi)}}{\chi-\alpha}\right]_{\chi=-P_1}^{\chi=P_1}\,. \end{aligned} \tag{86}$$

For the elliptic integrals we use the symmetric integrals of Carlson [36], which are $R_F(a,b,c) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\,[(\tau+a)(\tau+b)(\tau+c)]^{-1/2}$ and $R_J(a,b,c,p) = \frac{3}{2}\int_0^\infty \mathrm{d}\tau\,(\tau+p)^{-1}[(\tau+a)(\tau+b)(\tau+c)]^{-1/2}$. With $r_i = \sqrt{P_1 - e_i}$ and $\bar r_i = \sqrt{-P_1 - e_i}$ we define

$$U_{ij} \;=\; \frac{r_i\,r_j\,\bar r_k\,\bar r_l + \bar r_i\,\bar r_j\,r_k\,r_l}{2P_1}\,, \tag{87}$$

where i, j, k and l are distinct. The integral of the first kind is

$$J_F \;=\; \int_{-P_1}^{P_1}\frac{\mathrm{d}\chi}{4\,Y} \;=\; 2\,R_F\big(U_{12}^2, U_{13}^2, U_{14}^2\big)\,. \tag{88}$$

Since $Y = y_{23}\,y_{34}$, the integral of the third kind is $L_\Pi = 4\,(J_\Pi - J_F)/(\alpha - e_1)$ with

$$J_\Pi \;=\; \int_{-P_1}^{P_1}\frac{(\chi-e_1)\,\mathrm{d}\chi}{4\,(\chi-\alpha)\,Y} \;=\; \frac{2\,d_{12}\,d_{13}\,d_{14}}{3\,d_{15}}\,R_J\big(U_{12}^2, U_{13}^2, U_{14}^2, U_{15}^2\big) + \frac{1}{w}\,\ln\frac{S_5+w}{S_5-w}\,, \tag{89}$$

where $d_{ij} = e_j - e_i$ with $e_5 = \alpha$, and

$$\begin{aligned} U_{15}^2 &= U_{12}^2 - \frac{d_{13}\, d_{14}\, d_{25}}{d_{15}}\,, \qquad w^2 = S_5^2 + \frac{(P_1^2 - \alpha^2)\, U_{15}^2}{r_1^2\, \bar r_1^2}\,, \\ S_5 &= \frac{1}{2P_1}\left[\frac{r_2\, r_3\, r_4}{r_1}\,(-P_1 - \alpha) + \frac{\bar r_2\, \bar r_3\, \bar r_4}{\bar r_1}\,(P_1 - \alpha)\right]. \end{aligned} \tag{90}$$

Equations (83) to (90) are the complete result. We determine the branches of the logarithms and of the Carlson integrals by continuation from $\chi = -P_1$. The formulas in terms of R_F and R_J hold on a part of the segment on which the real quantities U_{ij} are positive. If one of them changes sign on the segment, we divide the segment at that point and add the integrals of the parts. Poles at the end points $\alpha = \pm P_1$ contribute their finite parts, and real poles inside the segment contribute principal values. The ancillary files contain a numerical implementation of these rules.

For momenta in a plane the factor $P_3 + y_{23} + y_{34}$ has a double zero at a point $\chi = \alpha_*$ on one sheet. On this sheet the integrand has the expansion $a_*/(\chi - \alpha_*)^2 + b_*/(\chi - \alpha_*)$ up to regular terms. We write $F(\alpha) = L_0 + \eta\, L_{23} + \eta'\, L_{34} + \eta\, \eta'\, L_\Pi$ for the bracket of eq. (83), where η and η' are the values of the square roots at $\chi = \alpha$ on the sheet. Since $\int \mathrm{d}\chi\, g(\chi)/(\chi - \alpha)^2$ is the derivative of $\int \mathrm{d}\chi\, g(\chi)/(\chi - \alpha)$ with respect to α, the double pole contributes

$$\rho\, F(\alpha) \;\longrightarrow\; a_*\, \frac{\mathrm{d}F}{\mathrm{d}\alpha}(\alpha_*) + b_*\, F(\alpha_*)\,, \qquad \frac{\rho\, \eta\, \eta'}{\alpha - e_1} \;\longrightarrow\; a_*\, \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\eta\, \eta'}{\alpha - e_1}\right]_{\alpha_*} + b_*\, \frac{\eta\, \eta'}{\alpha_* - e_1} \tag{91}$$

in eqs. (83) and (85). The derivatives are taken along the sheet. The ancillary implementation includes this term.

We obtain the monodromy of D_0 of section 6.6 by numerical continuation in one of the lengths of the quadrilateral. For two quadrilaterals that do not lie in a plane it contains elements with integer traces 4, 6, 18 and 20. The differential system of the residue (46) has rank 6 on one family of quadrilaterals and rank 4 on a second family. In both cases its solutions include the two periods of the curve. The system of a logarithmic form on the same curve has rank 1.

B.4 The elliptic curves on the line

On the line of section 6.4 three elliptic curves appear. They are the curves on which two partial energies vanish. The two partial energies are those of two adjacent sites, of two opposite sites, or of one site and of the pair of adjacent sites that contains it. We denote the curves of these three cases by E_{adj}, E_{opp} and E_{nest}, in this order.

The periods of all three are hypergeometric functions of the j-invariant. We write a curve in the Weierstrass form $Y_E^2 = 4X_E^3 - g_2\, X_E - g_3$. Under the rescaling $g_2 \to \nu^4 g_2$ and $g_3 \to \nu^6 g_3$ its periods are multiplied by ν^{-1}. A period is therefore $g_2^{-1/4}$ times a function of the invariant ratio

$$J \;=\; \frac{1728}{j} \;=\; 1 - \frac{27\, g_3^2}{g_2^3}\,. \tag{92}$$

This function obeys the hypergeometric equation with parameters $\frac{1}{12}$, $\frac{5}{12}$ and 1. For $0 < J < 1$ and $g_3 > 0$ the real period is

$$\varpi_E \;=\; \frac{2\pi}{(12\, g_2)^{1/4}}\; {}_2F_1\big(\tfrac{1}{12}, \tfrac{5}{12}; 1; J\big)\,. \tag{93}$$

For $g_3 < 0$ the same expression is the absolute value of the imaginary period. In both cases it is the period of smallest absolute value.

On the line j is a rational function of degree 24 in t for each curve. We denote by c_4 the polynomial of degree eight whose cube is the numerator of j. In a Weierstrass form with polynomial coefficients, g_2 is proportional to c_4. The Picard–Fuchs operator $\mathcal{L}_E$ of each curve is therefore the hypergeometric operator in the variable $J(t)$, conjugated by $c_4^{1/4}$,

$$\mathcal{L}_E \;=\; c_4^{-1/4} \Big[J(1-J)\,\frac{\mathrm{d}^2}{\mathrm{d}J^2} + \Big(1 - \frac{3J}{2}\Big)\frac{\mathrm{d}}{\mathrm{d}J} - \frac{5}{144} \Big]\, c_4^{1/4}\,, \tag{94}$$

with both sides normalized to unit leading coefficient in $\mathrm{d}/\mathrm{d}t$. Equation (94) holds exactly for the three operators of the reduction. The point $J = 0$ is $j = \infty$, where a curve degenerates. This happens at $t = 0$ for all three curves and at $t = -1+\sqrt{3/2}$ for E_{opp}. The variable t is not a modular function for any of the three curves. Each map $t \mapsto j$ has critical points away from $j = 0$, 1728 and ∞. These critical points are the apparent singularities of $\mathcal{L}_E$. We give the three j-invariants in the ancillary files.

B.5 The fifth-order operator

The order of the differential equation of the periods is 5 for the regular tetrahedron, 6 for tetrahedra whose opposite edges have equal lengths, and 8 for generic momenta. The fifth-order operator of section 6.3 arises in three ways. It is the operator of the periods of the surface as a function of $\Sigma = y_{41} + y_{12}$. It is the operator of the same surfaces along the line $X_v = 1 + t$, on which the pole of site 1 lies at $\Sigma = -(1 + t)$. It is also the irreducible factor of order 5 of the reduction on that line. With the substitution $\Sigma \to 1 + t$ the three operators are identical. The solutions have no logarithms at infinite Σ.

The operator is $\mathcal{L}_5 = \sum_{i=0}^{5} q_i(\Sigma)\,\mathrm{d}^i/\mathrm{d}\Sigma^i$ with

$$\begin{aligned}
q_5 &= \Sigma^2(\Sigma^2-1)^2(3\Sigma^2+1)(\Sigma^4-4\Sigma^2+12)(26\Sigma^4-41\Sigma^2+47)\,,\\
q_4 &= \Sigma\,(\Sigma^2-1)\big(2184\Sigma^{12}-10788\Sigma^{10}+30503\Sigma^8-38330\Sigma^6+33139\Sigma^4-2272\Sigma^2+1692\big)\,,\\
q_3 &= 19578\Sigma^{14}-93867\Sigma^{12}+232539\Sigma^{10}-318461\Sigma^8+274863\Sigma^6-70908\Sigma^4-3964\Sigma^2+1692\,,\\
q_2 &= 3\,\Sigma^3\big(21918\Sigma^{10}-83741\Sigma^8+168795\Sigma^6-189111\Sigma^4+115763\Sigma^2-7896\big)\,,\\
q_1 &= 18\,\Sigma^2\big(4082\Sigma^{10}-12389\Sigma^8+22570\Sigma^6-19379\Sigma^4+5392\Sigma^2+1316\big)\,,\\
q_0 &= 90\,\Sigma^5\big(182\Sigma^6-447\Sigma^4+846\Sigma^2-509\big)\,.
\end{aligned} \tag{95}$$

In a suitable basis the five solutions $\Pi_1, \ldots, \Pi_5$ obey

$$\Pi_1\,\Pi_2 + \Pi_3\,\Pi_4 + \Pi_5^2 \;=\; 0\,. \tag{96}$$

The symmetric square of $\mathcal{L}_5$ therefore has order 14 instead of 15. A relation of this form holds for the periods of a K3 surface whose transcendental lattice has rank 5.

The solutions of the fifth-order operator enter the discontinuity of V_4. For momenta that form a regular tetrahedron of unit side these solutions have branch points at the roots Σ_c of $\Sigma^4 - 4\Sigma^2 + 12$. A function that contains them has a term proportional to $(\Sigma - \Sigma_c)^{5/2}$. Its coefficient is fixed by the residue (46) at the point where two singular points of the surface coincide. With one leg per site the discontinuity of V_4 has this branch point, with a numerical exponent of 2.507 and with the predicted coefficient. With several legs per site the leading term of the residue at large site energies generates all five solutions, whereas a form of the opposite parity generates none of them. With one leg per site the differential system of the loop integrand of V_4 has rank 36, and that of the correlator has rank 34. The quotient of the former by the other forms at the single-site pole has rank 5, and its operator is the fifth-order operator.

B.6 The solutions on the line

The reduction on the line gives first-order systems for the master integrals and their coefficients in the expansion in ε. The master integrals have double poles in ε. The systems that include the double poles have dimension 81 for V_4 and 67 for C_4. The coefficient of the double pole of each master integral is a residue at infinite loop momentum, and it equals 0, $-\pi/8$ or $-\pi/16$. With these values V_4 and C_4 are solutions of quotient systems of dimension 34 and 28. Each second-order operator of an elliptic curve occurs twice, once for each order of the pole in ε. Only the one at the simple pole contributes to V_4 and C_4.

We compare the systems with numerical values of both functions. A fit of all boundary constants to values on a short interval reproduces the values of any smooth function, so it does not test the systems. We therefore use values close to the singular points and at large t, together with the conditions on the double poles. The solutions then reproduce the values that were not used in the fit, and the same fit to a different function fails.

An exact rational change of basis brings the system of dimension 28 to a form with simple poles only. The poles lie at the zeros of the 19 polynomials

$$\begin{aligned}
&3X^2+1, \quad 10X^2-1, \quad X^4-4X^2+12, \quad 16X^4-10X^2+3, \\
&X, \quad X+2, \quad X-2, \quad 3X+1, \quad 3X-1, \quad 2X^2+1, \quad 13X^2-4, \quad 12X^4-20X^2+9, \\
&X+1, \quad X-1, \quad 2X+1, \quad 2X-1, \quad 2X^2-1, \quad 2X^2-3, \quad 4X^4-4X^2+3\,,
\end{aligned} \tag{97}$$

and the system has no other singular points at finite X. At the zeros of the polynomials in the first line the solutions have square-root branch points, and at those of the second line they have logarithms. At those of the third line both occur.

The loop integral converges for $t > -1$, so C_4 is single-valued around the singular points of the system in that range. In the system of dimension 28 the solutions that are single-valued around the six points

$$t \;=\; 1\,, \quad -1+\sqrt{\tfrac{3}{2}}\,, \quad 0\,, \quad -1+\tfrac{1}{\sqrt{2}}\,, \quad -1+\tfrac{2}{\sqrt{13}}\,, \quad -\tfrac{1}{2} \tag{98}$$

form a space of dimension 4. Those that are also single-valued around the logarithmic point $t = -\frac{2}{3}$ form a space of dimension 3. For V_4 the corresponding dimensions are 8 and 7. The remaining singular point in this range is $t = -1 + 1/\sqrt{10}$. Exactly one solution in these spaces has no logarithms at large X. This solution is the function v, which we normalize to begin with X^{-4}. It is the elementary function of eq. (54), and its expansion is even in $1/X$ with rational coefficients,

$$v \;=\; X^{-4} + \frac{29}{80}\,X^{-6} + \frac{923}{12800}\,X^{-8} - \frac{1531}{204800}\,X^{-10} + O\big(X^{-12}\big)\,. \tag{99}$$

The function u has $\alpha_4 = 0$ and the coefficients β_4 to β_8 and α_5 of appendix B.7. Its three boundary constants at the double poles in ε equal $-\pi/8$. None of these conditions constrains the constant that multiplies v, and the condition at $t = -\frac{2}{3}$ does not constrain it either.

The condition at $t = -1 + 1/\sqrt{10}$ determines this constant. At this point $X^2 = \frac{1}{10}$, and v has a square-root branch point. Continuation around the point gives $v \to -v$ and $u \to u + 2c_v\, v$ with a number c_v. The sum $u + \Omega\, v$ is single-valued if and only if $c_v = \Omega$. We obtain c_v from a numerical continuation of the system around the point, and it agrees with the value of the integral in eq. (52) to 36 digits. The continuation around a root of $16X^4 - 10X^2 + 3$ gives the same number.

B.7 The expansion at large site energy

We now derive the leading coefficients of eq. (53). At large X the loop integral receives contributions from two regions, which we separate in $d = 3-2\bar{\varepsilon}$ dimensions. The regulator $\bar{\varepsilon} = -\varepsilon$ has the opposite

sign to that of eq. (13). In the soft region the loop momentum is of order one. We expand the integrand of the correlator in y_e/X,

$$\frac{1}{y_{12}\,y_{23}\,y_{34}\,y_{41}}\,\sum_{m\geq 0} X^{-4-m}\,\kappa_m(y)\,,\qquad \kappa_0=1\,,\qquad \kappa_1=-\tfrac{3}{2}\,\big(y_{12}+y_{23}+y_{34}+y_{41}\big)\,, \tag{100}$$

where κ_m is a polynomial of degree m. In the hard region the loop momentum is of order X. The integrand is homogeneous of degree -8, so that this region starts at order $X^{-5-2\bar\varepsilon}$. Poles in $\bar\varepsilon$ cancel between the two regions and leave logarithms of X.

The leading term comes from the soft region alone, and it is $\Omega\,X^{-4}$ with Ω defined in eq. (52). Its numerical value is $\Omega = 21.444295139357745195031667653426556$. To reduce Ω we introduce one Feynman parameter x_e for each factor $1/y_e = 1/|\vec\ell - \vec A_e|$, where the $\vec A_e$ are the vertices of the regular tetrahedron. The denominator becomes

$$\sum_e x_e\,(\vec\ell-\vec A_e)^2 \;=\; \Big(\vec\ell-\sum_e x_e\vec A_e\Big)^2+\sum_{e<f} x_e\,x_f \tag{101}$$

on the simplex $\sum_e x_e = 1$, because all distances between the vertices equal one. The Feynman-parameter formula for four factors of power one half has the prefactor $1/\pi^2$. With $\int \mathrm{d}^3\ell\,(\ell^2+\Delta)^{-2} = \pi^2\,\Delta^{-1/2}$ the integral over $\vec\ell$ gives

$$\Omega \;=\; \int_{\sum x=1} \frac{\mathrm{d}^3x}{\sqrt{x_1x_2x_3x_4}\,\sqrt{E_2(x)}}\,,\qquad E_2(x)=\sum_{e<f} x_e\,x_f\,. \tag{102}$$

The integral over the simplex can also be reduced to one dimension. With $\varrho = x(1-x)$ and $h = 1+6\varrho$ we define

$$m_\pm \;=\; \frac{1}{2}\Big(1\pm\frac{x^4-(1-x)^4}{h^2}-\frac{8\varrho\sqrt{(1+2\varrho)(4+9\varrho)}}{h^2}\Big)\,, \tag{103}$$

and find

$$\Omega \;=\; 2\sqrt{2}\,\pi^2\int_0^1 \frac{\mathrm{d}x}{\sqrt{h}}\;{}_2F_1\big(\tfrac14,\tfrac34;1;m_+\big)\;{}_2F_1\big(\tfrac14,\tfrac34;1;m_-\big)\,. \tag{104}$$

Each hypergeometric function is a complete elliptic integral of the first kind. We know no simpler form of eq. (104). To derive it, we write $x_1 = x\,\theta$, $x_2 = x\,(1-\theta)$, $x_3 = (1-x)\,\theta'$ and $x_4 = (1-x)\,(1-\theta')$. The integral over θ and θ' at fixed x is an Appell function,

$$\int_0^1\!\int_0^1 \frac{\mathrm{d}\theta\,\mathrm{d}\theta'}{\sqrt{\theta(1-\theta)\,\theta'(1-\theta')\,E_2}} \;=\; \frac{\pi^2}{\sqrt{\varrho}}\,F_2\Big(\tfrac12;\tfrac12,\tfrac12;1,1;-\frac{x}{4(1-x)},-\frac{1-x}{4x}\Big)\,. \tag{105}$$

Two identities for Appell functions [39] then give eq. (104). The first is the quadratic transformation

$$F_2(a;b,b';2b,2b';\xi,\zeta) \;=\; \Big(1-\tfrac{\xi+\zeta}{2}\Big)^{-a}F_4\Big(\tfrac{a}{2},\tfrac{a+1}{2};b+\tfrac12,b'+\tfrac12;\frac{\xi^2}{(2-\xi-\zeta)^2},\frac{\zeta^2}{(2-\xi-\zeta)^2}\Big)\,, \tag{106}$$

and the second is the product formula

$$F_4\big(a,b;c,a+b-c+1;m_+(1-m_-),m_-(1-m_+)\big) \;=\; {}_2F_1(a,b;c;m_+)\;{}_2F_1(a,b;a+b-c+1;m_-)\,. \tag{107}$$

At order X^{-5} the soft region contributes $-6\,T_3$, where T_3 is the integral of eq. (52) without the factor $1/y_{41}$. The steps that led to eq. (102) now give

$$T_3 \;=\; \int \mathrm{d}^d\ell\, \frac{1}{y_{12}\,y_{23}\,y_{34}} \;=\; \pi^{-\bar\varepsilon}\,\Gamma(\bar\varepsilon) \int_{\sum x=1} \frac{\mathrm{d}^2x}{\sqrt{x_1x_2x_3}}\, E_2(x)^{-\bar\varepsilon} \;=\; \frac{2\pi}{\bar\varepsilon} - 2\pi\big(\gamma_E + \ln\pi\big) - J_1 + O(\bar\varepsilon)\,, \tag{108}$$

with

$$\int_{\sum x=1} \frac{\mathrm{d}^2x}{\sqrt{x_1x_2x_3}} = 2\pi\,, \qquad J_1 = \int_{\sum x=1} \frac{\mathrm{d}^2x}{\sqrt{x_1x_2x_3}}\, \ln E_2(x) = 4\pi\ln 3 - 8\pi\,. \tag{109}$$

The pole of the soft region is therefore $-12\pi/\bar\varepsilon$. The hard region has the opposite pole, multiplied by $X^{-2\bar\varepsilon}$. The sum is finite and contains the logarithm $-24\pi\,X^{-5}\ln X$, which is the coefficient β_5 of eq. (53). The finite parts combine to

$$\alpha_5 \;=\; \frac{\pi}{12}\big(22 + 291\ln 3 - 108\ln 2\big)\,. \tag{110}$$

The higher orders follow in the same way. Each soft term is a one-loop integral with simple poles only, so that no higher power of the logarithm appears. A logarithm requires an ultraviolet pole of a soft term, which exists only at odd orders, so that $\beta_n = 0$ for even n. We find

$$\alpha_6 = -\frac{95}{3}\,\pi + 8\,\widetilde\Omega\,, \qquad \beta_7 = -\frac{158}{3}\,\pi\,, \qquad \alpha_7 = \pi\Big(\frac{22519}{360} + \frac{423}{8}\ln 3 - 16\ln 2\Big)\,, \tag{111}$$

where $\widetilde\Omega = -3.31425722084688564119181206 17$ is the dimensionally regularized value of the integral in eq. (52) with y_{12}^{-1} replaced by y_{12}. In terms of eq. (99) the coefficients are $\alpha_n = a_n\,\Omega + u_n$, where a_n and u_n are the coefficients of X^{-n} in v and in u, and the β_n come from u alone. The next two logarithms are $\beta_9 = -\frac{627}{10}\pi$ and $\beta_{11} = \frac{10247}{840}\pi$, which agree with the numerical values to 12 and 8 digits. The series converges for $|X| > 2$, beyond the outermost singular point of the system. We have found no relation between $\widetilde\Omega$ and known constants.

B.8 The wavefunction coefficient on the line

For the coefficient V_4 the integrand has no factor $1/\prod_e y_e$, and the expansion at large X starts at X^{-5},

$$V_4 \;=\; \sum_{n\geq 5} X^{-n}\big(\alpha_n^V + \beta_n^V \ln X\big)\,. \tag{112}$$

Through X^{-11} the coefficients are rational combinations of π, $\pi\ln 2$ and $\pi\ln 3$. The first logarithm appears at X^{-11} with $\beta_{11}^V = -\frac{11}{21}\pi$. In the space of dimension 8 these conditions leave two solutions without logarithms at large X, both of which begin at X^{-12}. Two exact conditions determine their coefficients. The first is single-valuedness at $t = -1 + 1/\sqrt{10}$, which is the condition that determines Ω in the correlator. The second is the absence of a pole at $t = 0$, where the loop integral is finite. With these conditions V_4 contains no undetermined constant, and it agrees with the numerical values to 32 digits. The conditions determine in particular the coefficients α_{12}^V and α_{14}^V of the expansion. The soft region gives the same two coefficients as integrals in dimensional regularization,

$$\alpha_{12}^V = 16\int \mathrm{d}^d\ell\; y_{12}\,y_{23}\,y_{34}\,y_{41} - \frac{11596}{5103}\,\pi\,, \qquad \alpha_{14}^V = 256\int \mathrm{d}^d\ell\; y_{12}^3\,y_{23}\,y_{34}\,y_{41} - \frac{2718675343}{220449600}\,\pi\,, \tag{113}$$

and the two determinations agree to 36 digits.

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